Global existence and stability results for shear flows of viscoelastic fluids

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Supported by National Science Foundation
Global existence of solutions

Existence proofs for initial value problems have two parts:
1. An argument for local existence, typically based on proving convergence of some approximation scheme.
2. A priori estimates showing solutions do not blow up and can be continued.

Example: Navier-Stokes equations

Assume, for simplicity, periodic boundary conditions.

\[
\rho (v_t + (v \cdot \nabla)v) = \eta \Delta v - \nabla p,
\]

\[
\text{div } v = 0.
\]

If we multiply by \( v \) and integrate we find

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |v|^2 \, dx = - \int_{\Omega} \eta |\nabla v|^2 \, dx.
\]
This is enough to guarantee global existence (but not uniqueness) of a weak solution. In two dimensions, we can do more. Take the curl of the equation of motion, and let $\omega$ be the vorticity. We find

$$\rho(\omega_t + (\mathbf{v} \cdot \nabla)\omega) = \eta \Delta \omega,$$

and hence

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho \omega^2 \, d\mathbf{x} = - \int_{\Omega} \eta |\nabla \omega|^2 \, d\mathbf{x}.$$

This suffices to prove global existence of smooth solutions.
How about non-Newtonian fluids?

CAPTAIN POINTY. by Stik

Win an argument against Prof Plums and he'd prove you didn't exist.
Global existence results in viscoelastic fluids are restricted to simple flows.
Viscoelastic fluids

Only one-dimensional results for global existence (other than for small data). We shall consider parallel shear flows (reduces to the heat equation for Newtonian case).

Governing equations:

\[ \rho u_t = \tau_y + \eta u_{yy} + f(y, t), \]

\[ T = \begin{pmatrix} \sigma & \tau \\ \tau & \psi \end{pmatrix}, \]

\[ T_t = G(T, u_y). \]

Initial conditions for \( u \) and \( T \), Dirichlet boundary conditions for \( u \).
A local existence result is easy, for instance by using an iterative construction like the following:

\[ \rho u_t^{n+1} = \tau_y^n + \eta u_y^{n+1} + f(y, t), \]

\[ T_t^{n+1} = G(T^n, u_y^{n+1}). \]

Global continuation (of a solution as smooth as the data will allow) is possible if we can get a priori bounds on the L$^1$ norms of $u_y$ and $T$. This is possible if we make certain assumptions that are satisfied for a number of constitutive laws.

Positive definiteness conditions for the stress tensor play an essential role in the arguments.
Assumptions sufficient for global existence

(A1) There is $p<1$ such that

$$|G(T, u_y)| \leq C(|u_y| + |T|)^p.$$  

(A2) There is $q<1$ and $\nu<1$ such that every solution of

$$T_t = G(T, u_y)$$

satisfies the bounds

$$|\tau| \leq C(1 + \max_{s \in [0,t]} |u_y(s)|^q),$$

$$|\sigma| + |\psi| \leq C(1 + \max_{s \in [0,t]} |u_y(s)|^\nu),$$

where $C$ depends only on $t$ and the initial data.
White-Metzner model

\[ \tau_t = u_y - \frac{\lambda}{\mu(u_y)} \tau, \]

\[ \sigma_t = 2\tau u_y - \frac{\lambda}{\mu(u_y)} \sigma, \]

\[ \psi = 0, \]

\[ \lambda > 0, \mu(u_y) > 0, \mu(u_y) \sim |u_y|^{-\gamma} \]

for large \( |u_y|, \ 0 < \gamma < 1. \)

Assumption (A2) holds with \( q=1-\gamma \) and \( v=2-2\gamma. \)
Phan-Thien Tanner model

\[ \sigma_t = 2\tau u_y - \lambda \sigma - \kappa \sigma^2, \]

\[ \tau_t = -\lambda \tau - \kappa \sigma \tau + \mu u_y, \]

\[ \psi = 0. \]

We can derive that

\[ \frac{d}{dt}(\mu \sigma - \tau^2) = -(\lambda + \kappa \sigma)(\mu \sigma - \tau^2) + (\lambda + \kappa \sigma)\tau^2. \]

This implies positive definiteness:

\[ \begin{vmatrix} \sigma & \tau \\ \tau & \mu \end{vmatrix} = \mu \sigma - \tau^2 \geq 0. \]
It follows that

\[
\frac{d}{dt} (\tau)^2 \leq -2\lambda \tau^2 - 2\kappa \tau^4/\mu + 2\mu \tau u_y.
\]

This implies (A2) with \(q=1/3\).
Johnson-Segalman model

\[\sigma_t = -\lambda \sigma + (1 + a)\tau u_y,\]
\[\tau_t = -\lambda \tau + \left(\frac{a}{2}(\sigma + \psi) + \frac{1}{2}(\psi - \sigma) + \mu\right)u_y,\]
\[\psi_t = -\lambda \psi + (a - 1)\tau u_y.\]

Here \(\lambda, \mu > 0\) and \(-1 < a < 1\). It is convenient to introduce new variables:

\[Y = (1-a)\sigma + (1+a)\psi, \quad Z = \frac{a}{2}(\sigma+\psi) + \frac{1}{2}(\sigma-\psi).\]

The equations transform to \(Y_t + \lambda Y = 0\) and

\[\tau_t = -\lambda \tau + (Z + \mu)u_y,\]
\[Z_t = -\lambda Z + (a^2 - 1)\tau u_y.\]
With

$$\Phi = \frac{1}{2}Z^2 + \mu Z + \frac{1}{2}(1 - \alpha^2)\tau^2,$$

we find

$$\Phi_t = -2\lambda \Phi + \lambda \mu Z.$$

This leads to a priori bounds on $\tau$ and $Z$.

Note:

$$\begin{vmatrix} \sigma + \mu/a & \tau \\ \tau & \psi + \mu/a \end{vmatrix} = -\frac{2}{1 - \alpha^2} \Phi + \frac{\mu^2}{a^2}.$$
Giesekus model

\[
\sigma_t = -\lambda \sigma - \kappa (\sigma^2 + \tau^2) + 2\tau u_y,
\]
\[
\tau_t = -\lambda \tau - \kappa (\sigma + \psi) \tau + (\mu + \psi) u_y,
\]
\[
\psi_t = -\lambda \psi - \kappa (\tau^2 + \psi^2).
\]

Here \(\lambda, \mu, \kappa > 0\), \(\kappa \mu < \lambda\).

We set

\[
\chi = \kappa \mu (\sigma - \psi) + \kappa (\sigma \psi - \tau^2) + \lambda \psi,
\]

and find

\[
\chi_t = -\left(\lambda + \kappa (\sigma + \psi)\right) \chi.
\]

We shall now assume \(\chi = 0\).
Next, consider

\[ d = \sigma (\psi + \mu) - \tau^2. \]

It can be shown that \( \sigma \geq 0, \ d \geq 0 \) if this is the case initially, since

\[ d_t = \frac{d^2 \kappa^2}{\lambda - \kappa \mu} - d(2\lambda + \kappa (\sigma - \mu)) + \mu \sigma (\lambda - \kappa \mu). \]

Moreover,

\[ d = \frac{(\lambda - \kappa \mu)(\mu \sigma - \tau^2)}{\lambda - \kappa \mu + \kappa \sigma}. \]

Hence \( \tau^2 \leq \mu \sigma \). Moreover, from \( \chi = 0 \) and \( \sigma \geq 0 \) it follows that \( \psi > -\mu \).
Now consider the equation

\[ \tau_t = -\lambda \tau - \kappa (\sigma + \psi) \tau^2 + (\mu + \psi) u_y. \]

We have \( \sigma \geq \tau^2/\mu \) and \( 0 \geq \psi > -\mu \). We can conclude (A2) with \( q=1/3 \).
Nonlinear dumbbell models

These do not fit into the preceding framework, but for creeping flow a priori bounds can be found by other means.

Creeping flow:

\[ \tau_y + \eta u_{yy} = 0. \]

An immediate consequence is

\[ |u_y(y, t)| \leq C + \frac{1}{\eta} \max_{y \in [0, L]} |\tau(y, t)|, \]

where the constant depends only on the boundary conditions.
Constitutive law:

\[
C_t = (\nabla u)C + C(\nabla u)^T + \gamma I - \delta f(\text{tr} \ C)C,
\]

\[
T = f(\text{tr} \ C)C.
\]

Here \( C \) is called a configuration tensor. \( \gamma \) and \( \delta \) are positive constants. For the function \( f \), we assume it is monotone and one of the following:

\[
f(c) \sim c^\mu, \quad f'(c) \sim c^{\mu-1}, \quad \text{for} \ c \to \infty, \ \mu > 0;
\]

\[
f(c) \sim (L-c)^{-\mu}, \quad f'(c) \sim (L-c)^{-\mu-1}, \quad \text{for} \ c \to L, \ \mu > 0.
\]
In shear flow, we have

\[
C = \begin{pmatrix}
A & D & 0 \\
D & E & 0 \\
0 & 0 & E
\end{pmatrix},
\]

\[
A_t = 2D u_y + \gamma - \delta f(A + 2E)A,
\]
\[
D_t = Eu_y - \delta f(A + 2E)D,
\]
\[
E_t = \gamma - \delta f(A + 2E)E.
\]
For physically acceptable initial data, $C$ is positive definite, and $E < \gamma/\delta f(0)$. We have

$$(A + 2E)_t = 2Du_y + 3\gamma - \delta f(A + 2E)(A + 2E).$$

Now let

$$Q = \max_{y \in [0, L]} (A + 2E), \quad R = \max_{y \in [0, L]} |D|,$$

$$S = \max_{y \in [0, L]} |u_y|.$$
We find

\[ Q_t \leq 2RS + 3\gamma - \delta f(Q)Q, \]

\[ R \leq \sqrt{Q\gamma/(\delta f(0))}, \]

\[ S \leq C + \frac{f(Q)R}{\eta}. \]

By combining these, we obtain

\[ Q_t \leq f(Q)Q\left(\frac{2\gamma}{\eta \delta f(0)} - \delta\right) + 3\gamma + C\sqrt{Q}. \]

This yields an a priori bound for \( Q \) if

\[ \eta > \frac{2\gamma}{\delta^2 f(0)}. \]
Global stability of the rest state

Assume a constitutive law of the form

\[ T_t = G(T, u_y). \]

Make the following assumptions:

1. \( G(0)=0 \) and polynomial growth of \( G \) and its derivatives.
2. A priori estimates which imply (for some \( p \geq 1 \))
   \[ \lim_{t \to \infty} \| T \|_p = 0. \]
3. Assumption (A2) for global existence as before.

Then \( \| \tau + \eta u_y \|_\infty \) tends to zero.
PTT model

\[ \sigma_t = 2\tau u_y - \lambda \sigma - \kappa \sigma^2, \]

\[ \tau_t = -\lambda \tau - \kappa \sigma \tau + \mu u_y, \]

\[ \rho u_t = \tau_y + \eta u_{yy}. \]

This yields (assuming homogeneous Dirichlet conditions for \( u \))

\[
\frac{d}{dt} \int \sigma + \frac{\tau^2}{2} + \rho u^2 (1 + \frac{\mu}{2}) \, dy
\]

\[ = -\int \lambda \sigma + \kappa \sigma^2 + (\lambda + \kappa \sigma) \tau^2 + (2 + \mu) \eta u_y^2 \, dy. \]

Consequently

\[ \|\sigma\|_1 + \|\tau\|_2 \to 0. \]

Note that the a priori information that \( \sigma \) is positive is essential here!
Similar arguments work for Johnson-Segalman and Giesekus. For the nonlinear dumbbell model and large enough $\eta$, we can prove global stability of the rest state by exploiting a refinement of the positive definiteness condition. Recall the set of equations

\[
\begin{align*}
A_t &= 2D u_y + \gamma - \delta f(A + 2E)A, \\
D_t &= E u_y - \delta f(A + 2E)D, \\
E_t &= \gamma - \delta f(A + 2E)E.
\end{align*}
\]

We derive

\[
\frac{d}{dt}((A-E)E - D^2) = -2\delta f(A+2E)((A-E)E - D^2) + \gamma(A-E).
\]

Positive definiteness of the conformation tensor implies that $A, E$ and $AE-D^2$ are nonnegative. We find the stronger condition that $A-E$ and $(A-E)E-D^2$ are nonnegative.
Questions?