Electromagnetic Self-force and Overcharging a Reissner-Nordström Black Hole

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The Weak Cosmic Censorship Conjecture posits that spacetime singularities are hidden behind event horizons, making them invisible to all observers in the external universe.

Counterexamples:
1. Critical collapse of a massless scalar field with infinitely fine-tuned initial conditions [Choptuik 1993].
Infalling test-charges in Reissner-Nördstrom

- The spacetime of a spherically symmetric charged black hole is described by the Reissner-Nördstrom (RN) line element

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2dΩ^2, \quad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]

- We consider a particle of mass \( m \) and charge \( q \gg m \) following a radial path towards a RN black hole having mass \( M \) and charge \( Q = M(1 - 2\epsilon^2) \) for small positive \( \epsilon \).

- The RN spacetime admits a timelike Killing vector \( t^\alpha = \frac{\partial x^\alpha}{\partial t} \) that gives rise to a conserved energy \( E_0 = -t^\alpha(mu_\alpha + qA_\alpha) \).
Infalling Test-charges in Reissner-Nordström

- Overcharging occurs when $Q + q > M + E_0$.
- A test charge with an open set of allowed parameters \( \{ q, m, E_0 \} \) moving according to $m^2 \dot{r}^2 = (E_0 - qQ/r)^2 - m^2 f(r)$ can cross the event horizon and overcharge a near-extreme black hole.
Overcharging Conditions

The overcharging conditions are

1. $\dot{r}^2 > 0$, $\forall r \geq r_+$
2. $Q + q > M + E_0$

By setting $M \equiv 1$, $Q \equiv 1 - 2\epsilon^2$, Hubeny showed that the parameter space for overcharging is the three-parameter family characterized by

\[
\begin{align*}
q &= a\epsilon \\
E_0 &= a\epsilon - 2b\epsilon^2 \\
m &= c\epsilon
\end{align*}
\]

\[
\begin{align*}
a &> 1, \\
1 < b < a, \\
c &< \sqrt{a^2 - b^2}.
\end{align*}
\]
Including Self-Force Effects

- Hubeny’s overcharging condition fails to account for energy lost by the particle due to radiation, which is also $O(\epsilon^2)$.
- The full $O(\epsilon^2)$ overcharging condition reads

$$q + Q > M + E_0 - E_{\text{rad}},$$

where $E_{\text{rad}}$ is the energy radiated to null future infinity.
- In addition, the radial acceleration of a Hubeny orbit at the event horizon is $O(\epsilon^2)$. Therefore self-force corrections, which are also of order $\epsilon^2$, must be included in the infall condition.
- The self-force is incorporated by computing the work it does on the particle as it moves inward

$$E(r) = E_0 - q \int_r^{\infty} F_{tr}^{\text{self}} \, dr$$

- The infall condition in the presence of the self-force is

$$(E(r) - qQ/r)^2 > m^2 f(r), \quad \forall r \geq r_+$$
The Initial Data Struggle

- We evolve the multipoles of the retarded field using a $1 + 1$ time-domain scheme with trivial initial data.
- The use of unphysical/inconsistent initial data creates a burst of “junk radiation”.
- Typical overcharging orbits are very high speed. In these cases, where the particle and the ingoing junk radiation are traveling in close proximity, the self-force at the particle is severely contaminated.
- We’ve tried several things to minimize the effect of the junk: static initial data, adiabatic transition of the source, transition of the trajectory, and direct excision of the ingoing null ray. Direct excision proved to be the most effective.
Grid Excision

- Without excision, clean self-force data is unavailable until after the ingoing junk reflects from the peak of the potential, interacts with the particle, and finally clears away.
- To deal with the burst and reflection of ingoing junk, we excise grid points covering the ingoing null ray from the domain.
Excision Results

\[ r^2 (F_{tr})_\ell \]

\[ r^2 (F_{tr})_\ell - A \]

\[ r^2 (F_{tr})_\ell - A - B \]

\[ r^2 (F_{tr})_\ell - A - B - D \]

\[ 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \quad 10^2 \]

\[ 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \]

\[ \ell \]

\[ 10^{-5} \quad 10^{-4} \quad 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^0 \quad 10^1 \]

\[ 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \]

\[ \ell \]
Overall, the local self-force $f_{loc}^\alpha = \frac{2}{3} q^2 \frac{D\alpha}{d\tau}$ captures the general features of the full self-force.
Searching for Overcharging Candidates

To search for overcharging candidates, we randomly generate parameters in the \( \{a, b, c\} \) space and implement a two-step selection procedure:

1. We use the local self-force to select only the orbits for which the particle crosses the event horizon.
2. From these orbits, we select only those whose parameters satisfy the overcharging condition

\[
q + Q > E_0 - E_{\text{rad}} + M
\]

where \( E_{\text{rad}} \) is approximated using the Larmor formula

\[
\frac{dE_{\text{rad}}}{dt} \approx \frac{2}{3} \frac{q^2}{m^2} F_{\mu}^\mu F_{\text{BH} \mu}
\]
For this candidate the full self-force is not sufficient to turn the particle around. The flux is required to determine if the particle actually overcharges the BH.
Conclusions and Future Work

- A Monte Carlo scan of the parameter space, based on crude estimates of the flux and the self-force, reveals candidate trajectories that may overcharge the BH.
- These cases (and others) require fuller scrutiny with exact calculations of the flux and self-force.
- Thanks to the excision trick, techniques to carry out the computations are in hand.
- Flux calculations are underway and answers will be forthcoming in the next weeks.
- An argument, based on extended bodies and the third law of BH mechanics is being devised which suggests that the self-force must enforce cosmic censorship.
Appendix I: Particle Motion

The equation of motion for a particle of mass $m$ and charge $q$ moving under the influence of an external electromagnetic field $F_{\alpha\beta}^{\text{ext}}$ is given by

$$ma_\alpha = qF_{\alpha\beta}^{\text{ext}} u^\beta.$$ 

Radial, accelerated motion in Reissner-Nordström spacetime is described by the equation

$$\frac{d^2r^*}{dt^2} = -\frac{m^2 f}{(E_0r - qQ)^3} \left( r ME_0 + qQ(M - r) - Q^2E_0 \right),$$

where

$$E_0 = \sqrt{r^2 + f + qQ/r}$$

in Schwarzschild coordinates $x^\alpha = \{t, r, \theta, \phi\}$. 
Appendix II: Retarded Field

▶ The retarded field \( F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} \) obeys the sourced Maxwell Equation

\[
g^{\mu\beta} \nabla_{\beta} F_{\alpha\mu} = 4\pi j_{\alpha},
\]

where \( A_{\alpha} \) is the electromagnetic potential and

\[
j_{\alpha} = q \int u_{\alpha} \delta_{4}(x, z(\tau)) d\tau.
\]

▶ Spacetime tensors are written as multipole expansions in spherical harmonics which decouple the radial and temporal tensor components from angular ones.

▶ Radial motion is imposed which sets both the \( m \)-modes and the angular current components to zero.
Appendix III: Retarded Field

- For radial motion, the relevant field multipole is

\[ F_{tr}^\ell = -\frac{1}{r^2} \psi^\ell(t, r) Y^\ell(\theta^A), \]

and the non-zero source components are given by

\[ j^\ell_t = -q \sqrt{\frac{2\ell + 1}{4\pi}} \frac{f_0}{r_0^2} \delta(r - r_0(t)), \]

\[ j^\ell_r = q \sqrt{\frac{2\ell + 1}{4\pi}} \frac{dr_0/dt}{f_0 r_0^2} \delta(r - r_0(t)). \]

- The field \( \psi^\ell \) obeys the wave equation

\[ 4\pi f \left[ \partial_t (r^2 j^\ell_r) - \partial_r (r^2 j^\ell_t) \right] = -\partial_t^2 \psi^\ell + f \partial_r (f \partial_r \psi^\ell) - \frac{\ell(\ell + 1)}{r^2} f \psi^\ell. \]
Appendix IV: Retarded Field - Numerical Method

- The wave equation for $\psi^\ell$ is written in standard form

$$\left[ \partial^2_{r^*} - \partial^2_t - V^\ell(r) \right] \psi^\ell(t, r) = S^\ell(r_0(t), r).$$

- The source cell is divided into sub-areas $A_{i=1...4}$ based on the locations where the particle enters and leaves the cell: $(t_1, r_{1}^*)$ and $(t_2, r_2^*)$. 

![Diagram of sub-areas](image)
Appendix V: Retarded Field - Numerical Method

- The field is evolved according to the second-order algorithm of Lousto and Price

\[
\psi_{\text{vac}}(t + h, r^*) = -\psi(t - h, r^*) + [\psi(t, r^* + h) \\
+ \psi(t, r^* - h)] \left[1 - \frac{1}{2} h^2 V(r^*)\right] + O(h^4),
\]

\[
\psi_{\text{source}}(t + h, r^*) = -\psi(t - h, r^*) \left[1 + \frac{V(r^*)}{4} (A_2 - A_3)\right] \\
+ \psi(t, r^* + h) \left[1 - \frac{V(r^*)}{4} (A_3 + A_4)\right] \\
+ \psi(t, r^* - h) \left[1 - \frac{V(r^*)}{4} (A_1 + A_3)\right] \\
- \frac{1}{4} \left[1 - \frac{V(r^*)}{4} A_4\right] \int \int_{\text{cell}} S \, du \, dv + O(h^3).
\]
Appendix VI: Singular Field

- The potential $A_\alpha$ contains a piece which is singular on the world line.
- The multipole coefficients of the singular field constitute the regularization parameters $A, B, C,$ and $D$.
- The regularization parameters are computed perturbatively using a local expansion about the world line. Their values are

\[
A = -\frac{1}{r_0^2}\text{sign}(\Delta), \\
B = -\frac{E}{2mr_0^2} + \frac{qQ}{mr_0^3}, \\
C = 0, \\
D = -\frac{3}{16} \frac{E(E - m)(E + m)}{m^3r_0^2} + \frac{3}{4mr_0^3} \left( \frac{qQ(m^2 + E^2)}{2m^2} - ME \right).
\]