Near-Cloaking by Change of Variables at Finite Frequency, I: An Approach using Lossy Layers

Robert V. Kohn
Courant Institute, NYU

CSCAMM, September 2008

Collaborators: D. Onofrei, M. Vogelius, M. Weinstein

This talk: framework and theory
Onofrei: examples and numerics
What is cloaking?

- the cloaked region should be invisible
- even the cloak itself should be invisible
- our cloaks will be coatings with heterogeneous, anisotropic dielectric properties

In what sense invisible?

- this talk: Helmholtz at fixed frequency
What is cloaking?

- the cloaked region should be invisible
- even the cloak itself should be invisible
- our cloaks will be coatings with heterogeneous, anisotropic dielectric properties

In what sense \textit{invisible}?

- this talk: Helmholtz at fixed frequency
(1) Cloaking by change of variables
   - The basic idea
   - Approximate cloaks and inclusion problems

(2) Does it work?
   - At frequency 0: yes
   - At frequency \( \neq 0 \): problem due to resonance
   - Resolution: damping

(3) How well does it work?
   - 2D case (is \( 1/|\log \rho| \) small?)
   - 3D case (much better)

Change-of-variable scheme introduced by:
   - Greenleaf, Lassas, Uhlmann (2003, freq 0 = impedance tomography)
   - Pendry, Schurig, Smith (2006, finite freq = electromag scattering)

Just one approach to cloaking; others include
   - anomalous localized resonance (Milton, Nicorovici)
   - optical conformal mapping (Leonhardt)
Basic definitions

\[ \sum \frac{\partial}{\partial x_i} \left( A_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \omega^2 q(x) u = 0 \quad \text{in } \Omega \]

Neumann-to-Dirichlet map characterizes “boundary measurements” (invertible if \( \omega^2 \) is not an eigenvalue)

\[ \Lambda_{A,q} : (A\nabla u) \cdot \nu|_{\partial \Omega} \rightarrow u|_{\partial \Omega} \]

Same DN map \( \iff \) same scattering data.

Cloaking in this setting: \( A_c(x) \) and \( q_c(x) \), defined on \( \Omega \setminus D \), cloak \( D \) if resulting bdry measurments “look uniform,” indep of content of \( D \).

\( (A,q) = (1,1) \)

uniform case: \( \Lambda_{1,1} \)

\( A(x), q(x) = \begin{cases} 
A_c(x), q_c(x) & \text{for } x \in \Omega \setminus D \\
\text{arbitrary} & \text{for } x \in D 
\end{cases} \)

arbitrary

same: \( \Lambda_{A,q} = \Lambda_{1,1} \)
Scattering seeks knowledge of interior properties, based on response to plane waves.

Exterior sees \( \Omega \) only via Cauchy data ("bdry meas" or "DN map").

We say \( A_c(x), q_c(x) \) (defined in \( \Omega \setminus D \)) cloaks \( D \) if the Cauchy data at \( \partial \Omega \) are (a) indep of content of \( D \), and (b) same as for uniform case \( A = q = 1 \).

Name is apt, since extn of \( A_c, q_c \) by 1 to larger domain is also a cloak.

\[(A,q) = (1,1) \]

unif case: \( \Lambda_{1,1} \)

arbitrary \( A_c, q_c \)

\[(A,q) = (1,1) \]

same as unif case: \( \Lambda_{A,q} = \Lambda_{1,1} \)
Invariance under change of variables

Basic observation: bdry meas determine material properties at most “up to change of variables.”

If \( F : \Omega \to \Omega \) is invertible and \( F(x) = x \) on \( \partial \Omega \) then \( A, q \) and \( F^*A, F^*q \) produce the same boundary measurements, where

\[
F^*A(y) = \frac{1}{\det(DF)(x)}DF(x)A(x)(DF(x))^T \quad \text{and} \quad F^*q(y) = \frac{1}{\det(DF)(x)}q(x)
\]

with \( y = F(x) \).

Sketch: write PDE in weak form, then change variables.

- weak form: \( \int_{\Omega} \langle A\nabla_x u, \nabla_x \phi \rangle - \omega^2 q u \phi \, dx = 0 \) if \( \phi = 0 \) near \( \partial \Omega \)
- change vars: \( \int_{\Omega} \langle F^*(A)\nabla_y u, \nabla_y \phi \rangle - \omega^2 F^*(q)u \phi \, dy = 0 \)
- \( F = \text{id} \) at bdry \( \Rightarrow \) chg of vars doesn’t affect bdry data
Invariance under change of variables

Basic observation: bdry meas determine material properties at most “up to change of variables.”

If $F : \Omega \rightarrow \Omega$ is invertible and $F(x) = x$ on $\partial \Omega$ then $A, q$ and $F^*A, F^*q$ produce the same boundary measurements, where

$$F^*A(y) = \frac{1}{\det(DF)(x)} DF(x)A(x)(DF(x))^T \quad F^*q(y) = \frac{1}{\det(DF)(x)} q(x)$$

with $y = F(x)$.

Sketch: write PDE in weak form, then change variables.
- weak form: $\int_\Omega \langle A\nabla_x u, \nabla_x \phi \rangle - \omega^2 qu\phi \, dx = 0$ if $\phi = 0$ near $\partial \Omega$
- change vars: $\int_\Omega \langle F^*(A)\nabla_y u, \nabla_y \phi \rangle - \omega^2 F^*(q)u\phi \, dy = 0$
- $F = \text{id}$ at bdry $\Rightarrow$ chg of vars doesn’t affect bdry data
Invariance under change of variables

Basic observation: bdry meas determine material properties at most “up to change of variables.”

If \( F : \Omega \to \Omega \) is invertible and \( F(x) = x \) on \( \partial \Omega \) then \( A, q \) and \( F^* A, F^* q \) produce the same boundary measurements, where

\[
F^* A(y) = \frac{1}{\det(DF) (x)} DF(x) A(x)(DF(x))^T \quad F^* q(y) = \frac{1}{\det(DF) (x)} q(x)
\]

with \( y = F(x) \).

Sketch: write PDE in weak form, then change variables.

- weak form: \( \int_\Omega \langle A \nabla_x u, \nabla_x \phi \rangle - \omega^2 q u \phi \, dx = 0 \) if \( \phi = 0 \) near \( \partial \Omega \)
- change vars: \( \int_\Omega \langle F^*(A) \nabla_y u, \nabla_y \phi \rangle - \omega^2 F^*(q) u \phi \, dy = 0 \)

\( F = \text{id} \) at bdry \( \Rightarrow \) chg of vars doesn’t affect bdry data
Invariance under change of variables

Basic observation: bdry meas determine material properties at most “up to change of variables.”

If $F : \Omega \to \Omega$ is invertible and $F(x) = x$ on $\partial \Omega$ then $A$, $q$ and $F^*A$, $F^*q$ produce the same boundary measurements, where

$$F^*A(y) = \frac{1}{\det(DF)(x)} DF(x)A(x)(DF(x))^T \quad F^*q(y) = \frac{1}{\det(DF)(x)} q(x)$$

with $y = F(x)$.

Sketch: write PDE in weak form, then change variables.

- weak form: $\int_{\Omega} \langle A \nabla_x u, \nabla_x \phi \rangle - \omega^2 q u \phi \, dx = 0$ if $\phi = 0$ near $\partial \Omega$
- change vars: $\int_{\Omega} \langle F^*(A) \nabla_y u, \nabla_y \phi \rangle - \omega^2 F^*(q) u \phi \, dy = 0$
- $F = id$ at bdry $\Rightarrow$ chg of vars doesn’t affect bdry data

Robert V. Kohn  Courant Institute, NYU
Near-cloaking by change of variables
Radial version, for simplicity only: domain is $B_2$, cloaked region is $B_1$.

Choose properties of the cloak to be $A_c = F^*1$ and $q_c = F^*1$, where $F$ “blows up” the origin to $B_1$:

$$F(x) = (1 + \frac{1}{2}|x|) \frac{x}{|x|}$$

Formally $B_1$ is cloaked. In fact, if

$$(A(y), q(y)) = \begin{cases} F^*(1, 1) & \text{for } y \in B_2 \setminus B_1 \\ \text{arbitrary} & \text{for } y \in B_1 \end{cases}$$

we have, using $F^{-1}$ as our change of variable,

$$\int_{B_2} \langle A(y) \nabla_y u, \nabla_y \phi \rangle - \omega^2 q(y) u \phi \, dy = \int_{B_2} \langle \nabla_x u, \nabla_x \phi \rangle - \omega^2 u \phi \, dx$$

since $F^{-1}$ shrinks $B_1$ (the region being cloaked) to a point.

Is this correct? $F$ and $F^{-1}$ are very singular.
Remarks on the singular cloak

- This scheme requires exotic materials. Recall that

\[(A_c(y), q_c(y)) = F_*(1, 1)\]

at \(y = F(x)\)

where \(F\) blows up a point to the region being cloaked. The material is anisotropic and singular: as \(|y| \downarrow 1\), \(A_c(y)\) has

- radial eigenvector with eigenvalue \(\sim (|y| - 1)^{n-1}\)
- tangential eigenspace with eigenvalue \(\sim (|y| - 1)^{n-3}\)

and \(q_c(y) \sim (|y| - 1)^{2(n-1)}\).

- Analysis is possible, but requires suitable notion of “weak solution” (Greenleaf, Kurylev, Lassas, Uhlmann, CMP 2008).

- The singular cloak makes me uncomfortable. We usually deal with singularities by smoothing them. Why not here?
A regularized version

Same idea, with more regular $F$. Domain $B_2$, cloaked region $B_1$.

Approx cloak uses $(A_c, q_c) = F_*(1, 1)$, where $F = F_\rho$ is less singular:

- $F$ is cont's and piecewise smooth
- it expands $B_\rho$ to $B_1$ while preserving $B_2$
- $F(x) = x$ at the outer bdry $|x| = 2$.

Impact of contents of $B_1$ on bdry data becomes, via change of vars, effect of small inclusion with uncontrolled properties. In fact, if

$$(A(y), q(y)) = \begin{cases} 
F_* (1, 1) & \text{for } y \in B_2 \setminus B_1 \\
A_D(y), q_D(y) & \text{for } y \in B_1
\end{cases}$$

then, using $F^{-1}$ as change of variable,

$$\int_{B_2} \langle A(y) \nabla_y u, \nabla_y \phi \rangle - \omega^2 q(y) u \phi \, dy = \int_{B_2 \setminus B_\rho} \langle \nabla_x u, \nabla_x \phi \rangle - \omega^2 u \phi \, dx + \int_{B_\rho} \langle F^{-1} (A_D) \nabla_x u, \nabla_x \phi \rangle - \omega^2 F_*^{-1} (q_D) u \phi \, dx.$$

Approximate cloaking $\iff$ small inclusion with uncontrolled content has little effect on bdry meas.
A regularized version

Same idea, with more regular $F$. Domain $B_2$, cloaked region $B_1$.

Approx cloak uses $(A_c, q_c) = F_*(1, 1)$, where $F = F_\rho$ is less singular:

- $F$ is cont's and piecewise smooth
- it expands $B_\rho$ to $B_1$ while preserving $B_2$
- $F(x) = x$ at the outer bdry $|x| = 2$.

Impact of contents of $B_1$ on bdry data becomes, via change of vars, effect of small inclusion with uncontrolled properties. In fact, if

$$(A(y), q(y)) = \begin{cases} 
F_*(1, 1) & \text{for } y \in B_2 \setminus B_1 \\
A_D(y), q_D(y) & \text{for } y \in B_1
\end{cases}$$

then, using $F^{-1}$ as change of variable,

$$
\int_{B_2} \langle A(y) \nabla_y u, \nabla_y \phi \rangle - \omega^2 q(y) u_\phi \, dy = \int_{B_2 \setminus B_\rho} \langle \nabla_x u, \nabla_x \phi \rangle - \omega^2 u_\phi \, dx + \int_{B_\rho} \langle F^{-1}_*(A_D) \nabla_x u, \nabla_x \phi \rangle - \omega^2 F^{-1}_*(q_D) u_\phi \, dx.
$$

Approximate cloaking $\Leftrightarrow$ small inclusion with uncontrolled content has little effect on bdry meas.
Frequency 0 is OK

Singular cloak works at frequency 0 (Greenleaf, Lassas, Uhlmann 2003)
Explanation via regularization (Kohn, Shen, Vogelius, Weinstein 2008):

$$\nabla \cdot (A \nabla u) = 0 \quad \text{in } \Omega, \quad \Lambda_A = \text{DN map}$$

Theorem: If $A \equiv 1$ outside $B_\rho$, then

$$\| \Lambda_A - \Lambda_1 \| \leq C \rho^n \quad \text{in space dim } n.$$ 

- Use operator norm, $\Lambda_A : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)$. Natural choice, since finite-energy solutions of $\nabla \cdot (A \nabla u) = 0$ have Dirichlet data in $H^{1/2}$ and Neumann data in $H^{-1/2}$.

- Estimate is well-known when inclusion has constant conductivity – even for the extreme cases, when $A = 0$ or $A = \infty$ in $B_\rho$.

- Variational principle implies that effect of any inclusion is bracketed by effect of extreme inclusions.

So: our regularized scheme almost cloaks $B_1$, if $\rho$ is small.
Finite frequency is different

Recall: approx cloaking achieved ⇔ small inclusion with uncontrolled content has little effect on bndry meas.

But: at finite frequency a small inclusion can have huge effect, due to resonance. Consider radial setting:

\[(A, q) = \begin{cases} (1, 1) & \text{in } B_2 \setminus B_\rho \\ (A_\rho, q_\rho) & \text{in } B_\rho \end{cases}\]

Separate variables:

\[u = \sum \alpha_k J_k \left( \omega r \sqrt{q_\rho / A_\rho} \right) e^{ik\theta} \quad \text{for } r < \rho\]

\[u = \sum \left( \beta_k J_k (\omega r) + \gamma_k H_k^{(1)} (\omega r) \right) e^{ik\theta} \quad \text{for } \rho < r < 2\]

At freq \(k\): 3 unknowns \(\alpha_k, \beta_k, \gamma_k\) and 3 eqns:

1 eqn at \(r = 2\) to match Neumann data
2 eqns at \(r = 2\) to impose transmission bdry cond

Hence unique solution if eqns are not redundant. But eqns are redundant at special \(A_\rho, q_\rho\) (resonances).
Greenleaf, Kurylev, Lassas, Uhlmann (CMP 2008) studied cloaking for 3D Helmholtz by (singular) change of variables. Their conclusion: if

\[(A, q) = \begin{cases} F^*(1, 1) & \text{in } \Omega \setminus D \\ (A_D, q_D) & \text{in } D \end{cases} \]

then \( \nabla \cdot (A \nabla u) + \omega^2 qu = 0 \) exactly when

- outside the cloaked region, \( u(y) = v(x) \) where \( y = F(x) \) and \( \Delta v + \omega^2 v = 0 \) in \( \Omega \).
- inside the cloaked region, \( u \) solves given PDE with Neumann data 0

Indicates cloaking (since \( v \) is indep of inclusion). But clearly problematic if Neumann problem for cloaked region has a resonance.
Before mapping:
uncontrolled inclusion of size $\rho$
coated by isotropic lossy shell of width $\rho$

After mapping:
uncontrolled inclusion of size $\frac{1}{2}$
coated by isotropic lossy shell of width $\frac{1}{2}$

$$A, q = \begin{cases} (1, 1) & \text{for } |x| > 2\rho \\ (1, 1 + i\beta) & \text{for } \rho < |x| < 2\rho \\ \text{arbitrary} & \text{for } |x| < \rho \end{cases}$$

$$A, q = \begin{cases} F_*(1, 1) & \text{for } |y| > 1 \\ F_*(1, 1 + i\beta) & \text{for } \frac{1}{2} < |y| < 1 \\ \text{arbitrary} & \text{for } |y| < \frac{1}{2} \end{cases}$$

Successful $\iff$ presence of inclusion has little effect on DN map, regardless of inclusion contents.

Our results:
- best choice of damping is $\beta \sim \rho^{-2}$
- effect of inclusion is $1/|\log \rho|$ in 2D, $\sqrt{\rho}$ in 3D.

Suboptimal in 3D? Intuition and numerics suggest $\rho$ not $\sqrt{\rho}$. 
Claim: an arbitrary but small inclusion, coated by a lossy layer, has little effect on bdry meas, if loss parameter is $\beta \sim \rho^{-2}$.

$$A, q = \begin{cases} 
(1, 1 + i\rho^{-2}) & \text{for } \rho < |x| < 2\rho \\
\text{arbitrary pos} & \text{for } |x| < \rho
\end{cases}$$

Theorem. When embedded in a uniform medium ($A = 1, q = 1$), the effect of such an inclusion is bounded by

$$\|\Lambda_{A,q} - \Lambda_{1,1}\| \leq C_\omega / |\log \rho|.$$ 

LHS is operator norm from $H^{-1/2}$ to $H^{1/2}$ (natural norms for Neumann and Dirichlet data of finite-energy solutions). If $f = \sum a_k e^{ik\theta}$,

$$\|f\|_{H^{-1/2}}^2 = \sum |k|^{-1} |a_k|^2, \quad \|f\|_{H^{1/2}}^2 = \sum |k| |a_k|^2.$$
For 2D Helmholtz, cloaking error was $C/|\log \rho|$. Linked to fund soln of Laplacian.

For 3D Helmholtz, obvious guess is $C \rho$. Supported by numerics. However our method gives only $C \sqrt{\rho}$: for

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega \subset \mathbb{R}^3$$

with

$$\begin{cases}
A = 1, q = 1 & \text{ in } \Omega \setminus B_{2\rho} \\
A = 1, q = 1 + i\rho^{-2} & \text{ in } B_{2\rho} \setminus B_\rho \\
\text{arbitrary real, positive} & \text{ in } B_\rho.
\end{cases}$$

we get

$$\|\Lambda_{A,q} - \Lambda_{1,1}\| \leq C \omega \sqrt{\rho}.$$
Overview of analysis

Recall eqn:
\[ \nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega \]

where
\[
\begin{aligned}
A = 1, & \quad q = 1 & \text{ in } \Omega \setminus B_{2\rho} \\
A = 1, & \quad q = 1 + i\beta & \text{ in } B_{2\rho} \setminus B_\rho \\
\text{arbitrary real, positive} & \quad & \text{ in } B_\rho.
\end{aligned}
\]

I. Compare Helmholtz in shell \( \Omega \setminus B_{2\rho} \) to Helmholtz in \( \Omega \).

Show that inclusion has little effect on boundary measurements, unless something wild is happening at \( \partial B_{2\rho} \).

II. Obtain global control using lossiness of \( B_{2\rho} \setminus B_\rho \).

Make good choice of lossiness \((\beta \sim \rho^{-2})\). Show that nothing wild can happen at \( \partial B_{2\rho} \), regardless of content of \( B_\rho \).

Estimate holds even when lossless problem is resonant.
I. Compare Helmholtz in shell $\Omega \setminus B_{2\rho}$ to Helmholtz in $\Omega$.

Consider

$$\begin{align*}
\Delta u_0 + \omega^2 u_0 &= 0 \text{ in } \Omega \\
\Delta u_\rho + \omega^2 u_\rho &= 0 \text{ in } \Omega \setminus B_{2\rho}
\end{align*}$$

with same Neumann data $\psi$ at $\partial \Omega$, and Dir data $\phi$ for $u_\rho$ at $\partial B_{2\rho}$. Then

$$\|u_\rho - u_0\|_{H^{1/2}(\partial \Omega)} \leq C e(\rho) \left( \|\psi\|_{H^{-1/2}(\partial \Omega)} + \|\phi(2\rho \cdot)\|_{H^{-1/2}(\partial B_1)} \right)$$

where

$$e(\rho) = \begin{cases} 
1/|\log \rho| & \text{in dim 2} \\
\rho & \text{in dim 3}.
\end{cases}$$

Main idea: if behavior at inclusion edge is uniform, then effect is like a small hole with a Dirichlet bdry condition.

If behavior at inclusion edge is oscillatory in $\theta$, influence decays faster.
II. Control $u_\rho$ on $\partial B_{2\rho}$, if annulus $\rho < |x| < 2\rho$ is lossy. Let

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega,$$

\[
\begin{align*}
A &= 1, \ q = 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\
A &= 1, \ q = 1 + i\beta & \text{for } \rho < |x| < 2\rho \\
\text{any real, pos values} & & \text{for } |x| < \rho.
\end{align*}
\]

using Neumann data $\psi$ at $\partial \Omega$. Then (in dim $n$)

$$\|u_\rho(2\rho \cdot)\|_{H^{-1/2}(\partial B_1)} \leq C(1 + (1 + \beta)\rho^2) \frac{1}{\rho^{n/2} \sqrt{\beta}} \left( \|\psi\|_{H^{-1/2}(\partial \Omega)} + \|u_\rho\|_{H^{1/2}(\partial \Omega)} \right)$$

Main ideas:

1) Imaginary part of energy identity gives

$$\omega^2 \beta \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2 \leq \left( \|\psi\|_{H^{-1/2}(\partial \Omega)} + \|u_\rho\|_{H^{1/2}(\partial \Omega)} \right)^2$$

2) Elliptic estimate for $\Delta u + \omega^2 (1 + i\beta)u = 0$ on $B_{2\rho} \setminus B_\rho$ gives:

$$\|u_\rho(2\rho \cdot)\|_{H^{-1/2}(\partial B_1)}^2 \leq C(1 + (1 + \beta)\rho^2)^2 \rho^{-n} \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2$$
II. Control $u_\rho$ on $\partial B_{2\rho}$, if annulus $\rho < |x| < 2\rho$ is lossy. Let

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega,$$

$$
\begin{cases}
A = 1, q = 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\
A = 1, q = 1 + i\beta & \text{for } \rho < |x| < 2\rho \\
\text{any real, pos values} & \text{for } |x| < \rho.
\end{cases}
$$

using Neumann data $\psi$ at $\partial \Omega$. Then (in dim $n$)

$$\|u_\rho(2\rho \cdot)\|_{H^{-1/2}(\partial B_1)} \leq C(1 + (1 + \beta)\rho^2) \frac{1}{\rho^{n/2} \sqrt{\beta}} \left( \|\psi\|_{H^{-1/2}(\partial \Omega)} + \|u_\rho\|_{H^{1/2}(\partial \Omega)} \right)$$

Main ideas:

1) Imaginary part of energy identity gives

$$\omega^2 \beta \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2 \leq \left( \|\psi\|_{H^{-1/2}(\partial \Omega)} + \|u_\rho\|_{H^{1/2}(\partial \Omega)} \right)^2$$

2) Elliptic estimate for $\Delta u + \omega^2 (1 + i\beta) u = 0$ on $B_{2\rho} \setminus B_\rho$ gives:

$$\|u_\rho(2\rho \cdot)\|_{H^{-1/2}(\partial B_1)}^2 \leq C(1 + (1 + \beta)\rho^2)^2 \rho^{-n} \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2$$

Robert V. Kohn Courant Institute, NYU Near-cloaking by change of variables
II. Control $u_\rho$ on $\partial B_{2\rho}$, if annulus $\rho < |x| < 2\rho$ is lossy. Let

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega,$$

with

$$\begin{aligned}
A = 1, q = 1 & \quad \text{for } x \in \Omega \setminus B_{2\rho} \\
A = 1, q = 1 + i\beta & \quad \text{for } \rho < |x| < 2\rho \\
\text{any real, pos values} & \quad \text{for } |x| < \rho.
\end{aligned}$$

using Neumann data $\psi$ at $\partial \Omega$. Then (in dim $n$)

$$\| u_\rho (2\rho \cdot) \|_{H^{-1/2}(\partial B_1)} \leq C (1 + (1 + \beta)\rho^2) \frac{1}{\rho^{n/2}\sqrt{\beta}} \left( \| \psi \|_{H^{-1/2}(\partial \Omega)} + \| u_\rho \|_{H^{1/2}(\partial \Omega)} \right)$$

Main ideas:

1) Imaginary part of energy identity gives

$$\omega^2 \beta \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2 \leq \left( \| \psi \|_{H^{-1/2}(\partial \Omega)} + \| u_\rho \|_{H^{1/2}(\partial \Omega)} \right)^2$$

2) Elliptic estimate for $\Delta u + \omega^2 (1 + i\beta)u = 0$ on $B_{2\rho} \setminus B_\rho$ gives:

$$\| u_\rho (2\rho \cdot) \|_{H^{-1/2}(\partial B_1)}^2 \leq C (1 + (1 + \beta)\rho^2)^2 \rho^{-n} \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2$$
Putting it together

Goal: compare solutions of

\[ \Delta u_0 + \omega^2 u_0 = 0 \quad \text{and} \quad \nabla (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega \]

with same Neumann data \( \psi \) at \( \partial \Omega \).

Step 1 gave

\[ \| u_\rho - u_0 \|_{H^{1/2}(\partial \Omega)} \leq C e(\rho) \left( \| \psi \|_{H^{-1/2}(\partial \Omega)} + \| u_\rho (2 \rho \cdot) \|_{H^{-1/2}(\partial B_1)} \right). \]

Step 2 with \( \beta \sim \rho^{-2} \) gives

\[ \| u_\rho (2 \rho \cdot) \|_{H^{-1/2}(\partial B_1)} \leq C \rho^{1-\frac{n}{2}} \left( \| \psi \|_{H^{-1/2}(\partial \Omega)} + \| u_\rho \|_{H^{1/2}(\partial \Omega)} \right) \]

Combining gives

\[ \| u_\rho - u_0 \|_{H^{1/2}(\partial \Omega)} \leq C e(\rho) \left( \rho^{1-\frac{n}{2}} \| \psi \|_{H^{-1/2}(\partial \Omega)} + \rho^{1-\frac{n}{2}} \| u_\rho \|_{H^{1/2}(\partial \Omega)} \right) \]

Eliminate last RHS term using \( \| u_\rho \| \leq \| u_\rho - u_0 \| + \| u_0 \| \) to get

\[ \| u_\rho - u_0 \|_{H^{1/2}(\partial \Omega)} \leq C e(\rho) \rho^{1-\frac{n}{2}} \| \psi \|_{H^{-1/2}(\partial \Omega)} \]

Thus: perturbation of boundary operator is at most

\[ \leq C e(\rho) \rho^{1-\frac{n}{2}} = \begin{cases} C/|\log \rho| & n = 2 \\ C\sqrt{\rho} & n = 3 \end{cases} \]
Putting it together

**Goal:** compare solutions of

\[ \Delta u_0 + \omega^2 u_0 = 0 \quad \text{and} \quad \nabla(A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega \]

with same Neumann data \( \psi \) at \( \partial \Omega \).

**Step 1** gave \( \|u_\rho - u_0\|_{H^{1/2}(\partial \Omega)} \leq C_\varepsilon(\rho) \left( \|\psi\|_{H^{-1/2}(\partial \Omega)} + \|u_\rho(2^\rho \cdot)\|_{H^{-1/2}(\partial B_1)} \right) \).

**Step 2** with \( \beta \sim \rho^{-2} \) gives

\[ \|u_\rho(2^\rho \cdot)\|_{H^{-1/2}(\partial B_1)} \leq C_\rho^{1-\frac{n}{2}} \left( \|\psi\|_{H^{-1/2}(\partial \Omega)} + \|u_\rho\|_{H^{1/2}(\partial \Omega)} \right) \]

Combining gives

\[ \|u_\rho - u_0\|_{H^{1/2}(\partial \Omega)} \leq C_\varepsilon(\rho) \left( \rho^{1-\frac{n}{2}} \|\psi\|_{H^{-1/2}(\partial \Omega)} + \rho^{1-\frac{n}{2}} \|u_\rho\|_{H^{1/2}(\partial \Omega)} \right) \]

Eliminate last RHS term using \( \|u_\rho\| \leq \|u_\rho - u_0\| + \|u_0\| \) to get

\[ \|u_\rho - u_0\|_{H^{1/2}(\partial \Omega)} \leq C_\varepsilon(\rho) \rho^{1-\frac{n}{2}} \|\psi\|_{H^{-1/2}(\partial \Omega)} \]

Thus: perturbation of boundary operator is at most

\[ \leq C_\varepsilon(\rho) \rho^{1-\frac{n}{2}} = \begin{cases} 
  C/|\log \rho| & n = 2 \\
  C\sqrt{\rho} & n = 3 
\end{cases} \]
Conclusions

How well does the change-of-variable-based cloaking scheme work?

- Equivalent to: how much can a small inclusion affect bdry meas?
- At freq 0: error estimate $\rho^n$ in dim $n$ (no damping)
- At freq $\neq 0$:
  - complete failure if object to be cloaked is resonant
  - difficulty fixed by introducing lossy shell
  - error estimate $1/|\log \rho|$ in 2D, $\sqrt{\rho}$ in 3D.

Examples and numerics to be presented by Onofrei.