Self-Force for Comparable Mass Binaries

Alexandre Le Tiec

University of Maryland

Based on collaborations with:
L. Barack, E. Barausse, L. Blanchet, A. Buonanno, A. H. Mroué, H. P. Pfeiffer, N. Sago, A. Taracchini, and B. F. Whiting
Methods to compute GW templates for compact binaries

\[ \log_{10}(r_{12}/m) \]

\[ \log_{10}(m_2/m_1) \]

Mass Ratio

Post-Newtonian Theory

Perturbation Theory

Numerical Relativity

\(\frac{m_2}{m_1}\)
Methods to compute GW templates for compact binaries

\[ v_{12}^2 \sim \frac{m}{r_{12}} \ll 1 \]
Methods to compute GW templates for compact binaries

\[ q \equiv \frac{m_1}{m_2} \ll 1 \]
Methods to compute GW templates for compact binaries

Gravitational waveforms
Periastron advance
First law of binary mechanics
Energy and angular momentum

\[ \log_{10} \left( \frac{m_2}{m_1} \right) \]

Mass Ratio

Post-Newtonian Theory
Perturbation Theory
Numerical Relativity

\[ \log_{10} \left( \frac{r_{12}}{m} \right) \]

Compactness

15th Capra Meeting — June 13, 2012
Methods to compute GW templates for compact binaries
Methods to compute GW templates for compact binaries

- Gravitational waveforms
- Periastron advance
- First law of binary mechanics
- Energy and angular momentum

\[ \log_{10} \left( \frac{m_2}{m_1} \right) \]

\[ \log_{10} \left( \frac{r_{12}}{m} \right) \]

(Compactness)

- Mass Ratio
- Perturbation Theory
- Post-Newtonian Theory
- Numerical Relativity

15th Capra Meeting — June 13, 2012
**Beware of confusing mass conventions**

<table>
<thead>
<tr>
<th></th>
<th>SF</th>
<th>PN/NR</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass of the “particle”</td>
<td>$\mu$</td>
<td>$m_1$</td>
</tr>
<tr>
<td>mass of the “black hole”</td>
<td>$M$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>total mass</td>
<td>$\mu + M \simeq M$</td>
<td>$m = m_1 + m_2$</td>
</tr>
<tr>
<td>reduced mass</td>
<td>$\frac{\mu M}{\mu + M} \simeq \mu$</td>
<td>$\mu = \frac{m_1 m_2}{m}$</td>
</tr>
<tr>
<td>symmetric mass ratio</td>
<td>$\frac{\mu M}{(\mu + M)^2} \simeq \frac{\mu}{M}$</td>
<td>$\nu = \frac{m_1 m_2}{m^2}$</td>
</tr>
<tr>
<td>(asymmetric) mass ratio</td>
<td>$\frac{\mu}{M} \ll 1$</td>
<td>$q = \frac{m_1}{m_2}$</td>
</tr>
</tbody>
</table>

We shall use the PN/NR mass conventions.
Outline

① Gravitational waveforms

② Periastron advance in black hole binaries

③ First law of binary black hole mechanics

④ Binding energy and angular momentum
Outline

1. Gravitational waveforms
2. Periastron advance in black hole binaries
3. First law of binary black hole mechanics
4. Binding energy and angular momentum
Head-on collision of two black holes

[Smarr (1979); Detweiler (1979)]

Numerical Relativity

\[ m_1 = m_2 \]

Perturbation Theory

\[ m_1 \ll m_2 \]

Rescaling \( m_1 \rightarrow \mu, m_2 \rightarrow m \)

Figure 3. The curvature \( \psi_{4, \text{RM}} \) in the equatorial plane crossing the 2-sphere at \( r = 2M \) as a function of time. This is for the two black hole collision Run II.

Figure 4. The same quantity as in Figure 3 except from the perturbation calculation of a particle of mass \( \mu \) falling into a black hole of mass \( M \). The abscissa is retarded time. The vertical scales are explained in the text. Only the quadrupole contribution is shown here.
Head-on collision for a mass ratio 1:100

[Sperhake, Cardoso et al. (2011)]
Head-on collision for a mass ratio 1:10

[Sperhake, Cardoso et al. (2011)]
Head-on collision for a mass ratio 1:4

[Sperhake, Cardoso et al. (2011)]
Outline

1. Gravitational waveforms
2. Periastron advance in black hole binaries
3. First law of binary black hole mechanics
4. Binding energy and angular momentum
Relativistic perihelion advance of Mercury

- Observed anomalous precession of Mercury’s perihelion of $\sim 43'' /\text{cent}$.
- Accounted for by the leading-order relativistic angular advance per orbit

$$\Delta \Phi_{\text{GR}} = \frac{6\pi G M_\odot}{c^2 a (1 - e^2)}$$

- One of the first successes of Einstein’s general theory of relativity
- Relativistic periastron advance of $\sim \circ /\text{yr}$ now measured in binary pulsars
Periastron advance in black hole binaries

- **Conservative** part of the dynamics only
- Generic non-circular orbit parametrized by the two frequencies

\[
\Omega_r = \frac{2\pi}{P}, \quad \Omega_\varphi = \frac{1}{P} \int_0^P \dot{\varphi}(t) \, dt
\]

- Periastron advance per radial period

\[
K \equiv \frac{\Omega_\varphi}{\Omega_r} = 1 + \frac{\Delta \Phi}{2\pi}
\]

- In the circular orbit limit \(e \to 0\), the relation \(K(\Omega_\varphi)\) is coordinate invariant
Early results in numerical relativity

[Mroué, Pfeiffer, Kidder & Teukolsky (2010)]
Tentative comparison with self-force results

[Barack & Sago (2011)]

\[
\frac{\Omega_{\phi}}{\Omega_{\gamma}} = (M+\mu)\hat{\Omega}_{\phi}^{\frac{1}{q}}
\]

0.015 0.02 0.025 0.03 0.035 0.04 0.045

1.3
1.35
1.4
1.45
1.5
1.55
1.6
1.65
1.7
1.75

q=0
q=1/6
q=1/4
q=1/3
q=1/2

\[
\frac{\Omega_{\phi}}{\Omega_{\gamma}} = (M+\mu)\hat{\Omega}_{\phi}^{\frac{1}{q}}
\]
Extensive comparison for a mass ratio 1:1

[Le Tiec, Mroué et al. (2011)]
Extensive comparison for a mass ratio $1:8$

[Le Tiec, Mroué et al. (2011)]
Variation with respect to the mass ratio

[Le Tiec, Mroué et al. (2011)]
Outline

① Gravitational waveforms

② Periastron advance in black hole binaries

③ First law of binary black hole mechanics

④ Binding energy and angular momentum
Generalized first law of mechanics

[Friedman, Uryū & Shibata (2002)]

- Spacetimes with black holes + perfect fluid matter sources
- One-parameter family of solutions \( \{g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)\} \)
- Globally defined Killing vector field \( K^\alpha \rightarrow \) conserved charge \( Q \)

\[
\delta Q = \sum_i \frac{\kappa_i}{8\pi} \delta A_i + \int_\Sigma \left[ \bar{h} \Delta(dM_b) + \bar{T} \Delta(dS) + v^\alpha \Delta(dC_\alpha) \right]
\]
Application to compact binaries on circular orbits

- For circular orbits, the geometry has a helical Killing vector
  \[ K^\alpha \rightarrow (\partial_t)^\alpha + \Omega (\partial_\varphi)^\alpha \quad \text{(when } r \to +\infty) \]

- For asymptotically flat spacetimes [Friedman et al. (2002)]
  \[ \delta Q = \delta M - \Omega \delta J \]

- In the exact theory, helically symmetric spacetimes are not asymptotically flat [Gibbons & Stewart (1983); Klein (2004)]

- Asymptotic flatness can be recovered if gravitational radiation can be “turned off”, e.g.
  - Conformal Flatness Condition
  - Post-Newtonian theory
Application to compact binaries on circular orbits

[Le Tiec, Blanchet & Whiting (2012)]

- **Conservative** dynamics only → no gravitational radiation
- Non-spinning compact objects modeled as **point masses** $m_A$:

\[
T^{\alpha\beta} = \sum_{A=1}^{2} m_A z_A u^\alpha_A u^\beta_A \frac{\delta(x - y_A)}{\sqrt{-g}}
\]

- For two point masses on a **circular orbit**, the first law becomes

\[
\delta M - \Omega \delta J = z_1 \delta m_1 + z_2 \delta m_2
\]
First integral associated with the variational law

[Le Tiec, Blanchet & Whiting (2012)]

- Variational first law: \( \delta M - \Omega \delta J = z_1 \delta m_1 + z_2 \delta m_2 \)
- Since \( \{M, J, z_A\} \) are all functions of \( \{\Omega, m_A\} \), we have
  \[
  \frac{\partial M}{\partial \Omega} = \Omega \frac{\partial J}{\partial \Omega} \quad \text{and} \quad z_A = \frac{\partial(M - \Omega J)}{\partial m_A}
  \]
- After a few algebraic manipulations, we obtain
  \[
  M - 2\Omega J = m_1 z_1 + m_2 z_2
  \]
- Alternative derivations based on:
  - Euler’s theorem applied to the function \( M(J^{1/2}, m_1, m_2) \)
  - The combination \( M_K - 2\Omega J_K \) of the Komar quantities
Outline

1. Gravitational waveforms
2. Periastron advance in black hole binaries
3. First law of binary black hole mechanics
4. Binding energy and angular momentum
Binding energy beyond the test-mass approximation

[Le Tiec, Barausse & Buonanno (2012)]

- The binding energy $E \equiv M - m$ is a function of $x \equiv (m\Omega)^{2/3}$
- In the “small” mass ratio limit $\nu \to 0$:
  \[
  z_1 = \sqrt{1 - 3x} + \nu z_{\text{GSF}}(x) + \mathcal{O}(\nu^2)
  \]
  \[
  \frac{E}{\mu} = \left( \frac{1 - 2x}{\sqrt{1 - 3x}} - 1 \right) + \nu E_{\text{GSF}}(x) + \mathcal{O}(\nu^2)
  \]
- The self-force contribution $z_{\text{GSF}}(x)$ is known numerically
  [Detweiler (2008); Sago, Barack & Detweiler (2008); Shah et al. (2011)]
- The first law provides a relationship $E \leftrightarrow z_1$, which implies
  \[
  E_{\text{GSF}}(x) = \frac{1}{2} z_{\text{GSF}}(x) - \frac{x}{3} z'_{\text{GSF}}(x) + f(x)
  \]
GSF correction to the Schwarzschild ISCO frequency

- The orbital frequency of the Schwarzschild ISCO is shifted under the effect of the conservative self-force:

\[ m \Omega_{\text{ISCO}} = 6^{-3/2} \left\{ 1 + \nu \frac{C_\Omega}{m} + \mathcal{O}(\nu^2) \right\} \]

- A stability analysis of slightly eccentric orbits near the ISCO yields [Barack & Sago (2009)]

\[ C_{\Omega}^{\text{BS}} = 1.2512(4) \]

- Strong-field benchmark used for comparison with PN/NR/EOB
GSF correction to the Schwarzschild ISCO frequency

- The angular frequency of the minimum energy circular orbit (MECO) is solution of

\[ \frac{\partial E}{\partial \Omega} \bigg|_{\Omega_{\text{MECO}}} = 0 \]

- Hamiltonian system: ISCO ⇔ MECO [Buonanno et al. (2003)]

- Our result for the energy \( E_{\text{GSF}}(x) \) yields [Le Tiec et al. (2012)]

\[ C_\Omega = \frac{1}{2} + \frac{1}{4\sqrt{2}} \left\{ \frac{1}{3} z''_{\text{GSF}}(1/6) - z'_{\text{GSF}}(1/6) \right\} \]

- Using accurate numerical self-force data for \( z_{\text{GSF}}(x) \), we find

\[ C_\Omega = 1.2510(2) \quad [C_{\Omega}^{\text{BS}} = 1.2512(4)] \]
NR/EOB comparison for an equal mass binary

[Damour, Nagar, Pollney & Reisswig (2012)]
NR/GSF comparison for an equal mass binary

[Le Tiec, Barausse & Buonanno (2012)]
Why do the GSF$_\nu$ results perform so well?

- In perturbation theory, one traditionally expands as

$$\text{GSF}_q: \sum_{n=0}^{n_{\text{max}}} A_n(m_2\Omega) q^n \quad \text{where} \quad q \equiv m_1/m_2 \in [0, 1]$$

- However, the relations $K(\Omega; m_A)$, $E(\Omega; m_A)$, and $J(\Omega; m_A)$ must be symmetric under exchange $m_1 \leftrightarrow m_2$

- Hence, a better-motivated expansion is

$$\text{GSF}_\nu: \sum_{n=0}^{n_{\text{max}}} B_n(m\Omega) \nu^n \quad \text{where} \quad \nu \equiv m_1 m_2/m^2 \in [0, 1/4]$$

- In a PN expansion, we have $B_n = \mathcal{O}(1/c^{2n}) = n\text{PN} + \cdots$
Perturbation theory for comparable mass binaries
How about spins?

- Calculation of $z_{\text{GSF}}(\Omega; S)$ for a particle on a circular equatorial orbit in a Kerr background [Shah, Friedman & Keidl (in progress)]

- Generalization of the first law to spinning point particles [Blanchet, Buonanno & Le Tiec (in progress)]

$$\delta M - \Omega \delta L = \sum_{A=1}^{2} (z_A \delta m_A + \Omega_A \delta S_A)$$

- Exact spin effects at linear order in $\nu$ in binding energy $E$ and total angular momentum $J$

- Shift of the Kerr ISCO under the effect of the conservative SF

- Spin-orbit and spin-spin contributions to EOB potentials
How about orbital evolution?

- Consider a binary on a **quasicircular** orbit with frequency $\Omega(t)$
- Binding energy $E[\Omega(t)]$ known to $\mathcal{O}(\nu)$ [Le Tiec et al. (2012)]
- Compute the **second order** metric perturbation at $\mathcal{I}^+$:
  \[ \mathcal{O}(\nu) \text{ corrections in } h_+[\Omega(t)], h_\times[\Omega(t)], \mathcal{F}[\Omega(t)] \]
- Apply **energy balance** in the adiabatic approximation:
  \[ \frac{dE}{dt} = \mathcal{F} \implies \Omega(t) \text{ accurate to } \mathcal{O}(\nu) \]
- The resulting templates should model the adiabatic inspiral and GW emission from EMRIs and IMRIs accurately
Summary and prospects

- Combined with the first law of mechanics, the redshift $z_1(\Omega)$ provides crucial information about the orbital dynamics.
Summary and prospects

- Combined with the first law of mechanics, the redshift \( z_1(\Omega) \) provides crucial information about the orbital dynamics.
- The GSF results with \( q \to \nu \) compare remarkably well to the NR results, even for binaries with \( m_1 \simeq m_2 \).
Summary and prospects

- Combined with the **first law** of mechanics, the redshift $z_1(\Omega)$ provides crucial information about the orbital dynamics.

- The **GSF results** with $q \rightarrow \nu$ compare remarkably well to the NR results, even for binaries with $m_1 \approx m_2$.

- **Perturbation theory** may be helpful to model the GW emission from IMRIs, or even binaries with comparable masses.
Summary and prospects

• Combined with the **first law** of mechanics, the redshift $z_1(\Omega)$ provides crucial information about the orbital dynamics

• The **GSF results** with $q \rightarrow \nu$ compare remarkably well to the NR results, even for binaries with $m_1 \simeq m_2$

• **Perturbation theory** may be helpful to model the GW emission from IMRIs, or even binaries with comparable masses

• Some directions for future research include:
Summary and prospects

- Combined with the **first law** of mechanics, the redshift $z_1(\Omega)$ provides crucial information about the orbital dynamics.

- The **GSF results** with $q \rightarrow \nu$ compare remarkably well to the NR results, even for binaries with $m_1 \sim m_2$.

- **Perturbation theory** may be helpful to model the GW emission from IMRIs, or even binaries with comparable masses.

- Some directions for future research include:
  - Extending the first law to **spinning** point particles.
Summary and prospects

- Combined with the first law of mechanics, the redshift $z_1(\Omega)$ provides crucial information about the orbital dynamics.
- The GSF results with $q \rightarrow \nu$ compare remarkably well to the NR results, even for binaries with $m_1 \simeq m_2$.
- Perturbation theory may be helpful to model the GW emission from IMRIs, or even binaries with comparable masses.
- Some directions for future research include:
  - Extending the first law to spinning point particles.
  - Adiabatic waveforms using energy balance at relative $O(\nu)$. 
Summary and prospects

- Combined with the first law of mechanics, the redshift $z_1(\Omega)$ provides crucial information about the orbital dynamics.

- The GSF results with $q \rightarrow \nu$ compare remarkably well to the NR results, even for binaries with $m_1 \simeq m_2$.

- Perturbation theory may be helpful to model the GW emission from IMRIs, or even binaries with comparable masses.

- Some directions for future research include:
  - Extending the first law to spinning point particles.
  - Adiabatic waveforms using energy balance at relative $O(\nu)$.
  - Redshift at second order $\rightarrow O(\nu^2)$ corrections in $E(\Omega), J(\Omega)$.
Summary and prospects

- Combined with the first law of mechanics, the redshift $z_1(\Omega)$ provides crucial information about the orbital dynamics.
- The GSF results with $q \rightarrow \nu$ compare remarkably well to the NR results, even for binaries with $m_1 \sim m_2$.
- Perturbation theory may be helpful to model the GW emission from IMRIs, or even binaries with comparable masses.
- Some directions for future research include:
  - Extending the first law to spinning point particles
  - Adiabatic waveforms using energy balance at relative $O(\nu)$
  - Redshift at second order $\rightarrow O(\nu^2)$ corrections in $E(\Omega), J(\Omega)$
  - Non-quasicircular orbits?