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CRITICAL THRESHOLDS IN A CONVOLUTION MODEL FOR NONLINEAR CONSERVATION LAWS*

HAILIANG LIU† AND EITAN TADMOR†

Abstract. In this work we consider a convolution model for nonlinear conservation laws. Due to the delicate balance between the nonlinear convection and the nonlocal forcing, this model allows for narrower shock layers than those in the viscous Burgers' equation and yet exhibits the conditional finite time breakdown as in the damped Burgers' equation. We show the critical threshold phenomenon by presenting a lower threshold for the breakdown of the solutions and an upper threshold for the global existence of the smooth solution. The threshold condition depends only on the relative size of the minimum slope of the initial velocity and its maximal variation. We show the exact blow-up rate when the slope of the initial profile is below the lower threshold. We further prove the L^1 stability of the smooth shock profile, provided the slope of the initial profile is above the critical threshold.

Key words. wave breakdown, critical threshold, shock profile, stability

AMS subject classifications. Primary, 35L65; Secondary, 35B30

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1. Introduction.

Consider the scalar equation of the form

$$u_t + uu_x = Q * u - u, \tag{1.1}$$

where Q is a regular symmetric kernel, monotonically decreasing on $\mathbb{R}^+$, subject to initial data

$$u(0,x) = u_0(x), \quad u_0 \in C_b^1(\mathbb{R}). \tag{1.2}$$

We are concerned with the critical threshold phenomenon supported by the balance between the nonlinear convection and the nonlocal source term in (1.1).

For the kernel Q, we make the following assumption:

(H1) $Q \in C^1(\mathbb{R})$, $Q(-r) = Q(r) \geq 0$, $\int Q(y)dy = 1$, $\int Q(y)|y|dy < \infty$, and $Q'(x) \leq 0$ for $x \geq 0$.

To clarify the effect of the nonlocal term on the right-hand side of (1.1), we make a hyperbolic scaling

$$(t,x) \to \left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \quad \epsilon > 0,$$

which leads to

$$u_t + uu_x = \frac{1}{\epsilon}[Q_\epsilon * u - u], \tag{1.3}$$

where $Q_\epsilon := \frac{1}{\epsilon}Q(\frac{x}{\epsilon})$ and is converging to a delta function $\delta(x)$ as the scaled parameter ϵ tends to zero.

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†Department of Mathematics, UCLA, Los Angeles, CA 90095-1555 (hliu@math.ucla.edu, tadmor@math.ucla.edu).

A typical example of the kernel Q is $\frac{1}{2}e^{-|x|}$; with this specific kernel, (1.3) can be written as

$$u_t + uu_x = \mathcal{F}^{-1}\left[\frac{-\epsilon\xi^2}{1+\epsilon^2\xi^2}\hat{u}(t,\xi)\right] = \epsilon\mathcal{F}^{-1}\left[\frac{1}{1+\epsilon^2\xi^2}\hat{u}(t,\xi)\right]_{xx}, \tag{1.4}$$

which is called an R-C-E model after Rosenau's regularized version of the Chapman–Enskog expansion for hydrodynamics [17]. The operator on the right-hand side of (1.4) looks like the usual viscosity term ϵu_{xx} at low wave-number ξ, while for higher wave numbers it is intended to model a bounded approximation of a linearized collision operator, thereby avoiding the artificial instabilities that occur when the Chapman–Enskog expansion is truncated after a finite number of terms [17]. This idea has been greatly advanced recently by Slemrod and his collaborators. A renormalization procedure was introduced in [19] to eliminate the truncation instability and to produce the desired dissipation; the corresponding applications can be found in [20, 21, 22]. The regularization of the Burnett equations via relaxation was investigated by Jin and Slemrod [5, 6]. The rigorous analysis of the model (1.4), including the existence of the shock profiles, the smoothness, as well as the upper-Lipschitz continuity, has been studied by Schochet and Tadmor [23]. We remark that, as observed in [23], the solution sequence $\{u^\epsilon\}$ of (1.4) does not satisfy the Kružkov entropy inequality. The convergence of the solution u^ϵ of (1.4) to the entropy solution of the inviscid Burgers' equation was proved in [23] via the L^1 contraction argument.

(1.3) with $Q = \frac{1}{2}e^{-|x|}$ can also be written as a hyperbolic-elliptic system

$$u_t + uu_x = \phi_x, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.5}$$

$$\epsilon^2\phi_{xx} - \phi + \epsilon u_x = 0. \tag{1.6}$$

It is easy to see that (1.6) enables one to express ϕ in terms of u formally as

$$\phi = (1-\epsilon^2\partial_x^2)^{-1}\epsilon u_x = \epsilon Q_\epsilon * u_x,$$

which in turn gives the right-hand side of (1.3),

$$\phi_x = \epsilon Q_\epsilon * u_{xx} = \frac{1}{\epsilon}[Q_\epsilon * u - u].$$

The system of equations (1.5)–(1.6) is derived as the third-order approximation of the full system describing the motion of radiating gas in therm-nonequilibrium, while the second-order approximation gives the viscous Burgers' equation $u_t + uu_x = \epsilon u_{xx}$, and the first-order approximation gives the inviscid Burgers' equation $u_t + uu_x = 0$. Hamer [4] studied these equations in the physical respect, especially for the steady progressive shock wave solutions. Noting that if ϵ in (1.6) is small, one has $\phi \sim \epsilon u_x$, which leads to the usual viscous Burgers' equation. The viscous Burgers' equation admits smooth shock wave profiles but does not allow the finite time breakdown. On the other hand, if the parameter ϵ is large, one finds from (1.6) that $\epsilon\phi_{xx} + u_x \sim 0$, which when combined with (1.5) gives the damped Burgers' equation $u_t + uu_x = -u/\epsilon$. This damped equation reflects the conditional breakdown in finite time but does not support monotone traveling waves (shock profiles).

The parameter ϵ in (1.3) does not play a role in our analysis and so will be set to 1 for convenience. Equation (1.3) with $\epsilon = 1$, i.e., (1.1), is a physical model that allows for the shock wave profile and yet exhibits the finite time breakdown. For stability of shock profiles via energy method we refer to [11, 8]. The global weak solution to (1.1) was studied in [23].

As is known, the typical well-posedness result asserts that either a solution of a time-dependent PDE exists for all time (global existence of the smooth solution) or else there is a finite time (called life span) such that some norm of the solution becomes unbounded as the life span is approached (called finite time breakdown). The natural question is whether there is a critical threshold for the initial data such that the global existence of the smooth solution or the finite time breakdown depends only on crossing such a critical threshold. This remarkable critical threshold phenomenon was first observed and studied in [3] for a class of Euler–Poisson equations. In this paper we confirm such a critical threshold phenomenon for (1.1)–(1.2) by giving an upper threshold for the global existence of the smooth solution and a lower threshold for the finite time breakdown. We also show the exact blow-up rate as the life span is approached.

In this paper we shall use the following notation for $g \in L^\infty(\mathbb{R})$ to denote the maximal variation:

$$V(g) := \max_{x\in\mathbb{R}} g(x) - \min_{x\in\mathbb{R}} g(x).$$

The first result tells us the critical threshold phenomenon in (1.1).

THEOREM 1.1. *Consider the Cauchy problem* (1.1)–(1.2) *with initial data* $u_0 \in C_b^1(\mathbb{R})$. *Let the kernel* Q *satisfy* (H_1); *then we have the following:*

- *If* $V(u_0) < \frac{1}{4Q(0)}$ *and*

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) > -\frac{1}{2}\left[1+\sqrt{1-4Q(0)V(u_0)}\right],$$

then the smooth solution exists for all time.

- *If*

$$\inf_{x} \partial_x u_0(x) < -\frac{1}{2}\left[1+\sqrt{1+4Q(0)V(u_0)}\right],$$

then the solution u *must break down at finite time* T. *Moreover,*

$$\lim_{t\to T}(\min_{x\in\mathbb{R}}\{u_x(t,x)\}) = -\infty$$

and the exact blow-up rate is

$$\lim_{t\to T}((T-t)\min_{x\in\mathbb{R}}\{u_x(t,x)\}) = -1.$$

Concerning this theorem, several remarks are in order.

Remarks. 1. The above results show that the solution behavior of (1.1)–(1.2) depends on the relative size of the minimum slope of the initial profile and its maximal variation. If either the maximal variation is too large or the initial velocity slope is too negative, the solution would lose smoothness in finite time. This peculiar phenomenon explains the result obtained in [23], in which additional constraints on the shock strength are imposed to ensure the smoothness of the shock profiles. Further relation between the smoothness of the shock profiles and the shock strength are given in [8]. The critical threshold phenomenon was already partially observed in previous studies; see [23] and [9].

2. As an example, we take $u_0^\theta(x) = \exp(-x^2/\theta)$ for $\theta > 0$. Note that

$$\inf_{x\in\mathbb{R}}[\partial_x u_0^\epsilon(x)] = -\sqrt{\frac{2}{e\theta}}, \quad V(u_0^\theta) = 1.$$

Therefore, choosing θ so small that

$$\theta < \frac{4}{e(1 + 2Q(0) + \sqrt{1 + 4Q(0)})},$$

we see that $\partial_x u_0^\theta$ is below the lower threshold, and thereby the corresponding solution $u^\theta(t, x)$ breaks down in finite time.

3. Note that at the blow-up time, the solution is still bounded, and the gradient of the solution becomes unbounded from below. Such a breakdown is referred to as wave breaking in the context of the shallow water waves. In [25] Whitham emphasized that wave breaking phenomena are some of the most intriguing long-standing problems of water theory. This issue was first settled recently in [15] for Whitham's equation. Another shallow water equation derived recently by Camassa and Holm [2] can be written as (1.5) coupled with the following equation:

$$\phi_{xx} - \phi - u^2 - \frac{1}{2}u_x^2 = 0.$$

This equation as a completely integrable system has a soliton solution and yet exhibits finite time breakdown phenomena for a large class of initial data, which has been observed and justified by Holm [2], Constantin and Escher [1], and McKean [14]. The main tool used in the above papers is to trace the solution gradient along a curve on which the minimum of the gradient is obtained. In this work we trace the dynamics of the solution gradient along the characteristics, which are well known in the context of the hyperbolic equations; see, e.g., [12, 7, 13]. For the global weak solution to the above shallow water equation, we refer to [24] and references therein.

4. From the results above we see that if the magnitude of the initial profile is small, both thresholds given in Theorem 1.1 are close to $\inf_{x\in\mathbb{R}} \partial_x u_0(x) = -1$, which is exactly the critical threshold for the damped Burgers' equation:

$$u_t + uu_x = -u.$$

Indeed, along the particle path $x(\alpha, t)$ defined by

$$\frac{d}{dt}x(\alpha, t) = u(t, x(\alpha, t)), \quad x(\alpha, 0) = \alpha, \quad \alpha \in \mathbb{R},$$

the gradient of the solution to the damped Burgers' equation above can be written explicitly as

$$u_x(t, x) = [e^t(1 + (\partial_x u_0(\alpha))^{-1}) - 1]^{-1},$$

which is bounded from below for all time if and only if

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) \geq -1.$$

This remarkable critical threshold phenomenon explains why (1.1) admits narrower shock layers than those in the viscous Burgers' equation. We now turn to discussing the asymptotic behavior of solutions, as the initial data are above the critical threshold. We shall concentrate on the case $u_0(-\infty) = u_- > u_+ = u_0(+\infty)$. As shown in [23], (1.1) with $Q = \frac{1}{2}e^{-|x|}$ admits a smooth shock profile $U(x-st)$ connecting u_+ to u_- if and only if the strength $|V(U)| = |u_+ - u_-| \le \sqrt{2}$. Considering the conservative form of the equation, the natural question is whether this shock profile is stable in $L^1(\mathbb{R})$.

Our stability result is summarized below.

THEOREM 1.2. *Let* $U(x-st)$ *be a continuous shock profile of* (1.1) *and* $S(t)u_0$ *be a solution to* (1.1)–(1.2) *with initial data* $u_0 \in U + L^1(\mathbb{R})$ *and* $u_0 \in [\inf U, \sup U]$. *If* $\partial_x u_0 \ge -\frac{1}{2}[1 + \sqrt{1 - 4Q(0)V(u_0)}]$, *then there exists a constant* k *such that*

$$\lim_{t\to\infty} \|S(t)u_0 - U(\cdot - st + k)\|_{L^1} = 0.$$

Remarks. 1. The $L^p(1 \le p < \infty)$ stability is immediate from the above L^1 stability result and the L^∞ boundedness of $S(t)u_0$. Consult [8] for the stability of traveling waves via the energy principle.

2. We assume that the initial data are above the upper critical threshold to ensure the regularity of the ω-limit set of the solution. This condition is expected to be relaxed since our upper threshold is not sharp.

We now conclude this section by outlining the rest of the paper. In section 2, we recall several properties of (1.1) and give the estimate of the nonlocal term in (1.1), which paves the way for the next sections. The lower threshold for finite time breakdown is given in section 3, in which we also prove the exact blow-up rate. The upper threshold for global existence of the smooth solution is carried out in section 4. The final section is devoted to the L^1 stability of the shock profiles.

2. Preliminaries. This section is devoted to some estimates which will be used in the next two sections.

In order to formulate the problem, we denote the solution operator of (1.1) as $S(t)$, indexed with $t \in [0, \infty)$,

$$S(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad t \ge 0,$$

such that the solution $u(t,x)$ of (1.1) with initial data a can be expressed as

$$u(t) = S(t)a.$$

We recall from [23] that the solution operator $S(t)$ satisfies the following properties:

- (translate invariance) $S(t)a(x+k) = (S(t)a)(x+k)$ for any $k \in \mathbb{R}$;
- (conservative) if $a - b \in L^1(\mathbb{R})$, then for all $t > 0$, $S(t)a - S(t)b \in L^1(\mathbb{R})$ and $\int(S(t)a - S(t)b) = \int(a-b)$;
- (L^1 contraction) if $a - b \in L^1(\mathbb{R})$, then $S(t)a - S(t)b \in L^1(\mathbb{R})$ and $\|S(t)a - S(t)b\|_1$ is nonincreasing for $t > 0$;
- (monotonicity) if $a(x) \ge b(x)$ for $x \in \mathbb{R}$, then $S(t)a \ge S(t)b$ for all $t > 0$.

The above monotonicity immediately gives us the following maximum principle.

LEMMA 2.1. *Let* $u_0 \in L^\infty(\mathbb{R})$. *Then the solution* $u(t,\cdot)$ *is also bounded with*

$$\min_{x\in\mathbb{R}} u_0(x) \le u(t,\cdot) \le \max_{x\in\mathbb{R}} u_0(x).$$

This maximum principle leads to the following bounds, which will be used in figuring out our threshold conditions.

LEMMA 2.2. *Let u be the smooth solution in $[0,T]$. Then it holds that*

$$\min_{x\in\mathbb{R}} u_0(x) \le Q * u(t,\cdot) \le \max_{x\in\mathbb{R}} u_0(x), \quad t \in [0,T], \tag{2.1}$$

$$-Q(0)V(u_0) \le Q * u_x(t,\cdot) \le Q(0)V(u_0). \tag{2.2}$$

Proof. The first inequality follows from the fact $Q * 1 = 1$ and the L^∞ bound $\min_{x\in\mathbb{R}} u_0(x) \le u(t,\cdot) \le \max_{x\in\mathbb{R}} u_0(x)$. We shall prove the second inequality as follows:

$$\begin{aligned}
Q * u_x &= \int_{\mathbb{R}} Q(x-y)u_y(t,y)dy \\
&= \int_{\mathbb{R}} Q_x(x-y)u(t,y)dy \\
&= \left[\int_{-\infty}^{x} Q_x(x-y)u(t,y)dy + \int_{x}^{+\infty} Q_x(x-y)u(t,y)dy\right] \\
&\le \min_{x\in\mathbb{R}} u_0(x)\int_{-\infty}^{x} Q_x(x-y)dy + \max_{x\in\mathbb{R}} u_0(x)\int_{x}^{+\infty} Q_x(x-y)dy \\
&\le Q(0)\left[-\min_{x\in\mathbb{R}} u_0(x) + \max_{x\in\mathbb{R}} u_0(x)\right] = Q(0)V(u_0).
\end{aligned}$$

The lower bound $-Q(0)V(u_0)$ is clear from the above estimate. ∎

The existence of T is ensured by the local existence theorem stated in the following lemma.

LEMMA 2.3. *Consider the Cauchy problem* (1.1)–(1.2) *with initial data $u_0 \in C_b^1(\mathbb{R})$. Then there exists a positive constant T, depending only on $\|u_0\|_{C_b^1(\mathbb{R})}$, such that* (1.1)–(1.2) *has a unique smooth solution in $C_b^1(\mathbb{R}\times[0,T])$.*

The proof of this local existence is standard via an iteration scheme; the details are omitted. This local existence provides a base for extending the solution or justifying the finite time breakdown.

3. Blow-up criterion—lower threshold. This section is devoted to a general discussion of wave breaking criteria.

THEOREM 3.1. *Consider the Cauchy problem* (1.1)–(1.2). *The maximal existence time T is finite if and only if the gradient of the solution becomes unbounded from below in finite time.*

Proof. From the local existence in Lemma 2.3 it follows that if the gradient of the solution becomes unbounded from below in finite time, then $T < \infty$.

Let the life span $T < \infty$ and assume that for some constant $M > 0$ we have

$$u_x(t,x) \ge -M, \quad (t,x) \in [0,T)\times\mathbb{R}. \tag{3.1}$$

On the other hand, by [23, Theorem 5.1] the solution $u(t,x)$ satisfies the one-sided Lipschitz condition, i.e.,

$$u_x(t,x) \le \frac{1}{(\max_{x\in\mathbb{R}} u_{0x})^{-1} + t} \le \max_{x\in\mathbb{R}} u_{0x} < \infty.$$

Therefore the standard continuation argument enables us to extend the solution to $[0, T+\delta)$ with $\delta > 0$, and thereby one must have $T = \infty$. This contradiction ensures that

$$\lim_{t\to T-}\left(\min_{x\in\mathbb{R}} u_x(t,x)\right) = -\infty. \qquad ∎$$

The lower threshold is given in the following theorem.

THEOREM 3.2. *Consider the Cauchy problem* (1.1)–(1.2) *with the initial profile* $u_0 \in C_b^1(\mathbb{R})$. *If* u_0 *is bounded and its gradient is negative with*

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) < -\frac{1}{2}\big[1+\sqrt{1+4Q(0)V(u_0)}\big],$$

then the life span T *must be finite. Moreover,*

$$T \le \left[-\frac{1}{2}\big(1+\sqrt{1+4Q(0)V(u_0)}\big) - \inf_{x\in\mathbb{R}} \partial_x u_0(x)\right]^{-1}$$

and

$$\lim_{t\to T}\big(\min_{x\in\mathbb{R}}\{u_x(t,x)\}\big) = -\infty.$$

Proof. Differentiation of (1.1) with respect to x leads to

$$d_t + ud_x + d^2 = Q * u_x - d, \quad t\in(0,T),$$

where $d := u_x(t,x)$. The smoothness of u ensures that there exists a smooth curve $x(\alpha,t)$ satisfying

$$\frac{d}{dt}x(\alpha,t) = u(t,x(\alpha,t)), \quad x(\alpha,0)=\alpha, \quad \alpha\in\mathbb{R}.$$

Evaluating the above $d-$ equation at $x(\alpha,t)$ and using $Q*u_x \le A := Q(0)V(u_0)$ stated in Lemma 2.2, we have

$$d' + d^2 = Q*u_x(t,x(\alpha,t)) - d \le A - d, \quad {}' := \partial_t + u\partial_x$$

for $t\in(0,T)$. That is,

$$d' \le -(d-M_1)(d-M_2), \quad t\in(0,T), \tag{3.2}$$

with

$$M_1 := -\frac{1}{2}[1+\sqrt{1+4A}], \quad M_2 := -\frac{1}{2}[1-\sqrt{1+4A}].$$

For a fixed $\alpha\in\mathbb{R}$, if $d_0(\alpha) := u_0'(\alpha) < M_1$, then we claim that

$$d(t) < d_0(\alpha), \quad t\in(0,T). \tag{3.3}$$

If this would not be true, there is some $t_0\in(0,T)$ with $d(t) < d_0$ on $[0,t_0)$ and $d(t_0) = d_0$ by the continuity of $d = u_x$ in time. But in this case

$$d' \le -(d_0-M_1)(d_0-M_2) < 0, \quad t\in(0,t_0).$$

An integration over $(0,t_0)$ yields

$$d(t_0) < d_0,$$

which contradicts our assumption that $d(t_0) = d_0$ for $t_0 < T$. This implies that (3.3) holds.

Combining (3.3) with (3.2), we obtain

$$d' \le -(d - M_1)^2, \quad t \in (0, T),$$

and integration yields

$$d(t) \le M_1 + \left[t - \frac{1}{M_1 - d_0}\right]^{-1}.$$

From this we find that $d(t) \to -\infty$ before t reaches $\frac{1}{M_1 - d_0}$. This proves that the solution breaks down in finite time once $\partial_x u_0 \ge M_1$ fails. ☐

The blow-up rate at the breaking time is summarized in the next theorem.

THEOREM 3.3. *Let T be the maximal existence time of* (1.1)–(1.2). *If the life span T is finite, then*

$$\lim_{t \to T} \left((T - t) \left(\min_{x \in \mathbb{R}} \{u_x(t, x)\} \right) \right) = -1.$$

Proof. By Theorem 3.1 one has

$$\lim_{t \to T} \left(\min_{x \in \mathbb{R}} \{u_x(t, x)\} \right) = -\infty.$$

For $t \in [0, T)$ the solution u is smooth and the curve $x(\alpha, t)$ is well defined by

$$\frac{d}{dt} x(\alpha, t) = u(t, x(\alpha, t)), \quad x(\alpha, 0) = \alpha, \quad \alpha \in \mathbb{R}.$$

This implies

$$\frac{\partial}{\partial \alpha} x(\alpha, t) = \exp\left(\int_0^t u_x(\tau, x(\alpha, \tau)) d\tau \right) > 0, \quad t \in (0, T),$$

and hence $x(\alpha, t)$ is a one-to-one mapping from $\mathbb{R}$ to $\mathbb{R}$. From these facts it follows that there exists an $\alpha \in \mathbb{R}$ such that

$$\min_{x \in \mathbb{R}} \{u_x(t, x)\} = u_x(t, x(\alpha, t)).$$

As done previously, we consider dynamics of $d = u_x$ along the curve $x(\alpha, t)$, using $-A \le Q * u_x \le A = Q(0)V(u_0)$ to obtain

$$-A - d \le d' + d^2 \le A - d, \quad t \in (0, T).$$

Let $\epsilon \in (0, 1)$ be suitably small. Since $\lim_{t \to T} d(t) = -\infty$, there exists $t_0 \in (0, T)$ such that

$$d(t) < B^-(\epsilon), \quad t \in [t_0, T), \tag{3.4}$$

with

$$B^-(\epsilon) = \frac{-2A}{\sqrt{1 + 4A\epsilon(2 - \epsilon)} - 1}$$

being the smaller root of $(\epsilon^2 - 2\epsilon)d^2 - d + A = 0$. Otherwise there exists $\delta > 0$ such that

$$d(t) < B^-(\epsilon), \quad t \in (t_0, t_0 + \delta),$$

and for $\delta < T - t_0$

$$d(t_0 + \delta) = B^-(\epsilon).$$

Hence for $d(t) < B^-(\epsilon)$ on $(t_0, t_0 + \delta)$,

$$\frac{d}{dt} d(t) \leq A - d - d^2 \leq -(1-\epsilon)^2 d^2 < 0, \quad t \in (t_0, t_0 + \delta).$$

Integration gives

$$d(t_0 + \delta) < d(t_0) < B^-(\epsilon).$$

This contradiction shows that

$$d \leq B^-(\epsilon), \quad t \in [t_0, T);$$

therefore

$$d' \leq -(1-\epsilon)^2 d^2, \quad t \in [t_0, T). \tag{3.5}$$

On the other hand, let

$$B^+(\epsilon) = \frac{-2A}{\sqrt{1 + 4A\epsilon(2+\epsilon)} + 1},$$

which is the bigger root of $(\epsilon^2 + 2\epsilon)d^2 - d - A = 0$. We find that $B^-(\epsilon) < B^+(\epsilon)$ and

$$d(t) < B^+(\epsilon), \quad t \in (t_0, T).$$

This gives $(\epsilon^2 + 2\epsilon)d^2 - d - A > 0$, yielding

$$d' \geq -(d^2 + d + A) \geq -(1+\epsilon)^2 d^2, \quad t \in (t_0, T). \tag{3.6}$$

A combination of (3.5) with (3.6) gives

$$-(1+\epsilon)^2 d^2 \leq d' \leq -(1-\epsilon)^2 d^2, \quad t \in (t_0, T).$$

Note that d is locally Lipschitz on (t_0, T) and so is $1/d$ on (t_0, T). The above inequality leads to

$$(1-\epsilon)^2 \leq \left(\frac{1}{d}\right)' \leq (1+\epsilon)^2, \quad t \in (t_0, T).$$

For $t \in (t_0, T)$, integrate the above over (t, T) to obtain

$$-(1-\epsilon)^2 (T - t) \leq \frac{1}{d(t)} \leq -(1+\epsilon)^2 (T - t), \quad t \in (t_0, T).$$

Optimizing the above in terms of ϵ, one then has

$$\lim_{t \to T} (T - t) d(t) = -1.$$

This completes the proof. ∎

4. Global smoothness—upper threshold. With the breakdown criterion in section 2, we are ready to discuss the upper threshold for the global existence of the smooth solution to (1.1)–(1.2).

THEOREM 4.1. *Consider the Cauchy problem* (1.1)–(1.2) *with the initial profile* $u_0 \in C_b^1(\mathbb{R})$. *If* u_0 *is bounded with the maximal variation* $V(u_0) \leq \frac{1}{4Q(0)}$ *and its gradient is above an upper threshold, i.e.,*

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) \geq -\frac{1}{2}\left[1+\sqrt{1-4Q(0)V(u_0)}\right],$$

then the smooth solution exists for all time and satisfies

$$\partial_x u(t,x) \geq -\frac{1}{2}\left[1+\sqrt{1-4Q(0)V(u_0)}\right].$$

Proof. To show the global existence of the smooth solution it suffices to establish an a priori lower bound for the gradient of solution u_x. As argued earlier, we evaluate $d := u_x$ along the particle path $x(\alpha,t)$ to obtain

$$d' + d^2 = Q * u_x(t, x(\alpha,t)) - d(t).$$

Noting that the lower bound of Qu_x is $-A = -V(u_0)Q(0)$, we find that

$$d' \geq -A - d - d^2 = -(d-A_1)(d-A_2),$$

where

$$A_1 = -\frac{1}{2}[1+\sqrt{1-4A}], \quad A_2 = -\frac{1}{2}[1-\sqrt{1-4A}].$$

Now let q solve the following problem:

$$\frac{d}{dt}q(t) = -(q-A_1)(q-A_2), \quad q(0) = d_0.$$

Then the comparison of the above differential relations yields

$$d - q \geq (d_0 - q(0)) \exp\left(-\int_0^t (d+q+1)d\tau\right) = 0, \quad t>0.$$

However, q can be solved explicitly as

$$q(t) = \left[A_1 - A_2 \frac{d_1 - A_1}{d_0 - A_2} \exp\left(A_2 - A_1\right)t\right]\left[1 - \frac{d_1 - A_1}{d_0 - A_2}\exp\left(A_2 - A_1\right)t\right]^{-1}.$$

Therefore for $A_2 > d_0 \geq A_1$ one has $d(t) \geq q(t) \geq A_1$; for $d_0 \geq A_2$ one has $d(t) \geq q(t) \geq A_2$. The possible breakdown occurs only when $d_0 < A_1$ because

$$q(t^*) = -\infty, \qquad t^* = \frac{1}{A_2 - A_1}\log\frac{d_1 - A_2}{d_0 - A_1} > 0.$$

The lower bound of d cannot be ensured for $d_0 < A_1$. However, $d_0 \geq A_1$ is sufficient to ensure the global existence of the smooth solution. ∎

5. L^1 stability of shock profiles. Let us rewrite (1.1) as

$$u_t + f(u)_x = Q * u - u, \quad f = u^2/2. \tag{5.1}$$

A shock wave with speed $s \in \mathbb{R}$ is a solution of (5.1) of the form $U(x - st)$, with U approaching two different shock states $u_\pm$ at far field. The function U formally satisfies the equation

$$-sU' + f(U)' = Q * U - U, \quad U(\pm\infty) = u_\pm.$$

The critical threshold phenomenon revealed in the previous sections suggests that the smooth shock profile is possibly subject to some constraints on the shock strength.

Indeed the existence of the shock profiles for (5.1) with convex flux function f has been proved [23, Theorem 3.1], which we state below, for $Q = \frac{1}{2}e^{-|x|}$, for the reader's convenience.

THEOREM 5.1. *Assume $f'' > 0$. Then the Lax shock condition*

$$f'(u_+) < s < f'(u_-) \tag{5.2}$$

and the Rankine–Hugoniot shock condition

$$H(u_+) = 0, \quad H(u) \equiv -s(u - u_-) + f(u) - f(u_-), \tag{5.3}$$

are necessary conditions for the existence of a traveling wave solution

$$U(z \equiv x - st), \quad \lim_{z\to\pm\infty} U(z) = u_\pm,$$

for (5.1). *Conversely, if* (5.2) and (5.3) *hold, then a sufficient condition for the existence of such a traveling wave is*

$$4 \sup_{u_+<u<u_-} \{-f''(u)H(u)\} \le 1,$$

and a necessary condition is

$$4\{-f''(u^*)H(u^*)\} \le 1.$$

Here u^* is defined by

$$f'(u^*) = s.$$

Note that for the Burgers' flux $f = u^2/2$, the shock speed by the Rankine–Hugoniot relation (5.3) becomes $s = \frac{u_+ + u_-}{2}$. If the shock condition (5.2), i.e.,

$$u_+ < u_-$$

holds, then there exists such a traveling wave if and only if

$$|u_+ - u_-| \le \sqrt{2}. \tag{5.4}$$

This shows that the traveling wave solutions of the R-C-E equation give narrower shock layers than those of the viscous Burgers' equation.

Recall that the solution operator

$$S(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad t \ge 0,$$

satisfies the nice properties listed in section 2, which ensures that $S(t)$ can be well extended to $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ and preserves all those properties.

To reformulate the stability problem, we introduce the following set:

$$A := U + L^1(\mathbb{R}),$$

which is a complete metric space with the metric

$$\rho(a_1, a_2) = \|a_1 - a_2\|_1.$$

We also set two subspaces of A,

$$A_1 := \{U(\cdot + k), \quad k \in \mathbb{R}\}$$

and

$$A_2 = \{a \in A : \lim_{t\to\infty} S(t)a \quad \text{exists and} \lim_{t\to\infty} S(t)a \in A_1\}.$$

Equipped with the above notations, we see that proving the stability result in Theorem 1.2 reduces to proving the relation

$$A \cap [u+, u_-] \subset A_2, \tag{5.5}$$

provided $S(t)a$ is smooth.

We introduce the ω-limit set of a as

$$\omega(a) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\{S(t)a\}}.$$

This ω-limit set is invariant for $S(t)$. In fact, the definition implies that $b \in \omega(a)$ if and only if there is a sequence $\{t_k\} \to \infty$ such that

$$\rho(S(t_k)a, b) \to 0.$$

The following lemma plays a critical role in proving (5.5).

Lemma 5.2. *If $a, b \in A \cap [u_+, u_-]$ and $a - b$ does not keep same sign on $\mathbb{R}$, then*

$$\|S(t)a - S(t)b\|_1 < \|a - b\|_1, \quad t > 0.$$

Proof. By Kružkov's argument [10] we have

$$\int_0^T \int_{\mathbb{R}} \{|u - v|\phi_t + sgn(u - v)[f(u) - f(v)]\phi_x\}dxdt$$
$$\geq \int_0^T \int_{\mathbb{R}} \{|u - v| - sgn(u - v)G * (u - v)\}\phi dxdt,$$

where ϕ is an arbitrary nonnegative test function. Thus, by taking $\phi(x,t) = \chi(t)\psi(x,t)$, letting $\psi = 1 - g_\epsilon(|x - x_0| - M(T - t))$ with $M = sup|f'|$ tend to the function that is identically one, and letting $\chi(t)$ approximate the indicator function of the interval $[0, t]$, we conclude

(5.6)

$$\|a - b\|_1 - \|S(t)a - S(t)b\|_1 \geq \int_{\mathbb{R}} |S(t)a - S(t)b| - sgn(a - b)G * (S(t)a - S(t)b)dx.$$

Using the monotonicity of $S(t)$ we see that if $a-b$ changes sign on $\mathbb{R}$, then so does $S(t)a - S(t)b$. Note that since $\|Q\|_1 = 1$, we find that

$$\int_{\mathbb{R}} |u| - sign(u) Q * u dx = 0$$

if and only if u does not change sign or $u \equiv 0$. This shows that the right-hand side of (5.6) is positive if $a-b$ changes sign on $\mathbb{R}$. ☐

Armed with the above lemma we proceed to complete the stability proof via the following steps, which have become standard since the work by Osher and Ralston [16] and Serre [18].

First we restrict our stability proof to the initial data in

$$N(U, k_1, k_2) := \{a \in A, \quad U(x+k_1) \le a(x) \le U(x+k_2), \quad \text{for some} \quad k_1, k_2 \in \mathbb{R}\},$$

and we can later extend our argument to a larger class using the following dense lemmas.

Step 1 (dense argument). We first show that both A_1 and A_2 are complete subspaces of A.

LEMMA 5.3. *Let U be the monotone shock profile; then A_i, $i = 1, 2$, are close in A.*

Proof. We first show the closeness of A_1. It is easy to see that for any $k \in \mathbb{R}$, $U(x+k) \in A$ since

$$\|U(\cdot + k) - U(\cdot)\|_1 = |k(u_+ - u_-)| < \infty.$$

We assume $U(x+k_n)$ converges in A; then it is a Cauchy sequence. Note that

$$\|U(\cdot + k_n) - U(\cdot + k_m)\|_{L^1} = |(k_n - k_m)(u_+ - u_-)|$$

implies k_n is also a Cauchy sequence in $\mathbb{R}$. Let its limit be k; then by letting $m \to \infty$ in the above equation, one finds that the limit of $U(x+k_n)$ is $U(x+k) \in A_1$.

We now turn to showing the closeness of A_2. Let $a_k \in A_2$ be a Cauchy sequence with its limit being $a \in A$. We need to show $a \in A_2$. Note that for each $a_k \in A_2$ we have that $\lim_{t\to\infty} S(t)a_k = \tilde{a}_k \in A_1$ exists. Hence $\tilde{a}_k$ is a Cauchy sequence in the complete metric space A_1, for

$$\|\tilde{a}_k - \tilde{a}_l\|_1 = \lim_{t\to\infty} \|S(t)a_k - S(t)a_l\|_1 \le \|a_k - a_l\|_1.$$

We denote the limit of $\tilde{a}_k$ by $\tilde{a}$ as $k \to \infty$, which, when combined with the closeness of A_1, implies that $\tilde{a} \in A_1$. Therefore $a \in A_2$ since

$$\|S(t)a - \tilde{a}\|_1 \le \|S(t)a - S(t)a_k\|_1 + \|S(t)a_k - \tilde{a}_k\|_1 + \|\tilde{a}_k - \tilde{a}\|_1 \to 0$$

as $k \to \infty$ and $t \to \infty$. ☐

LEMMA 5.4. *For any given $k_1, k_2 \in \mathbb{R}$, the set $N(U, k_1, k_2)$ is dense in $A \cap [u_+, u_-]$.*

The proof can be done as in [16]; the details are omitted.

Step 2 (compact criteria).

LEMMA 5.5. *For any $k_1, k_2 \in \mathbb{R}$, the ω-limit set $\omega(N(U, k_1, k_2))$ is not empty.*

Proof. It suffices to show that $\cup_{t\ge 0}\{S(t)a\}$ is precompact for any $a \in N(U, k_1, k_2)$. Indeed, due to $a - U \in L^1$ and the L^1 contraction of $S(t)$ we have

$$\|S(t)a - U\|_1 = \|S(t)a - S(t)U\|_1 \le \|a - U\|_1 < \infty, \quad t \ge 0.$$

The L^1 equicontinuity follows from the fact that

$$\|S(t)a(x+h) - S(t)a(x)\|_1 \le \|a(x+h) - a(x)\|_1 \to 0$$

uniformly in time as h goes to zero. Using the semigroup property of $S(t)$, we have

$$U(x+k_1) \le S(t)a \le U(x+k_2), \quad t \ge 0.$$

Hence

$$||S(t)a - U(x)||_{L^1(|x|>M)} \le \max\{\|U(\cdot+k_1) - U\|_{L^1(|x|>M)}, \|U(\cdot+k_2) - U\|_{L^1(|x|>M)}\} \to 0$$

uniformly in t as M goes to ∞.

When recalling the Frechet–Kolmogorov–Riesz compactness theorem, the above facts yield that $\cup_{t\ge 0}\{S(t)a\}$ is precompact. □

Step 3 (time-invariance).

LEMMA 5.6. *Let $b \in \omega(N(U, k_1, k_2))$. Then for any given $k \in \mathbb{R}$*

$$\|b - U(\cdot + k)\|_1 = \|S(t)b - U(\cdot + k)\|_1.$$

Proof. Since $b \in \omega(N(U, k_1, k_2))$, we see that there exists $a \in N(U, k_1, k_2)$ and a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \|S(t_n)a - b\|_1 = 0.$$

Given any $k \in \mathbb{R}$, by contraction of $S(t)$ we know that

$$\|S(t)a - U(x+k)\|_1 = \|S(t)a - S(t)U(x+k)\|_1$$

is decreasing in time and thus admits a limit $c_k \ge 0$ as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \|S(t)a - U(x+k)\|_1 = c_k \ge 0.$$

Letting $t = t_n$ in the above equation and passing to the limit, we have

$$\|b - U(\cdot + k)\|_1 = c_k.$$

Note that if $b \in \omega(a)$, then $S(t)b \in \omega(a)$ (ω is invariant under the flow); thereby

$$\|S(t)b - U(\cdot + k)\|_1 = c_k.$$

Therefore

$$\|S(t)b - U(\cdot + k)\|_1 = \|b - U(\cdot + k)\|_1 \quad \forall t > 0, \quad k \in \mathbb{R}. \qquad \square$$

We are now ready to prove (5.5). We first prove

$$N(U, k_1, k_2) \subset A_2.$$

By Lemma 5.5 we know that $\omega(N(U, k_1, k_2))$ is not empty. For $a \in N(U, k_1, k_2)$ and $b \in \omega(a)$, we need to show that there exists a $k \in \mathbb{R}$ such that

$$b = U(x+k).$$

Lemma 5.6 shows that

$$\|b - U(\cdot + k)\|_1 = \|S(t)b - U(\cdot + k)\|_1 = c_k.$$

Noting that $U(x+k)$ is the fixed point of $S(t)$, Lemma 5.2 shows that $b - U(x+k)$ must stay with one sign.

Therefore, choosing

$$k = \int_{\mathbb{R}} (a - U)dx/(u_+ - u_-)$$

gives

$$c_k = \int_{\mathbb{R}} [b - U(\cdot + k)] = \int_{\mathbb{R}} [a - U(\cdot + k)] = 0.$$

On the other hand, since the initial data a are assumed to be above the critical threshold, $\partial_x(S(t)a)$ is uniformly bounded with respect to t, and hence b is Lipschitz continuous. This regularity combined with the above fact yields

$$b = U(x+k).$$

We now conclude the proof of (5.5). Let $a \in A \cap [u_+, u_-]$. We need to show $a \in A_2$.

Using Lemma 5.4 shows that there exists $a_n \in N(U, k_1, k_2) \in A$ such that $\|a_n - a\|_1 \to 0$ as $n \to \infty$. By the above proved fact we see that there exists k_n such that

$$\lim_{t\to\infty} \|S(t)a_n - U(\cdot + k_n)\|_1 = 0.$$

This tells us that $a_n \in A_2$. Due to the closeness of A_2, the limit a also belongs to A_2. Therefore there exists a k such that

$$\lim_{t\to\infty} \|S(t)a - U(\cdot + k)\|_1 = 0;$$

as argued above, the constant k as the limit of $\int_{\mathbb{R}} (a_n - U)dx/(u_+ - u_-)$ is

$$\int_{\mathbb{R}} (a - U)dx/(u_+ - u_-)$$

since $|\int (a_n - a)dx| \le \|a_n - a\| \to 0$ as $n \to \infty$. This completes the proof of (5.5) and thereby of Theorem 1.2.

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