Self-force: Foundations and formalism

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Motivation

Extreme-mass-ratio inspirals

- solar-mass neutron star or black hole orbits supermassive black hole
- $m$ emits gravitational radiation, loses energy, spirals into $M$
- waveforms carry information about strong-field dynamics and structure of spacetime near black hole
- need to model motion of small body

![Diagram of extreme-mass-ratio inspirals](image-url)
**Linearized theory**

- treat \( m \) as point particle in background \( g_{\mu\nu} \)

\[
T^{\mu\nu}_{(1)} = \int_\gamma m u^\mu u^\nu \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau
\]

- linearized EFE \( \delta G^{\mu\nu}[h^{(1)}_{\rho\sigma}] = 8\pi T^{\mu\nu}_{(1)} \)

\[
\Rightarrow h^{(1)}_{\mu\nu} = m \int_\gamma G_{\mu\nu\mu'\nu'} u^{\mu'} u^{\nu'} d\tau
\]

**Tails**

- perturbation propagates within light cone

- also, caustics develop—light “cone” intersects itself

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\Rightarrow h^{(1)}_{\mu\nu} \text{ depends on entire past history of } \gamma
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Point particle picture

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Extreme-mass-ratio inspirals

The motion of small bodies in curved spacetime
Extreme-mass-ratio inspirals

The motion of small bodies in curved spacetime
**Self-force: geodesic motion in an effective metric**

**MiSaTaQuWa equation (Mino, Sasaki, Tanaka, & Quinn, Wald)**
- nonlocal tail acts as potential, exerts force $F^\mu \sim m \nabla^\mu \text{tail}$
- tail isn’t nice: non-differentiable, not a solution to a field equation

**Detweiler-Whiting decomposition**
- local field near particle split into two: $h^{(1)}_{\mu\nu} = h^{S(1)}_{\mu\nu} + h^{R(1)}_{\mu\nu}$
- $h^{S(1)}_{\mu\nu} \sim \frac{m}{r} + O(r^0)$; local bound field of particle
- $h^{R(1)}_{\mu\nu} \sim \text{tail} + \text{local terms}$; smooth solution to source-free EFE
- motion is geodesic in effective metric $g_{\mu\nu} + h^{R(1)}_{\mu\nu}$
Outline

1. Introduction
2. Motion of a small extended body
3. Point particle limits & matched asymptotic expansions
4. Equation of motion
5. Finding the global field
1 Introduction

2 Motion of a small extended body

3 Point particle limits & matched asymptotic expansions

4 Equation of motion

5 Finding the global field
A small extended body moving through spacetime

**Fundamental question**
- how does a body’s gravitational field affect its own motion?

**Regime: small body**
- examine spacetime $(\mathcal{M}, g_{\mu\nu})$ containing body of mass $m$ and external lengthscales $\mathcal{R}$
- seek representation of body’s motion when its mass and size are $\ll \mathcal{R}$
Non-perturbative approach [Harte 2011]

**Momentum**
- Assume the body is material, not a black hole
- Give body stress-energy \( T^{\mu \nu} \)
- Define momentum \( P \sim \int_{\text{body}} T^{\mu \nu} \)

**Motion**
- Choose representative worldline \( \gamma \) with coordinates \( z^{\mu}(\tau) \) inside body
- Relate \( u^\mu = \frac{dz^\mu}{d\tau} \) to \( P \)
  \[ \Rightarrow \frac{DP}{d\tau} \] determines acceleration of \( \gamma \)
Motion of a test body in an effective metric

Non-perturbative decomposition

- split metric into "self-field" generated by body and slowly varying remainder

\[
\text{full metric } \ g_{\mu\nu} \quad = \quad \text{"self field" } \ h^S_{\mu\nu} \quad + \quad \text{effective metric } \ g_{\mu\nu} + h^R_{\mu\nu}
\]

Equation of motion

- define multipole moments \( I \sim \int_{\text{body}} T^{\mu\nu} \)
- body moves as test body in effective metric \( g_{\mu\nu} + h^R_{\mu\nu} \): motion is geodesic except for coupling of multipole moments to curvature of effective metric
However...

**Material body**
- integrals over body’s interior preclude description of black hole

**Field**
- describing motion in terms of metric isn’t sufficient: we need a means of solving the EFE to obtain the metric (and a means of isolating the effective metric)
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Perturbation theory

- treat body as source of perturbation of external background spacetime \((\mathcal{M}_E, g_{\mu\nu})\):
  \[
g_{\mu\nu} = g_{\mu\nu} + \epsilon h_{\mu\nu}^{(1)} + \epsilon^2 h_{\mu\nu}^{(2)} + \ldots
\]

- \(\epsilon\) counts powers of \(m\)
- assume body is compact, so as \(m \to 0\), linear size \(\to 0\) at same rate
- seek representation of motion in \((\mathcal{M}_E, g_{\mu\nu})\)
Approach I [Gralla & Wald 2008]: power series

Expansion of EFE

- expand metric in Taylor series:
  \[ g_{\mu \nu}(x, \epsilon) = g_{\mu \nu}(x) + \epsilon h^{(1)}_{\mu \nu}(x) + \epsilon^2 h^{(2)}_{\mu \nu}(x) + \ldots \]

- solve EFE order by order outside body:
  \[ \delta G_{\mu \nu}[h^{(1)}] = 0 \]
  \[ \delta G_{\mu \nu}[h^{(2)}] = -\delta^2 G_{\mu \nu}[h^{(1)}] \]
  \[ \vdots \]

- motion determined by Bianchi identity
Representation of motion in power series

Expanded worldline
- Worldline $\gamma_0$ identified as remnant of body left at $\epsilon = 0$
- $\gamma_0$ is geodesic
- Corrections accounted for by deviation vector $\delta \gamma$

Problem
- As body drifts away from $\gamma_0$, $\delta \gamma$ grows large
- Representation of motion only meaningful and accurate for short time
Expansion of EFE

- allow $\gamma$ to depend on $\epsilon$ and assume expansion of form

$$\begin{align*}
g_{\mu\nu}(x, \epsilon) &= g_{\mu\nu}(x) + h_{\mu\nu}(x; \gamma_{\epsilon}) \\
&= g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x; \gamma_{\epsilon}) + \epsilon^2 h^{(2)}_{\mu\nu}(x; \gamma_{\epsilon}) + \ldots
\end{align*}$$

- need a method of systematically solving for each $h^{(n)}_{\mu\nu}$
  $\Rightarrow$ impose Lorenz gauge (or other wave gauge) on the total perturbation: $\nabla_\mu \bar{h}^{\mu\nu} = 0$

- $\delta G_{\mu\nu}$ becomes a wave operator and EFE outside body becomes
  weakly nonlinear wave equation:

$$\Box \bar{h}_{\mu\nu} + 2R_{\mu}^{\rho \nu \sigma} \bar{h}_{\rho\sigma} = 2\delta^2 G_{\mu\nu}[\bar{h}] + \ldots$$

- can be split into wave equations for each subsequent $h^{(n)}_{\mu\nu}[\gamma]$ and
  exactly solved for arbitrary $\gamma$

- gauge condition will then constrain $\gamma$
How to determine motion? Buffer region

- define buffer region by $m \ll r \ll R$
- because $m \ll r$, can treat mass as small perturbation of external background
- because $r \ll R$, can use information about small body

inner region ($r \sim m$)
buffer region
external universe ($r \sim R$)
Matched asymptotic expansions: *inner expansion*

**Zoom in on body**

- use scaled coords $\tilde{r} \sim r/\epsilon$ to keep size of body fixed, send other distances to infinity as $\epsilon \to 0$
- unperturbed body defines background spacetime $g_{I\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity $r \gg m$
  $\Rightarrow$ can define multipole moments without integrals over body
Representation of motion in self-consistent approximation

Enforce a relationship between the expansions
...to define a worldline for all time, even for black hole

- in buffer region, write metric in coordinates centered on $\gamma$
- make body at “center” of coordinates, in that its mass dipole vanishes
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Local coordinates

Fermi-Walker coordinates
- spatial coordinates $x^a$ span surface intersecting $z^\mu(\tau)$ orthogonally
- time $t$ on that surface $= \text{proper time } \tau$
- radial distance $r^2 = \delta_{ab}x^ax^b$ is geodesic distance from $\gamma$
Solving the EFE in buffer region

Expansion for small $r$

- allow all negative powers of $r$ in $h_{\mu\nu}^{(n)}$
- but inner expansion must not have negative powers of $\epsilon$
  $\Rightarrow$ most singular power of $r$ in $\epsilon^n h_{\mu\nu}^{(n)}$ is $\frac{\epsilon^n}{r^n} = \frac{\epsilon^n}{\tilde{r}^n} = \frac{1}{\tilde{r}^n}$

Therefore

$$h_{\mu\nu}^{(n)} = \frac{1}{r^n} h_{\mu\nu}^{(n,-n)} + r^{-n+1} h_{\mu\nu}^{(n,-n+1)} + r^{-n+2} h_{\mu\nu}^{(n,-n+2)} + \ldots$$

Information from inner expansion

- $1/\tilde{r}^n$ terms arise from asymptotic expansion of zeroth-order background in inner expansion
  $\Rightarrow$ $h_{\mu\nu}^{(n,-n)}$ is determined by multipole moments of isolated body
Form of solution in buffer region

What appears in the solution?

- throw expansion into $n$th-order wave equation, solve order by order in $r$
- expand each $h_{\mu\nu}^{(n,p)}$ in spherical harmonics
- given a worldline $\gamma$, the solution at all orders is fully characterized by
  1. body's multipole moments (and corrections thereto): $\sim \frac{Y_{\ell m}}{r^{\ell+1}}$
  2. smooth solutions to vacuum wave equation: $\sim r^{\ell} Y_{\ell m}$
- everything else made of (linear or nonlinear) combinations of the above

Self field and regular field

- multipole moments define $h_{\mu\nu}^{S(n)}$; interpret as bound field of body
- smooth homogeneous solutions define $h_{\mu\nu}^{R(n)}$; free radiation, determined by global boundary conditions
First order

- \( h_{\mu\nu}^{(1)} = h_{\mu\nu}^{S(1)} + h_{\mu\nu}^{R(1)} \)
- \( h_{\mu\nu}^{S(1)} \sim 1/r + O(r^0) \) defined by mass monopole \( m \)
- \( h_{\mu\nu}^{R(1)} \) is undetermined homogenous solution regular at \( r = 0 \)
- evolution equations (from gauge condition): \( \dot{m} = 0 \) and \( a_\mu^{(0)} = 0 \) (assuming \( a_\mu = a_\mu^{(0)} + \epsilon a_\mu^{(1)} + \ldots \))

Second order

- \( h_{\mu\nu}^{(2)} = h_{\mu\nu}^{S(2)} + h_{\mu\nu}^{R(2)} \)
- \( h_{\mu\nu}^{S(2)} \sim 1/r^2 + O(1/r) \) defined by
  1. mass correction \( \delta m \)
  2. mass dipole \( M_\mu \)
  3. spin dipole \( S_\mu \)
- evolution equations: \( \dot{S}_\mu = 0, \delta \dot{m} = \ldots, \) and \( \dot{M}_\mu = \ldots \)
A master equation of motion

Evolution of mass dipole

\[
\dddot{M}^\alpha - R^\alpha_{\beta\gamma\delta} u^\beta u^\gamma M^\delta = -ma^\alpha_{(1)} + \frac{1}{2} R^\alpha_{\beta\gamma\delta} u^\beta S^{\gamma\delta} - \frac{1}{2} m (g^{\alpha\delta} + u^\alpha u^\delta) \left( 2h^{R(1)}_{\delta\beta;\gamma} - h^{R(1)}_{\beta\gamma;\delta} \right) u^\beta u^\gamma
\]

Includes

- geodesic deviation
- first-order term in acceleration of $\gamma$
- Mathisson-Papapetrou spin force
- self-force (force due to regular field)
- this relationship between $a^\alpha$ and $M^\alpha$ is valid for any $\gamma$
Self-force in self-consistent expansion

- $\gamma$ defined by $M_\alpha(t) \equiv 0$. Therefore

$$a^{\alpha}_{(1)} = -\frac{1}{2} \left( g^{\alpha \delta} + u^\alpha u^\delta \right) \left( 2h^{R(1)}_{\delta \beta; \gamma} - h^{R(1)}_{\beta \gamma; \delta} \right) u^\beta u^\gamma$$

- through order $\epsilon$, small body moves on a geodesic of $g_{\mu \nu} + h^R_{\mu \nu}$

Self-force in power series expansion

- $\gamma$ is geodesic, so $a^\mu_{(n)} = 0$. Therefore

$$\partial_t^2 M^\alpha = R^\alpha_{\beta \gamma \delta} u^\beta u^\gamma M^\delta - \frac{1}{2} m \left( g^{\alpha \delta} + u^\alpha u^\delta \right) \left( 2h^{R(1)}_{\delta \beta; \gamma} - h^{R(1)}_{\beta \gamma; \delta} \right) u^\beta u^\gamma$$
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Effective interior metric

From self-field to singular field

- $h^S_{\mu\nu}$ and $h^R_{\mu\nu}$ derived only in buffer region
- simply extend them to all $r > 0$ (and $r = 0$, for $h^R_{\mu\nu}$)
- does not change field in buffer region or beyond

full metric $g_{\mu\nu}$  =  "self field" $h^S_{\mu\nu}$  +  effective metric $g_{\mu\nu} + h^R_{\mu\nu}$
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\[
\text{full metric } g_{\mu\nu} = \text{singular field } h^S_{\mu\nu} + \text{effective metric } g_{\mu\nu} + h^R_{\mu\nu}
\]
Obtaining global solution

**Puncture/effective source scheme**

- Define $h^P_{\mu\nu}$ as small-$r$ expansion of $h^S_{\mu\nu}$ truncated at order $r$ or higher.
- Define $h^R_{\mu\nu} = h_{\mu\nu} - h^P_{\mu\nu} \approx h^R_{\mu\nu}$.

The point...

- $h^S_{\mu\nu}$ found in buffer region suffices to determine effective metric “inside” body and full metric everywhere else.

\[ \delta G^{\mu\nu}[h_{\rho\sigma}] = -(\delta^2 G^{\mu\nu}[h_{\rho\sigma}] + \ldots) \]

\[ \delta G^{\mu\nu}[h^R_{\rho\sigma}] = 8\pi T^{\mu\nu}[\gamma] - (\delta^2 G^{\mu\nu}[h_{\rho\sigma}] + \ldots) - \delta G^{\mu\nu}[h^P_{\rho\sigma}] \]
What looks like the source of the perturbation?

- All terms in $h^S_{\mu\nu}$ are (linear and nonlinear) combinations of multipole moment terms $\sim Y^{\ell m}/r^{\ell+1}$
- Using $\partial^i \partial_i 1/r = -4\pi \delta^3(x^a)$, can show moments are effectively sourced by:
  $$T^{\mu\nu}[\gamma] = \sum_\ell \int_\gamma I^{\mu\nu\alpha_1 \cdots \alpha_\ell} \nabla_{\alpha_1} \cdots \nabla_{\alpha_\ell} \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau$$
- In buffer region and outside it, body looks like a skeleton of multipole moments on $\gamma$

Point particle picture recovered

- At first order, there is only the mass monopole
- $T^{\mu\nu}_{(1)}[\gamma] = \int_\gamma m u^\mu u^\nu \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau$
- All the early point-particle results hold true
Determining the motion of a small body

- a self-gravitating material body moves as a test body in an effective geometry $g_{\mu\nu} + h_{\mu\nu}^R$
- EFE solved perturbatively to find full field $h_{\mu\nu}$ outside body and the piece $h_{\mu\nu}^R$ that determines the motion
- singular field $h_{\mu\nu}^S$ calculated in buffer region outside body suffices to determine both $h_{\mu\nu}^R$ and $h_{\mu\nu}$

Current status

- point particle picture and MiSaTaQuWa equation have been justified
- for spherical body, analytical portion of problem now also complete at second order [Pound 2012, Gralla 2012]
- for more general body, we will require some model for evolution of body’s multipole moments