Convergence results for a class of spectrally hyperviscous models of 3-D turbulent flow

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Abstract

We consider the spectrally hyperviscous Navier–Stokes equations (SHNSE) which add hyperviscosity to the NSE but only to the higher frequencies past a cutoff wavenumber m0. In Guermond and Prudhomme (2003) [18], subsequence convergence of SHNSE Galerkin solutions to dissipative solutions of the NSE was achieved in a specific spectral-vanishing-viscosity setting. Our goal is to obtain similar results in a more general setting and to obtain convergence to the stronger class of Leray solutions. In particular we obtain subsequence convergence of SHNSE strong solutions to Leray solutions of the NSE by fixing the hyperviscosity coefficient µ while the spectral hyperviscosity cutoff m0 goes to infinity. This formulation presents new technical challenges, and we discuss how its motivation can be derived from computational experiments, e.g. those in Borue and Orszag (1996, 1998) [3,4]. We also obtain weak subsequence convergence to Leray weak solutions under the general assumption that the hyperviscous coefficient µ goes to zero with no constraints imposed on the spectral cutoff. In both of our main results the Aubin Compactness Theorem provides the underlying framework for the convergence to Leray solutions.

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1. Introduction

We obtain subsequence convergence results for solutions of the 3-D spectrally hyperviscous Navier–Stokes equations (SHNSE):

\[ u_t + νAu + μA_p u + (u \cdot \nabla) u + \nabla p = g, \] (1.1a)
\[ \nabla \cdot u = 0. \] (1.1b)

Here \( A = -Δ, u = (u_1, u_2, u_3) \) is the velocity field of the fluid, \( g = (g_1, g_2, g_3) \) is the external force, and \( p \) is the pressure. We have that \( u_i = u(x, t), g_i = g_i(x, t), i = 1, 2, 3, \) and \( p = p(x, t) \) where \( x \in Ω, \) a domain in \( \mathbb{R}^3. \) We assume that \( Ω \) is a periodic box; then “modding out” the constant vectors as in standard practice, \( A \) has eigenvalues \( 0 < λ_1 < λ_2 < \cdots \) with corresponding eigenspaces \( E_1, E_2, \ldots \). Let \( P_m \) be the projection onto \( E_1 \oplus \cdots \oplus E_m, \) let \( Q_m = I - P_m \) and let \( P_k \) be the projection onto each \( E_i. \) The general class of operators \( A_p \) considered in [1] satisfy \( A_p = \sum_{j=1}^{∞} a(λ_j) P_j \) such that (for a constant \( α > 1 \) ) \( A_p ≥ A_m \equiv Q_m A^α \) in the sense of quadratic forms, i.e., \( (A_p u, v) ≥ (A_m u, v). \)

Specializing to what would arise in typical computations we identify here an applicable distinguished class (ADC) in which \( A_p = \sum_{j=m_0+1}^{m} d_j λ_j^α P_j + Q_m A^α = \sum_{j=m_0+1}^{m} d_j P_j A^α + Q_m A^α \) where \( \{d_j\}_{j=m_0+1}^{m} \) is such that \( 0 < d_j ≤ 1 \) and \( d_j ↑ 1. \) Included in the ADC is the special case \( m_0 = m \) and \( d_{m_0+1} = d_{m+1} = 1 \) in which case \( a(λ) \) is a Heaviside function and \( A_p = A_m. \)

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The ADC derives much of its motivation from the basic idea in large-eddy-simulation (LES) of adding the divergence of a subgrid-scale (SGS) stress tensor to the Navier-Stokes equations (NSE) in order to model the effect of the subgrid scales on the resolvable scales. The most common LES approach models the SGS stress tensor using a scalar eddy viscosity. A number of standard implementations of this approach have well-known limitations, however (see e.g. [3,4,22]), and in response the concept of spectral-eddy viscosity (SEV) was introduced in [22]; therein it is demonstrated that for eddy viscosity to accurately model the effects of the subgrid scales, the eddy viscosity coefficient \( \nu_e \) must depend spectrally on \( \lambda \). As developed in [7,8] a working fit to the predictions of [22] has the non-dimensional form \( d_j = \nu_e(j) = \nu_0(j/N) = K_0^{-3/2} [0.441 + 15.2 \exp(-3.03N/j)] \) for \( j \leq N \) where \( K_0 = 2.1 \) denotes the Kolmogorov constant. It was suggested in [8] that the introduction of a hyperviscous term \( \nu_0 A^2 \) could approximate the spectral growth of \( \nu_e(j) \); see also [6] where arguments employing vector–calculus techniques and certain physical assumptions are used to show that a hyperviscous term of order \( \alpha = 2 \) can approximate the SGS tensor. The resulting technique has been widely employed; see the references in e.g. [6] and in particular the computational experiments in [3,4] which use the hyperviscosity term \( \nu^2 A^2 \) (and higher powers of it) in robust and effective application to high-Reynolds number regimes. Moreover, the computational results using hyperviscous terms are observed in [4] to be virtually identical to those using the full SGS stress tensor directly.

Since for high Reynolds numbers the hyperviscosity coefficient \( \mu = \nu^2 \) is small, it therefore makes sense that in some cases the hyperviscous term may be negligible when acting on the low-frequency components of \( u \). For higher frequencies the combined term \( \nu^2 \lambda_j \) would then become more significant due to the growth of \( \lambda_j \) with \( j \). Thus we can envision a cutoff \( m_0 \) such that \( \nu^2 \lambda_j \) is significant for \( j > m_0 \) and insignificant for \( j \leq m_0 \). In this way and for \( m = m_0 \) we can motivate the operator \( A_\nu = A_\nu^m = Q_m A^2 \) on physical grounds as a specific SGS modeling technique, and more subtle gradations of negligible vs. nonnegligible terms can be modeled by more general operators in the ADC. Such cases will be the focus of Theorem 2 below.

This idea of having a spectral cutoff \( m_0 > 0 \) in the ADC versions of the operators \( A_\nu \) is related to the spectral-vanishing-viscosity (SVV) method, introduced in the study of gas dynamics in [32] to obtain stable and accurate approximation to the correct entropy solution of conservation laws. In SVV the distribution of the coefficients \( d_j \) is similar to the SEV case but spectral viscosity is added only to the high wavenumber modes. SVV was first applied to the incompressible 3-D NSE in [19] for Reynolds numbers up to \( Re = 395 \), and was used in obtaining improved accuracy in long-term computations of 2-D flows past a circular cylinder in [31] with \( Re = 100 \) and \( Re = 500 \). Higher Reynolds numbers have since been treated effectively in 3-D using SVV, e.g. in [20] with \( Re = 1250 \) in a study of a triangular duct, in [29,28] in computing the turbulent wake of a cylinder with \( Re = 3900 \), and in [27] with \( Re = 768000 \) in a treatment of the Ahmed car–body problem. In these works SVV “can be thought of as using hyperviscous dissipation that will affect only the high Fourier modes” [19], and in fact the SVV terms qualitatively resemble spectrally-applied hyperviscous terms in truncation due to the generally exponential growth of the coefficients \( d_j \).

In fact spectral hyperviscosity, or superviscosity, has already been used in the modeling of gas dynamics in [33]; \( H^1 \)-stability was established, and convergence proven under the same \( L^\infty \)-stability assumption used in [31]. Applying spectral hyperviscosity directly to the 3-D NSE in computations was suggested in [19], discussed experimentally in [6], and advocated in [18,17]. With the coefficients \( \mu \) and \( m_0 \) chosen in (1.1) to depend on certain negative and positive powers, respectively, of the truncation order \( N \) according to a specific version of SVV methodology, it was shown in [18] that a subsequence of the resulting Galerkin solutions \( u_0 \) converges to a weak dissipative solution of the NSE using the definition of dissipative solution advanced in [5,12,30]. The choices of \( \mu \) and \( m_0 \) in this case are designed to tune the system (1.1) so as to “introduce the least possible dissipation while ensuring that the limit solution is dissipative” and the results represent the first of theoretical type (1.1) with \( 0 < m_0 < m \).

In [1] global regularity for the general class of \( A_\nu \) described above was established for \( \alpha \geq 5/4 \), generalizing the classical results in [26] which held for the case \( m_0 = m = 0 \). Estimates on the dimension of the attractor for (1.1) were obtained in [1] by adapting elements of the “CFT” framework [9,11,34,35] and the generalized Lieb–Thirring inequalities [34–36]. Additionally in [1] the machinery developed in [15,16] was adapted to establish the existence of an inertial manifold of dimension \( m_0 \) for \( A_\nu \) in the ADC case. In [2] the strong convergence of Galerkin solutions was obtained and continuous dependence on data was established with estimates optimized for the high frequencies.

In this paper we similarly show as in [18] that the system (1.1) approximates the NSE through subsequence convergence, but here the target solutions will be Leray solutions and we will consider a wider class of operators \( A_\nu \). First introduced and explored among the classical results in [23,25,24], we recall that \( u \) is a Leray solution of (1.1) if \( u \) solves an appropriate weak version of (1.1) (see (3.12) below) and also satisfies the energy inequality in a particular weak form (see (3.16) below). Let \( P \) be the Leray projection onto the solenoidal vectors; then setting as in standard practice \( H = PL^2(\Omega), V = PH^1(\Omega), \) and \( V' = V = PH^{-1}(\Omega) \), the following is the first of our main results.

**Theorem 1.** Let \( \{u_k\}_{k=1}^\infty \) be the corresponding strong solutions of (1.1) with \( u_k(0, 0) = u_0(0) \) and with \( \mu = \mu_k \) such that \( \mu_k \downarrow 0 \) as \( k \to \infty \). Assume that \( \alpha \geq 5/4 \), and let the corresponding \( A_\nu = A_{\mu_k} \) be in the applicable distinguished class. Then on each interval \([0, T]\) there exists a subsequence, also denoted \( \{u_k\}_{k=1}^\infty \), and a divergence-free vector \( u \in L^\infty([0, T]; H) \cap L^2([0, T]; V) \) such that \( u_k \to u \) strongly in \( L^2([0, T]; H) \), \( u_k \to u \) weakly in \( L^p([0, T]; V) \), \( \frac{d}{dt}u_k \to \frac{d}{dt}u \) weakly in \( L'([0, T]; PH^{-\alpha}) \), and \( u \) is a Leray weak solution of the NSE. Here \( r = 4/(5 - 2\alpha) \) if \( 5/4 \leq \alpha < 3/2 \) and \( r = 2 \) if \( \alpha \geq 3/2 \).

We note that the assumption \( \mu_k \downarrow 0 \) as \( k \to \infty \) is the only restriction we place on the \( \mu_k \) and there is no restriction placed on \( m_0 \) (in particular \( m_0 \) can be constant or arbitrarily depend on \( k \)). With \( A_\nu \) as in Theorem 1 we have that \( \mu_k \| A_{\mu_k} u \|_2 \leq \)
\[ \mu_k \| A^\alpha u \|_2^2 \sim \mu_k \| u \|_{H^{2\alpha}(\Omega)}^2 \] since \( 0 < \delta_1 \leq 1 \), so we also have the same sense of the hyperviscosity term \( A_\mu \) being “spectrally small” as in [18, Proposition 3.1]. In fact since Theorem 1 can be readily adapted to include Galerkin approximations to the solutions \( u_k \), it therefore strengthens the results in [18] by demonstrating that any scheme of dissipation represented by the \( \mu_k \) will, as long as \( \mu_k \downarrow 0 \) as \( k \to \infty \), guarantee subsequence convergence to a Leray solution. We also note that Leray solutions represent a significantly stronger subclass of the dissipative solutions.

We note also that \( \frac{d}{dt} u_k \rightarrow \frac{d}{dt} u \) in a different and weaker Banach space than in the classical results of Leray (see also [10] and the references contained therein), but in a stronger Banach space than in [18] if \( \alpha \geq 3/2 \). We also note that a result similar to Theorem 1 was established for the NS-alpha model in [13], along with global regularity and attractor results. In particular subsequence convergence to a weak solution in the sense of (3.12) below was established as \( \alpha \downarrow 0 \), and \( \frac{d}{dt} u \) in again a different and weaker Banach space than in the classical results of Leray.

We now examine the case in which \( \mu \) and the \( \delta_i \) are fixed (e.g. \( \mu = \nu^2 \) as in [4,5]), but \( m_0 \rightarrow \infty \); such a convergence case has not been considered theoretically from the practical point of view. Replacing \( m \) by \( k \) for a corresponding \( m = m_k \geq k \) where \( k \rightarrow \infty \), and denoting again the resulting \( A_\mu \) by \( A_{\mu,k} \), the following is our second main result.

**Theorem 2.** Let \( A_{\mu,k} \) be as above, let \( \{ u_k \}_{k=1}^\infty \) be the corresponding strong solutions of (1.1) such that \( k = m_0 \) and \( k \to \infty \), and assume that \( \alpha > 5/4 \). Then on each interval \( [0, T] \) there exists a subsequence, also denoted \( \{ u_k \}_{k=1}^\infty \), and a divergence-free vector \( u \in L^\infty([0, T]; H) \cap L^2([0, T]; V) \) such that \( u_k \rightarrow u \) strongly in \( L^2([0, T]; H) \), \( u_k \rightarrow u \) weakly in \( L^2([0, T]; V) \), \( \frac{d}{dt} u_k \rightarrow \frac{d}{dt} u \) weakly in \( L^{4/3}([0, T]; V^*) \) and \( u \) is an appropriate weak solution of (1.1). If we further assume that \( \alpha > 3/2 \) then \( u \) is a Leray weak solution of (1.1).

By an appropriate weak solution of (1.1) we mean that (3.12) is satisfied below. To obtain this we assume from the outset the same condition on \( \alpha \) as in [18]. To further obtain that \( u \) is a Leray weak solution, i.e. that the energy inequality (3.16) below is satisfied, we need the further assumption \( \alpha > 3/2 \), but this restriction leaves out no cases of practical interest that we are aware of.

In Theorem 2 we again have a sense in which adding the spectrally hyperviscous term produces a perturbation of theNSE which is suitably spectrally small. This will be evident in the Proof of Theorem 2 and will be discussed in detail in the conclusion; in particular, if \( P_k \) projects onto the first \( k \) eigenspaces of \( A \) and \( Q_k = I - P_k \), then we will show that a subsequence of \( P_k u_k \) converges weakly to \( u \) and that \( Q_k u_k \) vanishes so that in some sense \( P_k u_k \) is approximately a Galerkin solution. Also in Theorem 2 we have in this case that \( \frac{d}{dt} u_k \rightarrow \frac{d}{dt} u \) in exactly the same Banach space as in the classical result of Leray. In both Theorems 1 and 2 the Aubin Compactness Theorem plays a central role in establishing convergence to Leray solutions. After some preliminary observations and calculations in Section 2, we will prove Theorem 1 in Section 3 and Theorem 2 in Section 4.

**2. Preliminaries**

We express the Sobolev inequalities on \( \Omega \) in terms of the operator \( A = -\Delta \):

\[ \| u \|_{q} \leq M_1 \| A^{\alpha/2} u \|_p \] (2.1)

where \( q \leq 3p / (3 - 2\theta p) \) and \( M_1 = M_1 (\theta, p, q, \Omega) \). For the semigroup \( \exp(-tA) \) we have the decay estimate

\[ \| e^{-tA} u \|_2 \leq \| u \|_2 e^{-\lambda_1 t}. \] (2.2)

Like the standard NSE, (1.1) satisfies an energy inequality, which we derive as follows: taking the inner product of both sides of (1.1) with \( u \) we have that

\[ \frac{1}{2} \frac{d}{dt} \| u \|_2^2 + \nu \| A^{1/2} u \|_2^2 + \mu \| A^{1/2} u \|_2^2 = (g, u) \] (2.3)

where we note that since \( \text{div} \, u = 0 \) we have that \( (\nabla p, u) = 0 \) and \( (u \cdot \nabla) u, u) = -((\text{div} \, u) u, u) = 0 \). Now

\[ (g, u) = (A^{-1/2} g, A^{1/2} u) \leq \frac{v}{2} \| A^{1/2} u \|_2^2 + \frac{1}{2v} \| A^{-1/2} g \|_2^2; \] (2.4)

combining (2.4) with (2.3) and multiplying by 2 we have our basic energy inequality

\[ \frac{d}{dt} \| u \|_2^2 + \nu \| A^{1/2} u \|_2^2 + 2\mu \| A^{1/2} u \|_2^2 \leq \frac{1}{v \lambda_1} \| g \|_2^2 \] (2.5)

where we note that by Poincaré's inequality \( \| A^{-1/2} g \|_2 \leq \lambda_1^{-1/2} \| g \|_2 \); note that (2.5) reduces to the standard NSE energy inequality when \( \mu = 0 \). Integrating both sides of (2.5) we have that

\[ \| u \|_2^2 + \nu \int_0^T \| A^{1/2} u \|_2^2 \, ds + 2\mu \int_0^T \| A^{1/2} u \|_2^2 \, ds \leq \| u_0 \|_2^2 + \frac{1}{v \lambda_1} \int_0^T \| g \|_2^2 \, ds. \] (2.6)
Discarding the term \( \|A_{\psi}^{1/2}u\|_2^2 \) in (2.5) and again using Poincaré's inequality we obtain

\[
\frac{d}{dt} \|u\|_2^2 + \nu \lambda_1 \|u\|_2^2 \leq \frac{1}{\nu \lambda_1} \|g\|_2^2
\]  

so that, setting

\[ L_g = \sup_{t \geq 0} \|g\|_2 \]

we have that

\[
\frac{d}{dt} \|u\|_2^2 + \nu \lambda_1 \|u\|_2^2 \leq \frac{L_g^2}{\nu \lambda_1}.
\]  

(2.9)

Solving the differential inequality (2.9) we have that for \( u_0 = u(x, 0) \)

\[
\|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\nu \lambda_1 t} + \int_0^t \left( \frac{L_g^2}{\nu \lambda_1} \right) e^{-\nu \lambda_1 (t-s)} ds
\]

or, since \( \frac{L_g^2}{(\nu \lambda_1)} \) is a constant,

\[
\|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\nu \lambda_1 t} + \left( \frac{L_g}{\nu \lambda_1} \right)^2.
\]  

(2.10)

Thus, we have the a priori estimate

\[
\|u(t)\|_2^2 \leq \|u_0\|_2^2 + \left( \frac{L_g}{\nu \lambda_1} \right)^2 = U_g^2.
\]  

(2.11)

This concludes our preliminary observations and in the next section we will prove Theorem 1.

### 3. Proof of Theorem 1

For \( [t_0, t] \subset [0, T] \) we take the \( L^2([t_0, t]; H) \) inner product of both sides of (1.1) with a smooth (divergence-free) test function \( v = v(x, t) \) and using self-adjointness we obtain the variational formulation

\[
(u_k(t), v) + \int_{t_0}^t v \left( A_{\psi}^{1/2}u_k, A_{\psi}^{1/2}v \right) + \mu_k \left( A_{\psi}^{1/2}u_k, A_{\psi}^{1/2}v \right) + \left((u_k \cdot \nabla)u_k, v\right) ds = (u_k(t_0), v) + \int_{t_0}^t (g, v) ds
\]

(3.1)

where \((\cdot, \cdot)\) denotes the inner product in \( H \). Note that since the \( \alpha (\lambda_j) \) are positive real numbers for \( j \geq m_0 + 1 \) in the definition of \( A_{\psi} \), we can define \( A_{\psi}^{1/2} \) by \( A_{\psi}^{1/2} = \sum_{j=m_0}^{\infty} \alpha(\lambda_j)^{1/2} P_j \). We first establish suitable convergence of subsequences of \( \{u_k\}_{k=1}^{\infty} \). We first assume that \( \alpha \geq 3/2 \); applying \( A^{-\alpha/2} \) and the Leray projection \( P \) to both sides of (1.1), using (2.8) and (2.12), taking \( L^2 \)-norms, and noting that \( \alpha/2 \geq 3/4 \), we have for an appropriate \( M_1 \) from (2.1) that

\[
\|A^{-\alpha/2}(u_k)_t\|_2 \leq \nu \|A^{-\alpha/2}u_k\|_2 + \mu_k \|A^{-\alpha/2}A_{\psi}^{1/2}u_k\|_2 + \|PA^{-\alpha/2}(u_k \cdot \nabla u_k)\|_2 + \|A^{-\alpha/2}P g\|_2
\]

\[
\leq \nu \lambda_1^{1/2} \|A^{1/2}u_k\|_2 + \frac{\mu_k}{\lambda_1^{1/2}} \|A^{1/2}u_k\|_2 + \|A^{-\alpha/2}A_{\psi}^{1/2}u_k\|_2 + \|A^{-\alpha/2}A_{\psi}^{1/2}u_k\|_2 + \lambda_1^{-\alpha/2} \|g\|_2
\]

\[
\leq \frac{\nu \lambda_1^{1/2} \lambda_1^{1/2}}{\lambda_1^{1/2}} \|\nabla u_k\|_2 + \frac{\mu_k}{\lambda_1^{1/2}} \|A^{1/2}u_k\|_2 + \frac{1}{\lambda_1^{1/2}} M_1 \|u_k\|_2 \|\nabla u_k\|_2 + \lambda_1^{-\alpha/2} \|g\|_2
\]

\[
\leq \nu \lambda_1 \|\nabla u_k\|_2 + \mu_k \lambda_1^{-\alpha/2} \|A_{\psi}^{1/2}u_k\|_2 + \lambda_1^{-\alpha/2} M_1 \|u_k\|_2 \|\nabla u_k\|_2 + \lambda_1^{-\alpha/2} L_g^2
\]

(3.2)

where we have used Poincaré's inequality in the second line of (3.2). Squaring both sides of (3.2), there is a constant \( C_0 \) such that

\[
\|A^{-\alpha/2}(u_k)_t\|_2^2 \leq C_0 \left[ \nu^2 \lambda_1 \|\nabla u_k\|_2^2 + \mu_k^2 \lambda_1^{-\alpha} \|A_{\psi}^{1/2}u_k\|_2^2 + \lambda_1^{-\alpha/2} M_1 \|u_k\|_2 \|\nabla u_k\|_2^2 + \lambda_1^{-\alpha/2} L_g^2 \right].
\]  

(3.3)

From (3.3) and from \((\mu_k)^2 \leq \mu_k \) (without loss of generality we can assume that \( \mu_k \leq 1 \)) we have that there is a constant \( C_1 = C_1(\nu, \lambda_1, \alpha, M_1, U_k, L_g) \) such that

\[
\|A^{-\alpha/2}(u_k)_t\|_2^2 \leq C_1 \left[ \nu \|A_{\psi}^{1/2}u\|_2^2 + 2\mu_k \|A_{\psi}^{1/2}u\|_2^2 + \lambda_1^{-\alpha} L_g^2 \right].
\]  

(3.4)
Integrating both sides of (3.4) and using (2.6), (2.8) we have that
\[
\int_0^T \left\| A^{-\alpha/2} (u_k)_t \right\|_2^2 \, dt \leq C_1 \left[ \left\| u_0 \right\|_2^2 + \frac{1}{\nu \lambda_1} L_g^2 T \right] + \lambda_1^{-\alpha} L_g^2 T. \tag{3.5}
\]

From (3.5) we thus have that \( \frac{d}{dt} u_k \) is uniformly bounded in \( L^2([0, T]; PH^{-\alpha}) \). If \( 5/4 \leq \alpha < 3/2 \), then the third term in the second line of (3.2) can be estimated by \( \left\| A^{-\alpha/2} (u_k \cdot \nabla u_k) \right\|_2 \leq M' \left\| u_k \right\|_2^{5/2-\alpha} \left\| \nabla u_k \right\|_2^{5/2-\alpha} \leq M' U_g^{-1/2} \left\| u_k \right\|_2^{5/2-\alpha} \) for an appropriate constant \( M' \) (see e.g. [10, Chapter 6]); substituting the right-hand side of this for the third term in the last line of (3.2), it then follows from (2.6) that each term in the last line of (3.2) and hence \((u_k)_t\) is uniformly bounded in \(L'(([0, T]; PH^{-\alpha}))\) for \( r = 4/(5 - 2\alpha) \). For what follows next we need the Aubin Compactness Theorem; the following is the version stated in [14, p. 224]:

**Theorem 3 (Aubin Compactness Theorem).** Let \( X_0, X, X_1 \) be three Banach spaces and suppose that \( X_0 \) and \( X_1 \) are reflexive, \( X_0 \subset X \) with compact injection and \( X \subset X_1 \) with continuous injection. Let \( T > 0 \) and \( p_0, p_1 > 1 \). Consider the space
\[
Y = \left\{ v \in L^{p_0}([0, T]; X_0) : \frac{d}{dt} v \in L^{p_1}([0, T]; X_1) \right\}
\]
endowed with the norm \( \left\| v \right\|_Y = \left\| v \right\|_{L^{p_0}([0, T]; X_0)} + \frac{d}{dt} \left\| v \right\|_{L^{p_1}([0, T]; X_1)} \). Then the injection of \( Y \) into \( L^{p_0}([0, T]; X) \) is compact.

We set \( X_0 = V, X = H, X_1 = PH^{-\alpha}(\Omega) \), and set \( p_0 = 2, p_1 = r \), where as in Theorem 1 \( r = 4/(5 - 2\alpha) \) if \( 5/4 \leq \alpha < 3/2 \) and \( r = 2 \) if \( \alpha \geq 3/2 \). Note that \( X_1 \) has the norm \( \left\| v \right\|_{PH^{-\alpha}(\Omega)} = \left\| A^{-\alpha/2} v \right\|_2 \). Standard Sobolev theory shows that the hypotheses of Theorem 3 are met. From (2.6) and (3.5) we have that the sequence \( \{u_k\}_{k=1}^{\infty} \) is contained in closed balls in \( X_0, X, X_1 \), from which we conclude from Theorem 3 that there is a subsequence (also denoted \( \{u_k\}_{k=1}^{\infty} \) contained in \( L^2([0, T]; V) \) and a \( u \in L^2([0, T]; V) \) such that \( \|u_k - u\|_{L^2([0, T]; H)} \to 0 \), \( u_k \to u \) weakly in \( L^2([0, T]; V) \), and \((u_k)_t \to u_t \) weakly in \( L'(([0, T]; PH^{-\alpha}(\Omega)) \) as \( k \to \infty \). From the weak convergence in \( L^2([0, T]; V) \) it follows that
\[
v \int_0^t (A^{1/2} u_k, A^{1/2} v) \, ds \to v \int_0^t (A^{1/2} u, A^{1/2} v) \, ds \tag{3.6}
\]
as \( k \to \infty \), while using that \( |d_j| \leq 1 \) we have for each \( t \) that
\[
\left| \mu_k(A^{1/2} u_k(t), A^{1/2} v(t)) \right| = \mu_k \left( (u_k(t), A v(t)) \right) \leq \mu_k \left( u_k(t), A v(t) \right)_2 \leq \mu_k \|A^\alpha v(t)\|_2 \to 0 \tag{3.7}
\]
and thus by the Dominated Convergence Theorem we have that
\[
\int_0^t \mu_k(A^{1/2} u_k, A^{1/2} v) \, ds = \int_0^t \mu_k(u_k, A v) \, ds \to 0 \tag{3.8}
\]
as \( k \to \infty \). Note that (3.7) is independent of the choice of \( m_0 \) and in particular of any potential dependence of \( A_v \) on \( k \). Since \( u_k \to u \) weakly in \( L^2([0, T]; PH^1(\Omega)) \) we may also assume that \( u_k(t) \to u(t), u_k(t_0) \to u(t_0) \) weakly in \( PH^1(\Omega) \) for all \( t, t_0 \in [0, T] \setminus E \) where \( E \) has Lebesgue measure zero, so that
\[
(u_k(t_0), v) \to (u(t_0), v), (u_k(t), v) \to (u(t), v) \tag{3.9}
\]
as \( k \to \infty \) for all \( t, t_0 \in [0, T] \setminus E \). For the nonlinear term we can use the classical Galerkin-approximation proof of weak Leray solutions of theNSE or directly note that for the appropriate tensor product \( \otimes \) we have that
\[
\left| ((u_k \cdot \nabla) u_k, v) - ((u \cdot \nabla) u, v) \right| \leq \left| ((u_k - u) \cdot \nabla) u_k, v) \right| + \left| ((u \cdot \nabla) (u_k - u), v) \right| \leq \left| ((u_k - u) \cdot \nabla) u_k \right|_2 \|v\|_\infty + |(\nabla (u_k - u), u \otimes v) \right| \leq \|u_k - u\|_2 \|\nabla u_k\|_2 \|v\|_\infty + \|\nabla (u_k - u), u \otimes v) \right| \tag{3.10}
\]
where in the third line we use the estimate \( \left| ((u_k - u) \cdot \nabla) u_k, v) \right| \leq \left| ((u_k - u) \cdot \nabla) u_k \right|_1 \|v\|_\infty \) and observe that \( u \otimes v \) is in \( L^2(\Omega) \) since \( \|u \otimes v\|_2 \leq \|u\|_2 \|v\|_\infty \). From this it follows by (2.6), the Dominated Convergence Theorem, from \( u_k \to u \) weakly in \( L^2([0, T]; PH^1(\Omega)) \), and from \( u_k \to u \) strongly in \( L^2([0, T]; PL^2(\Omega)) \) that
\[
\int_0^t (g, v) + (u_k, (u_k \cdot \nabla) v) \, ds \to \int_0^t (g, v) + (u, (u \cdot \nabla) v) \, ds \tag{3.11}
\]
Combining (3.6), (3.8), (3.9), and (3.11) with (3.1) we have that
\[(u(t), v) + v \int_{t_0}^{t} (A^{1/2} u, A^{1/2} v) + ((u, \nabla) u, v) ds = (u(t_0), v) + \int_{t_0}^{t} (g, v) ds\]  
(3.12)
and thus u is a suitable weak solution of the NSE.

Meanwhile from (2.3) we have, neglecting the term involving $A^{1/2}_u u_k$, that
\[
\frac{1}{2} \|u_k(t)\|_2^2 + v \int_{t_0}^{t} \|A^{1/2} u_k\|_2^2 ds \leq \frac{1}{2} \|u_k(t_0)\|_2^2 + \int_{t_0}^{t} (g, u) ds.
\]
(3.13)
For $t_0 \in [0, T] \setminus \mathcal{E}$ we have that
\[
\frac{1}{2} \|u_k(t_0)\|_2^2 + \int_{t_0}^{t} (g, u) ds \to \frac{1}{2} \|u(t_0)\|_2^2 + \int_{t_0}^{t} (g, u) ds
\]
(3.14)
and we have that
\[
\limsup_{k \to \infty} \left[ \frac{1}{2} \|u_k(t)\|_2^2 + v \int_{t_0}^{t} \|A^{1/2} u_k\|_2^2 ds \right] \geq \limsup_{k \to \infty} \frac{1}{2} \|u_k(t)\|_2^2 + v \liminf_{k \to \infty} \int_{t_0}^{t} \|A^{1/2} u_k\|_2^2 ds
\geq \frac{1}{2} \|u(t)\|_2^2 + v \int_{t_0}^{t} \|A^{1/2} u\|_2^2 ds
\]
(3.15)
where we have used Fatou’s lemma in the last line of (3.15). Combining (3.14) with (3.15) we have that
\[
\frac{1}{2} \|u(t)\|_2^2 + v \int_{t_0}^{t} \|A^{1/2} u\|_2^2 ds \leq \frac{1}{2} \|u(t_0)\|_2^2 + \int_{t_0}^{t} (g, u) ds
\]
(3.16)
and hence u satisfies the standard energy inequality. Combining this with (3.12) we have that u is a Leray weak solution of the NSE. This completes the Proof of Theorem 1.

4. Proof of Theorem 2

Let $P_k$ be the projection onto the first $k = m_0$ eigenspaces of $A$ and let $Q_k = I - P_k$. With $A_\varphi = A_{\varphi,k}$ and with the decomposition $u_k = P_k u_k + Q_k u_k$ the variational formulation (3.1) becomes
\[
(P_k u_k(t), v) + v \int_{t_0}^{t} \left( A^{1/2} P_k u_k, A^{1/2} v \right) + \mu \left( A^{1/2} P_k u_k, A^{1/2} v \right) + ((P_k u_k \cdot \nabla) P_k u_k, v) ds
= (P_k u_k(t_0), v) + \int_{t_0}^{t} (g, v) ds - Q_k (u, v)
\]
(4.1)
where
\[
Q_k (u, v) = (Q_k u_k(t), v) + \int_{t_0}^{t} \left( A^{1/2} Q_k u_k, A^{1/2} v \right) + \mu \left( A^{1/2} Q_k u_k, A^{1/2} v \right) - (Q_k u_k(t_0), v) ds + Q \mathcal{N}_k (u, v)
\]
(4.2)
with
\[
Q \mathcal{N}_k (u, v) = \int_{t_0}^{t} \left( (Q_k u_k \cdot \nabla) P_k u_k, v \right) + \left( (Q_k u_k \cdot \nabla) Q_k u_k, v \right) + \left( (Q_k u_k \cdot \nabla) Q_k u_k, v \right) ds.
\]
(4.3)
We first show that $Q_k (u, v) \to 0$ as $k \to \infty$. For this it will be useful to show for large enough $\beta$ that $\|A^{\beta/2} Q_k u_k(t)\|_2 \to 0$ uniformly in $t$ as $k \to \infty$. In fact we will show this provided that $\alpha > 5/4$ and $0 \leq \beta < 2\alpha - 5/2$. Taking the $H$-inner product of both sides of (1.1) with $A^{\beta} Q_k u_k = Q_k A^{\beta} u_k$, and noting that $(u, \nabla) u_k, Q_k A^{\beta} u_k = (Q_k (u_k \cdot \nabla)) P_k u_k, A^{\beta} Q_k u_k + (Q_k (u_k \cdot \nabla) Q_k u_k, A^{\beta} Q_k u_k)$ where we have used that $Q_k^2 = Q_k$, we have that
\[
\frac{1}{2} \frac{d}{dt} \|A^{\beta/2} Q_k u_k\|_2^2 + v \|A^{(1+\beta)/2} Q_k u_k\|_2^2 + \mu \|A^{\beta/2} A^{1/2} Q_k u_k\|_2^2
\leq |(Q_k (u_k \cdot \nabla) P_k u_k, A^{\beta} Q_k u_k)| + |(Q_k (u_k \cdot \nabla) P_k u_k, A^{\beta} Q_k u_k)| + (Q_k g, A^{\beta} Q_k u_k)
\leq |(A^{(-\alpha/2)} Q_k (u_k \cdot \nabla) P_k u_k, A^{(\alpha+\beta)/2} Q_k u_k)| + |(A^{(-\alpha/2)} Q_k (u_k \cdot \nabla) Q_k u_k, A^{(\alpha+\beta)/2} Q_k u_k)|
+ |(A^{(1-\beta)/2} Q_k g, A^{(1+\beta)/2} Q_k u_k)|.
\]
(4.4)
Using that $A_{k+1}^{a/2}Q_{uk} \geq d_{k+1}A^{a/2}Q_{uk}$ in the sense of quadratic forms, setting $d = \mu d_{k+1}$, and using Young's inequality we have that

$$
\frac{1}{2} \frac{d}{dt}\|A^{a/2}Q_{uk}\|_2^2 + v\|A^{(1+\beta)/2}Q_{uk}\|_2^2 + d\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 \\
\leq \frac{1}{d}\|A^{-(\alpha-\beta)/2}Q_{uk}\|_2^2 + \frac{d}{4}\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 + \frac{1}{d}\|A^{-(\alpha-\beta)/2}Q_{uk}\|_2^2 \\
+ \frac{d}{4}\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 + \frac{1}{2v}\|A^{-(1+\beta)/2}Q_{uk}\|_2^2 + \frac{v}{2}\|A^{(1+\beta)/2}Q_{uk}\|_2^2 .
$$

(4.5)

After combining the terms in (4.5), multiplying by 2, using Poincaré on the term involving $g$, neglecting the term $v\|A^{(1+\beta)/2}Q_{uk}\|_2^2$, and using (2.8), we have that

$$
\frac{d}{dt}\|A^{\beta/2}Q_{uk}\|_2^2 + d\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 \leq \frac{2}{d}\|A^{-(\alpha-\beta)/2}Q_{uk}\|_2^2 \\
+ \frac{2}{d}\|A^{-(\alpha-\beta)/2}Q_{uk}\|_2^2 \|P_{uk}\|_2^2 + \frac{1}{v\lambda_{k+1}^{1+\beta}}\|Q_{g}\|_2^2 \\
\leq \frac{2}{d}\|A^{-(\alpha-\beta)/2}\|_2^2 \|P_{uk}\|_2^2 + \frac{2}{d}\|A^{-(\alpha-\beta)/2}Q_{uk}\|_2^2 \|Q_{g}\|_2^2 + \frac{1}{v\lambda_{k+1}^{1+\beta}}\|g\|_2^2 \\
\leq \frac{2}{d\lambda_{k+1}^{\alpha-\beta}}\|Q_{uk}\|_{\infty}^2 \|\nabla P_{uk}\|_{\infty}^2 + \frac{2}{d\lambda_{k+1}^{\alpha-\beta}}\|Q_{uk}\|_{\infty}^2 \|\nabla Q_{uk}\|_{\infty}^2 + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2.
$$

(4.6)

Using (2.1) with the appropriate constant $M_2$, we have from (2.12), (4.6), and Poincaré's inequality that

$$
\frac{d}{dt}\|A^{\beta/2}Q_{uk}\|_2^2 + d\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 \leq \frac{2M_2}{d\lambda_{k+1}^{\alpha-\beta}}U_{y_{k+1}}^2\|A^{5/4}P_{uk}\|_2^2 + \frac{2M_2}{d\lambda_{k+1}^{\alpha-\beta}}U_{y_{k+1}}^2\|A^{5/4}Q_{uk}\|_2^2 + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2 \\
\leq \frac{2M_2}{d\lambda_{k+1}^{\alpha-\beta}}U_{y_{k+1}}^2\|P_{uk}\|_2^2 + \frac{2M_2}{d\lambda_{k+1}^{\alpha-\beta}}U_{y_{k+1}}^2\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2 \\
\leq \frac{2M_2U_{y_{k+1}}^2}{d\lambda_{k+1}^{\alpha-\beta}} + \frac{2M_2U_{y_{k+1}}^2}{d\lambda_{k+1}^{\alpha-\beta}}\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2.
$$

(4.7)

Next we choose $k$ large enough so that

$$
\lambda_{k+1}^{2a-5/2} \geq 4M_2U_{y_{k+1}}^2d^{-2}
$$

(4.8a)

which guarantees that

$$
\frac{2M_2U_{y_{k+1}}^2}{d\lambda_{k+1}^{\alpha-\beta}} \leq \frac{d}{2}.
$$

(4.8b)

Using the condition (4.8) in (4.7) and combining terms, we obtain that

$$
\frac{d}{dt}\|A^{\beta/2}Q_{uk}\|_2^2 + (d/2)\|A^{(\alpha+\beta)/2}Q_{uk}\|_2^2 \leq \frac{2M_2U_{y_{k+1}}^4}{d\lambda_{k+1}^{\alpha-\beta}} + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2
$$

(4.9)

from which after applying Poincaré's inequality again we have that

$$
\frac{d}{dt}\|A^{\beta/2}Q_{uk}\|_2^2 + (d/2)\lambda_{k+1}^{2a}\|A^{\beta/2}Q_{uk}\|_2^2 \leq \frac{2M_2U_{y_{k+1}}^4}{d\lambda_{k+1}^{\alpha-\beta}} + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2.
$$

(4.10)

Integrating the inequality (4.10) we have that

$$
\|A^{\beta/2}Q_{uk}(t)\|_2^2 \leq \|A^{\beta/2}Q_{uk}\|_2^2 + \int_0^t \left[ \frac{2M_2U_{y_{k+1}}^4}{d\lambda_{k+1}^{\alpha-\beta}} + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2 \right] e^{-(d/2)\lambda_{k+1}^{2a}(t-s)} ds
$$

$$
\leq \|A^{\beta/2}Q_{uk}\|_2^2 + \int_0^t \left[ \frac{2M_2U_{y_{k+1}}^4}{d\lambda_{k+1}^{\alpha-\beta}} + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2 \right] e^{-(d/2)\lambda_{k+1}^{2a}(t-s)} ds
$$

$$
\leq \|A^{\beta/2}Q_{uk}\|_2^2 + \int_0^t \left[ \frac{2M_2U_{y_{k+1}}^4}{d\lambda_{k+1}^{\alpha-\beta}} + \frac{1}{v\lambda_{k+1}^{1+\beta}}l_{g}^2 \right] e^{-(d/2)\lambda_{k+1}^{2a}(t-s)} ds
$$
$$\parallel p \parallel (3.10)$$

that

$$\parallel x \parallel \beta/\parallel \gamma \parallel < \beta$$

so that also

$$\parallel A^{\gamma/2}Q_k u_k (t) \parallel _2 \rightarrow 0 \text{ as } k \rightarrow \infty \text{ uniformly in } t.$$
on $L^2(\Omega)$ with $\|A^{-1/2}\text{div}\|_2 \leq 1$, using (2.1) for an appropriate constant $M_3$, and using (4.11) with $\beta = 1/2$, we have that

$$\|(P (u_k \cdot \nabla) Q_k u_k, P_k u_k)\| = \|(u_k \cdot \nabla) Q_k u_k, P_k u_k)\| = \left\| (A^{-1/2} (u_k \cdot \nabla) Q_k u_k, A^{1/2} P_k u_k) \right\|$$

$$\leq \left\| A^{-1/2}\text{div}(u_k \otimes Q_k u_k) \right\|_2 \|P_k A^{1/2} u_k\|_2$$

$$\leq \left\| A^{-1/2}\text{div}(\|u_k\|_6) Q_k u_k \right\|_2 \|A^{1/2} u_k\|_2$$

$$\leq M_3 \|A^{1/2} u_k\|_2 \|A^{1/4} Q_k u_k\|_2 \|A^{1/2} u_k\|_2$$

$$\leq M_3 \left[ \|A^{1/4} Q_k u_k\|_2^2 + \frac{4M_3 U_k^4}{d^2 \lambda_k^{-3}} + \frac{2}{d^2 \lambda_k^{-1/2} L_k^2} \right]^{1/2} \|\nabla u_k\|_2^2.$$  (4.17)

If $\alpha = 3/2 > 0$ as in the second assumption on $\alpha$ in Theorem 2, then the term in brackets in (4.17) goes to zero for each $k$ as $k \to \infty$, so by (2.6), (4.17), and the Dominated Convergence Theorem the third term on the right-hand side of (4.16) goes to zero as $k \to \infty$. For a set $E$ of Lebesgue measure zero and for $t_0 \in [0, T) \setminus E$ we thus have that

$$\frac{1}{2} \left\| P_k u_k(t_0) \right\|_2^2 + \int_{t_0}^t (g, P_k u_k) \, ds - \int_{t_0}^t (P (u_k \cdot \nabla) Q_k u_k, P_k u_k) \, ds \to \frac{1}{2} \left\| u(t_0) \right\|_2^2 + \int_{t_0}^t (g, u) \, ds.$$  (4.18)

Since (3.15) holds as before with $P_k u_k$ replacing $u_k$, we thus have that (3.16) as well as (3.12) hold for the $u$ produced here in the development following (4.15). Hence $u$ is a Leray weak solution of the NSE and this completes the Proof of Theorem 2.

5. Conclusion

That the applicable distinguished class operators $A_\alpha$ in Theorem 2 represent a “small” hyperviscosity perturbation of the NSE is reflected both in terms of the estimate (4.11) and the mechanics of showing that (3.12) and (3.16) are satisfied. In particular the perturbation term $Q_k(u, v)$ goes to zero as $k \to \infty$ and similarly the term involving $Q_k u_k$ in (4.18) goes to zero as well. Taken together, the convergence behavior of these perturbative terms shows that in a tangible sense $P_k u_k$ for large $k$ is close to and behaves like a Galerkin solution for the NSE.

We observe also that (4.11) gives an estimate that streamlines the proof of global regularity for (1.1); in fact $\|A^{1/2} u(t)\|_2^2 = \|P_k u(t)\|_2^2 + \|A^{1/2} Q_k u(t)\|_2^2 \leq \lambda_k \|P_k u(t)\|_2^2 + \|A^{1/2} Q_k u(t)\|_2^2$ and the latter term is bounded by (4.11). We also note that if $k$ is larger than the Kolmogorov wavenumber $\lambda_k$ we have an explicit estimate in (4.11) which reflects the Kolmogorov theory [21] that wavenumbers in the dissipative range are insignificant dynamically; this is shown here in a concrete theoretical sense where (4.11) is used to demonstrate that $Q_k(u, v) \to 0$ as $k \to \infty$.

As we have noted Theorem 2 is designed to be applicable in the case that $\mu$ is fixed and in particular in the case $\mu = \nu^2$ as motivated by [4,5]. In this case we have argued that we can envision a cutoff $m_0$ such that $\nu^2 \lambda_j$ is significant for $j \geq m_0$ and insignificant for $j < m_0$; of future interest within the framework of this approach would be further theoretical or empirical investigation toward determining or estimating the optimal spectral cutoff $m_0$.

References


