An improved local blow-up condition for Euler–Poisson equations with attractive forcing

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**Abstract**

We improve the recent result of Chae and Tadmor (2008) \([10]\) proving a one-sided threshold condition which leads to a finite-time breakdown of the Euler–Poisson equations in arbitrary dimension \(n\).

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**1. Introduction**

The pressure-less Euler–Poisson (EP) equations in dimension \(n \geq 1\) are

\[
\rho_t + \text{div}(\rho \mathbf{u}) = 0 \tag{1.1a}
\]

\[
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \Delta^{-1}(\rho - c) \tag{1.1b}
\]

governing the unknown density \(\rho = \rho(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+\) and velocity \(\mathbf{u} = \mathbf{u}(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n\) subject to initial conditions \(\rho(0,x) = \rho_0(x)\) and \(\mathbf{u}(0,x) = \mathbf{u}_0(x)\). They involve two constants: (i) a fixed background state \(c \geq 0\) – typical cases include the case of zero background, \(c = 0\), or the case of a nonzero background given by the average mass, \(c = \int \rho(t,x)dx = \int \rho_0(x)dx\); and (ii) a constant \(k\) which parameterizes the repulsive \(k > 0\) or attractive \(k < 0\) forcing, governed by the Poisson potential \(\Delta^{-1}(\rho - c)\).

The EP system appears in numerous applications including semiconductors and plasma physics \((k > 0)\) and the collapse of stars due to self gravitation \((k < 0)\). In particular, the pressureless EP model becomes relevant in interstellar clouds where gravitational forces dominate pressure gradient. \([5]\), for example, or in the context of the Euler–Monge–Ampère systems and their quasi-neutral limits to the incompressible Euler equations \([6]\).

This paper is restricted to the attractive case, \(k < 0\). We begin by setting \(c = 1\), \(k = -1\) in \((1.1a), (1.1b)\) to arrive at the unit-free EP system,

\[
\rho_t + \text{div}(\rho \mathbf{u}) = 0 \tag{1.2a}
\]

\[
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \Delta^{-1}(\rho - 1) \tag{1.2b}
\]

Our discussion remains valid for the general physical parameters \(c \geq 0\), \(k < 0\) upon a simple rescaling and limiting arguments, outlined in Corollary 1.1 below for \(c > 0\) and Corollary 1.2 for the case of zero background \(c = 0\).

We are concerned here with the persistence of \(C^1\) regularity for solutions of the attractive EP system. Our Main theorem reveals a pointwise criterion on the initial data, a so-called critical threshold criterion \([7–9]\), that leads to finite time blow-up of \(\nabla \mathbf{u}\). It quantifies the balance between the two term \(\text{div} \mathbf{u}\) and \(\rho_t\), which govern two competing mechanisms that dictate the \(C^1\) regularity of EP flows. Our result also stands out as a generalization of several existing results \([7,10,11,9]\) for which further discussion is given after the Main theorem and its corollary.

**Main Theorem 1.1.** Consider the \(n\)-dimensional, attractive Euler–Poisson system \((1.2a), (1.2b)\) subject to initial data \(\rho_0, \mathbf{u}_0\). Then, the solution will lose \(C^1\) regularity at a finite time \(t = t_c < \infty\), if there exists a non-vacuum initial state \(\rho_0(x) > 0\) with vanishing initial...
Consider the Euler–Poisson system
\begin{align}
(1.1a) \quad \rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u &= -\nabla p + \nabla \phi, \\
(1.3b) \quad \nabla \cdot u &= 0,
\end{align}
where \( \phi \) is the external potential.

In particular, \( \min \text{div } u(t, x) \rightarrow -\infty \) and \( \max_{t} \rho(t, x) \rightarrow \infty \) as \( t \uparrow t_c \).

**Proof.** Combine Lemmas 3.1 and 4.2, while noting that the curve
\[
\text{div } u = \text{sgn}(\rho - 1) \sqrt{nF(\rho)},
\]
is the separatrix along the boundary of the blow-up region \( \Omega = \Omega_1 \cup \Omega_2 \) defined in (4.3) and illustrated in Fig. 4.1. \( \square \)

We note in passing that, by classical arguments, the force-free Euler system \( u_t + u \cdot \nabla u = 0 \) exhibits finite time blow-up if and only if there exists at least one negative eigenvalue of \( \nabla \cdot u(\rho, x) \).

In the above theorem, however, finite-time blow-up can occur solely depending on the initial profile of \( \text{div } u_0 \) and \( \rho_0 \) regardless of individual eigenvalues of \( \nabla \cdot u_0 \).

We also note that, by rescaling \( \rho \) to \( \rho/c, x \rightarrow \sqrt{-kc} x \) and \( t \rightarrow -\sqrt{-k} t \), the main theorem immediately applies to the EP system (1.1a), (1.1b) with physical parameters. Since the EP system with \( k < 0 \) models the collapse of interstellar cloud, the following corollary reveals a pointwise condition for mass concentration, \( \rho \rightarrow \infty \), which interestingly preludes the birth of new stars.

**Corollary 1.1.** Consider the Euler–Poisson system (1.1a), (1.1b) with \( c > 0, k < 0 \) subject to initial data \( \rho_0, u_0 \). Then, the solution will lose \( C^1 \) regularity at a finite time \( t_c < \infty \) if \( \text{div } u_0(\rho, x) = 0 \) with a vanishing initial vorticity, \( \nabla \times u_0(\rho) = 0 \), such that the super-critical condition is fulfilled,
\[
\text{div } u_0(\rho_0) < \text{sgn}(\rho_0 - c) \sqrt{nkc} \left( \frac{\rho_0(\rho_0)}{c} \right) \quad (1.4)
\]
where \( F() \) is given in (1.3b). In particular, \( \min_{x} \text{div } u(t, x) \rightarrow -\infty \) and \( \max_{x} \rho(t, x) \rightarrow \infty \) as \( t \uparrow t_c \).

In the limited regime as \( c \rightarrow 0^+ \), condition (1.4) converges to a super-critical condition which is summarized by the following result, the proof of which is given in Section 5.

**Corollary 1.2.** Consider the n-dimensional Euler–Poisson system (1.1a), (1.1b) with \( c > 0, k < 0 \) subject to initial data \( \rho_0, u_0 \). Assume a vanishing initial vorticity everywhere, \( \nabla \times u_0 = 0 \). Then, the solution will lose \( C^1 \) regularity at a finite time \( t_c < \infty \), if either (i) \( n = 1, 2 \) or (ii) \( n \geq 3 \) and there exists a non-vacuum initial state \( \rho_0(x) > 0 \) such that
\[
\text{div } u_0(\rho_0) < \sqrt{-\frac{2nk\rho_0(\rho)}{n-2}}, \quad n \geq 3.
\]

In other words, the pressureless and vorticity-free one- and two-dimensional attractive Euler–Poisson systems with zero background (\( c = 0 \)), inevitably collapse to singularity at a finite time. On the other hand, the complete characterization of finite-time breakdown in higher dimensions remains open, even for \( c = 0 \).

The concept of Critical Threshold and associated methodology is originated and developed in a series of papers by Engelberg, Liu and Tadmor [7], Liu and Tadmor [9,8] and more. It first appears in [7] regarding pointwise criteria for \( C^1 \) solution regularity of 1D EP system. The key argument in that paper is based on the convective derivative along particle paths \( \dot{x} = \dot{t} + u \cdot \nabla \). It makes it possible to obtain a 2-by-2 ODE system for \( u_0 \) and \( \rho_0 \) along particle paths—the so-called Lagrangian formulation. Phase plane analysis is then employed to study the finiteness of the ODE solutions and therefore \( C^1 \) regularity of the PDE solution. Similar results stay valid for Euler–Poisson systems with geometric symmetry in higher dimensions [3,8]. To treat genuinely multi-D cases, Liu and Tadmor introduce in [8] the method of spectral dynamics which relies on the ODE system governing eigenvalues of
\[
\lambda = \nabla u,
\]
which is the velocity gradient matrix, along particle paths. They identify if-and-only-if, pointwise conditions for global existence of \( C^1 \) solutions to restricted Euler–Poisson systems. Chae and Tadmor [10] further extend the Critical Threshold argument to multi-D full Euler–Poisson systems (1.2a), (1.2b) with attractive forcing \( k < 0 \). Their result, however, offers a blow-up region \( \nabla \times u_0 = 0, \text{div } u_0 \leq -\sqrt{-nk} \) which is only a subset of the blow-up region in (1.4). This subset is to the left of the solid line \( d \leq d^*= -\sqrt{-nk} \) depicted in Fig. 4.1. Finally, a recent paper by Tadmor and Wei [12] reveals the critical threshold phenomena in the 1D Euler–Poisson system with pressure.

When tracking other results on the well-posedness of Euler–Poisson equations, we find them commonly relying on (the vast family of) energy methods and thus fundamentally differ from our pointwise results obtained via the Lagrangian approach. With a repulsive force \( k > 0 \), we refer to [13,14] for the global existence of classical solutions with finite data and [15] for the nonexistence of global solutions. With attractive force \( k < 0 \), see [1] for local regularity of classical solutions and [16,17] for nonexistence results. Discussions on weak solutions of Euler–Poisson systems can be found in e.g. [18–20]. We also refer to [21–25] and references therein for steady-state solutions. The study of the Euler–Poisson system with damping relaxation can be found in e.g. [26–28].

The rest of this paper is organized as follows. In Section 2, we follow the idea of [10] to derive along particle paths an ODE system governing the dynamics of eigenvalues for \( S := \frac{1}{2} (M + M^T) \). This is a variation of the spectral dynamics for \( M \) introduced in [8]. We then derive in Section 3 a closed \( 2 \times 2 \) ODE system (3.1) at the cost of turning one equation into inequality. By the comparison principle, this inequality is in favor of blow-up. Thus, with the inequality sign being replaced with an equality sign, a modified ODE system is used to yield sub-solutions and to study a blow-up scenario for the original system. Section 4, devoted to the modified system, reveals the Critical Threshold for such a system. Consequently, a pointwise blow-up condition for the original system is identified. Finally, in Section 5 we prove Corollary 1.2 regarding the Euler–Poisson system with zero background using techniques developed in previous sections.

### 2. Spectral dynamics

We examine the gradient matrix \( M = \nabla u \) and its symmetric part, \( S = \frac{1}{2} (\nabla u + (\nabla u)^T) \). Both matrices are used to study the spectral dynamics of Euler systems (see e.g. [8] for \( M \) and [10] for \( S \)). The relation between the spectra of \( M \) and \( S \) is described in the following.

**Proposition 2.1.** Let \( \{\lambda_M\} \) denote the eigenvalues of \( M \) and \( \{\lambda_S\} \) for \( S \). Then
\[
\sum_{\lambda_M} \lambda_M = \sum_{\lambda_S} \lambda_S = \text{div } u, \quad (2.1)
\]
\[
\sum_{\lambda_M} \lambda_M^2 = \sum_{\lambda_S} \lambda_S^2 - \frac{1}{2} |\omega|^2 . \quad (2.2)
\]
Here, $\omega$ is the $\frac{n(n-1)}{2}$ vorticity vector which consists of the off-diagonal entries of $A := \frac{1}{2} \left( \nabla u - (\nabla u)^T \right)$.

**Proof.** Use identity $M = S + A$ and the skew-symmetry of $A$,

$$\sum_{\lambda_M} \lambda_M = \text{tr}(M) = \text{tr}(S + A) = \text{tr}(S) = \sum_{\lambda_S} \lambda_S.$$  

Squaring the last identity we have $M^2 = S^2 + A^2 + AS + SA$ and therefore,

$$\sum_{\lambda_M} \lambda_M^2 = \text{tr}(M^2) = \text{tr}(S^2 + A^2 + AS + SA) = \sum_{\lambda_S} \lambda_S^2 + \text{tr}(A^2).$$  

Note that $AS + SA$ is skew-symmetric and thus traceless. A simple calculation yields $\text{tr}(A^2) = -\frac{1}{2} |\omega|^2$. □

Following [8], we turn to study the dynamics of $M$ along particle paths. Take the gradient of (2.1b) to find

$$M' + M^2 \equiv M_t + u \cdot \nabla M + M^2 = -R(\rho - 1),$$  

where $R$ stands for the Riesz matrix, $R = \{R_{ij}\} := \{\partial_{x_i} A^{-1}\}$.

The trace of (2.3) then yields that the divergence, $d := \text{tr}(M)$, is governed by

$$d' = -\sum_{\lambda_M} \lambda_M^2 - (\rho - 1),$$

and in view of (2.2),

$$d' = -\sum_{\lambda_S} \lambda_S^2 + \frac{1}{2} |\omega|^2 - (\rho - 1).$$  

We now make the first observation regarding the invariance of the vorticity $\omega$: taking the skew-symmetric part of the $M$-equation (2.3),

$$A' + AS + SA = 0.$$  

(2.5)

It follows that if the initial vorticity vanishes, $\omega_0(\vec{x}) \mapsto \nabla \times u_0(\vec{x}) = 0$, then by (2.5), $\omega \mapsto \nabla \times u$ vanishes along the particle path which emanates from $\vec{x}$. This allows us to decouple the vorticity and divergence dynamics, and (2.4) implies

$$d' = -\sum_{\lambda_S} \lambda_S^2 - (\rho - 1), \quad \nabla \times u = 0.$$  

(2.6)

Finally, we use Cauchy–Schwartz $\sum \lambda_S^2 \geq \frac{1}{4} \left( \sum \lambda_S \right)^2 = \frac{1}{2} d^2$ and the fact that all $\lambda_S$ are real (due to the symmetry of $S$), to deduce the inequality,

$$d' \leq -\frac{1}{n} \rho^2 - (\rho - 1).$$  

(2.7a)

This, together with the mass equation (1.2a) which can be written along particle path

$$\rho' = -d \rho.$$  

(2.7b)

give us the desired closed system which dominates $(\rho, d)$ along particle paths.

**Remark 2.1.** The approach pursued in this paper will be based on the inequality (2.7a) and is therefore limited to derivation of a finite time breakdown. To argue the global regularity, one needs to study the underlying inequality (2.6), and to this end, to study the trace $\sum \lambda_S^2$. In the two-dimensional case, for example, one can use $\sum \lambda_S^2 = d^2/2 + \eta^2/2$ to replace (2.7a) with

$$d' = -\frac{1}{2} d^2 - \frac{1}{2} \eta^2 - (\rho - 1), \quad \eta := \lambda_{5,2} - \lambda_{5,1}.$$  

In this framework, global 2D regularity is dictated by the dynamics of the spectral gap, $\eta = \lambda_{5,2} - \lambda_{5,1}$, which in turn requires the dynamics of the Riesz transform $R(\rho - 1)$.

### 3. A comparison principle with a majorant system

The blow-up analysis, driven by the inequalities (2.7),

$$d' \leq -\frac{1}{n} d^2 - (\rho - 1),$$  

(3.1a)

$$\rho' = -d \rho.$$  

(3.1b)

is carried out by standard comparison with the majorant system

$$e' = -\frac{1}{n} e^2 - (\zeta - 1),$$  

(3.2a)

$$\zeta' = -\varepsilon \zeta.$$  

(3.2b)

The following proposition guarantees the monotonicity of the solution operator associated with (3.1).

**Lemma 3.1.** The following monotone relation between system (3.1) and system (3.2) is invariant forward in time,

$$\begin{align*}
|d(0) < e(0) & \text{ implies } |d(t) < e(t) \text{ for } t \geq 0, & (3.3)
0 < \zeta(0) < \rho(0) & \text{ implies } 0 < \zeta(t) < \rho(t) \text{ for } t \geq 0,
\end{align*}$$

as long as all solutions remain finite on the time interval $[0, t]$.

**Proof.** Invariance of positivity of $\zeta$ is a direct consequence of (3.2b) and finiteness of $e$. The rest can be proved by contradiction. Suppose $t_1$ is the earliest time when (3.3) is violated. Then,

$$\zeta(t_1) = \zeta(0) \exp \left( -\int_0^{t_1} e(t) dt \right) < \rho(0) \exp \left( -\int_0^{t_1} d(t) dt \right) = \rho(t_1).$$  

(3.4)
Therefore, we are left with only one possibility, namely, \( e(t_1) = d(t_1) \). Subtracting (3.1a) from (3.2a),

\[
(e - d)' \geq -\frac{1}{n}(e^2 - d^2) - (\zeta - \rho),
\]

and by (3.4), we find that at \( t = t_1 \),

\[
\text{RHS of (3.5) at } t = t_1 = 0 - [\zeta(t_1) - \rho(t_1)] > 0.
\]

However, this contradicts the negativity of the expression on the left of (3.5), since \( e(t) - d(t) > 0 \) for all \( t < t_1 \) and vanishes at \( t = t_1 \) which implies that

\[
\text{LHS of (3.5) at } t = t_1 = (e(t_1) - d(t_1))' \leq 0. \quad \square
\]

In the next section, we employ phase plane analysis on the modified system (3.2). When translated in terms of the original system (3.1), however, such analysis can only yield blow-up results and is insufficient for global existence results. In other words, estimate (3.3) is only useful for proving \( d \to \infty \) as \( t \to \infty \), the key mechanism for blow-up of C1 solutions.

4. Stability analysis of the majorant system

We shall prove the blow-up of the majorant system (3.2),

\[
e(t) \to -\infty \quad \text{as } t \uparrow t_c,
\]

which in turn, by Lemma 3.1 implies \( d(t) \to -\infty \). Abusing notations, we express the majorant system in terms of the original variables \( (e, \zeta) \mapsto (d, \rho) \):

\[
d' = -\frac{1}{n}d^2 - (\rho - 1),
\]

\[
\rho' = -d\rho.
\]

The (in-)stability analysis of (4.1) hinges on the path invariants of this system. To this end, we use the same q-transformation employed in [29]: setting \( q := d^2 \) and differentiate along the path \( (t, X(a, t)) \) and \( X(t, a) = a \), we find

\[
\frac{dq}{d\rho} = 2d\rho' = \frac{2}{n}\rho - 2\left(1 - \frac{1}{\rho}\right),
\]

which yields

\[
\frac{d}{d\rho}\left(q\rho^{\frac{2}{n}}\right) = 2(1 - \rho^{-1})\rho^{-\frac{2}{n}}.
\]

Upon integration, we arrive at the following key observation.

**Lemma 4.1.** The majorant system (4.1) is equipped with the path invariant,

\[
I(d(t), \rho(t)) = I(d_0, \rho_0),
\]

along each path \( (t, X) \) initiated with a non-vacuum state \((d_0, \rho_0) > 0\).


\[
I(d, \rho) := d^2 \rho^{-\frac{2}{n}} - 2 \int_1^\rho \left(1 - r^{-1}\right)r^{-\frac{2}{n}} dr
\]

\[
= \rho^{-\frac{2}{n}} \left(d^2 - nF(\rho)\right),
\]

where \( F(\cdot) \) is specified in (1.3b).

It is a simple calculation to show that the majorant system (4.1) admits three distinct critical points (see Fig. 4.1):

\[
(d^*, \rho^*) := (0, 1), \quad (d^{\pm}, \rho^{\pm}) := (\pm\sqrt{n}, 0),
\]

and that \((0, 1)\) is a saddle point, \((-\sqrt{n}, 0)\) a nodal source and \((\sqrt{n}, 0)\) a nodal sink. The separatrix is given by the zero level set \( I(d, \rho) = 0 \). Moreover, the right branch of the separatrix, \( d = \sqrt{n}F(\rho) \) connects critical points \((0, 1)\) and \((-\sqrt{n}, 0)\) while the left branch, \( d = -\sqrt{n}F(\rho) \) connects \((0, 1)\) and \((\sqrt{n}, 0)\).

By inspection of the phase plane in Fig. 4.1, we postulate the following invariant region of finite-time blow-up for the modified system (4.1),

\[
\Omega = \Omega_1 \cup \Omega_2 = \{(d, \rho) \mid d < \text{sgn}(\rho - 1)\sqrt{n}F(\rho)\}
\]

where

\[
\Omega_1 := \{(d, \rho) \mid I(d, \rho) > 0 \text{ and } d < \rho > 0\},
\]

\[
\Omega_2 := \{(d, \rho) \mid I(d, \rho) < 0 \text{ and } \rho > 1\}.
\]

**Lemma 4.2.** Consider the modified system (4.1), equipped with initial data \((d_0, \rho_0)\). If \((d_0, \rho_0) \in \Omega\), then \( \text{div } u \to -\infty \) and \( \rho \to -\infty \) at a finite time.

**Proof.** We begin by recalling (1.3b), consult (4.2).

\[
F(\rho) = \frac{2}{n}\rho^{\frac{2}{n}} - 2\left(1 - \frac{1}{\rho}\right)\rho^{-\frac{2}{n}}.
\]

Clearly, \( F(1) = F'(1) = 0 \) and a simple calculation shows that \( F'(\rho) = \frac{2}{n}\rho^{\frac{2}{n}-2} \), which implies that \( F(\rho) \) is a strictly convex function of \( \rho \) and attains its only minimum at \( \rho = 1 \),

\[
F(\rho) \geq F(1) = 0.
\]

We shall also utilize the invariance of (4.2)

\[
d^2 - nF(\rho) = \rho^{\frac{2}{n}} - 2 , \quad l_0 = l(d_0, \rho_0).
\]

We now turn to discuss the two possible blow-up scenarios, depending whether the initial data \((d_0, \rho_0)\) belong to the blow-up regions \(\Omega_1\) or \(\Omega_2\) given in (4.3).

**Case #1.** Assume that \((d_0, \rho_0) \in \Omega_1\) so that the invariant \( I \) remains a positive constant

\[
I > 0.
\]

In this case, \( d \) remains negative, for otherwise, setting \( d = 0 \) in (4.5) would result in \( F(\rho) = -\rho^{\frac{2}{n}}l/n < 0 \), violating (4.4). Thus, (4.5) and (4.4) yield an upper bound,

\[
d \leq -\rho^{1/2}\sqrt{l}.
\]

Then, by (4.1b), we have a Riccati type of equation \( \rho' \geq \sqrt{l}\rho^{1+1/n} \) for which the solution exhibits blow-up \( \rho \to +\infty \) and the divergence \( d = \text{div } u \) approaches \( -\infty \) at a finite time due to (4.5).

**Case #2.** Assume that \((d_0, \rho_0) \in \Omega_2\) so that the invariant \( I \) remains a negative constant

\[
I < 0.
\]

In this case, \( \rho - 1 \) remains positive, for otherwise setting \( \rho = 1 \) in (4.5) would result in \( F(1) = (d^2 - l)/n > 0 \) in contradiction to (4.4). Now, for \( \rho > 1 \) we have

\[
F(\rho) = \frac{2}{n}\rho^{2/n} - 2\int_1^\rho \left(1 - \frac{1}{r}\right)^{1/2} dr \leq \frac{2}{n}\rho^{2/n}(\rho - 1).
\]

This together with (4.5) yield

\[
\frac{2}{n}\rho^{2/n}(\rho - 1) \geq F(\rho) = \frac{2}{n}(d^2 - \rho^{2/n}) \geq -\frac{2}{n}\rho^{2/n}l
\]

and the lower bound, \( \rho = 1) \geq -1/2 \) follows. Thus, by (4.1a), we end up with a Riccati type of equation

\[
d' \leq \frac{d^2 - l}{n} + \frac{1}{2}.
\]

Since the invariant \( I \) remains a negative constant, the solution exhibits blow-up \( d = \text{div } u \to -\infty \) at a finite time even if initially \( d_0 > 0 \). The density \( \rho \) also approaches \( -\infty \) at a finite time due to (4.5). \( \square \)

The last step of proving the Main theorem is just to combine the comparison principle in Lemma 3.1 with the above lemma. We notice that \( \Omega \) is an open set and thus given any initial data \((d_0, \rho_0) \in \Omega\) for the original system, we can always find \( \epsilon > 0 \) and initial data \((d_0 + \epsilon, \rho_0 - \epsilon) \in \Omega\) for the modified system. This
latter initial data will lead to a finite time blow-up of the modified system and therefore, by Lemma 3.1, initial data \((d_0, \rho_0) \in \Omega\) will lead to finite time blow-up of the original system.

5. Critical threshold for zero background

We now turn to the attractive Euler–Poisson system \((1.1a), \quad (1.1b)\) with zero background \(c = 0\) and prove Corollary 1.2. For simplicity, we only show the case with \(k = -1\) since a straightforward rescaling argument, \(x \to \sqrt{-k} x\) and \(t \to \sqrt{-k} t\), will cover the case for general \(k < 0\).

Proof of Corollary 1.2. Following the same calculation that leads to the majorant system \((4.1a), \quad (4.1b)\), we arrive at a similar ODE system for the case \(c = 0, \quad k = -1\).

\[
d' = -\frac{d^2}{n} - \rho, \quad (5.1a)
\]

\[
\rho' = -d \rho. \quad (5.1b)
\]

Then, as an analogue to the invariant \((4.2)\), we find the corresponding invariant,

\[
I(d, \rho) := d^2 \rho^{-\frac{2}{n}} - 2 \int_a^\rho r^{-\frac{2}{n}} \mathrm{d}r.
\]

By choosing the constant

\[
a = \begin{cases} +\infty, & n = 1, \\ 1, & n = 2, \\ 0, & n \geq 3, \end{cases}
\]

we have

\[
I(d, \rho) = \begin{cases} d^2 \rho^{-2} + 2 \rho^{-1}, & n = 1 \\ d^2 \rho^{-1} - 2 \ln \rho, & n = 2 \\ d^2 \rho^{-\frac{2}{n}} - \frac{2}{n-2} \rho^{1-\frac{2}{n}}, & n \geq 3. \end{cases} \quad (5.2)
\]

Using the positivity of \(\rho\) and \(d^2\) in \((5.2)\), we have \(I \geq 2 \rho^{-1} > 0\) for \(n = 1\) and \(I \geq -2 \ln \rho > 0\) for \(n = 2\). Both estimates imply that \(\rho\) is bounded from below by a positive constant. In the case of \(n \geq 3\), the sup-critical condition \((1.5)\) implies \(I < 0\). Thus, by \((5.2)\), we have \(0 > I \geq -\frac{2n}{n-2} \rho^{1-\frac{2}{n}}\) which, again, implies \(\rho\) is greater than a positive constant.

Therefore, by \((5.1a)\), \(d\) satisfies a differential inequality

\[
d' \leq -\frac{d^2 + \alpha}{n}
\]

with positive constant \(\alpha\). Obviously, \(d(t)\) approaches \(-\infty\) at a time no later than \(\frac{\alpha}{2\sqrt{n}}\).

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References


