EnKF and Catastrophic filter divergence

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DAS 13, University of Maryland.
The set-up for EnKF

We have a deterministic model

$$\frac{dv}{dt} = F(v) \quad \text{with} \quad v_0 \sim N(m_0, C_0).$$

We will denote $v(t) = \Psi_t(v_0)$.

We want to estimate $v_j = v(jh)$ for some $h > 0$ and $j = 0, 1, \ldots, J$ given the observations

$$y_{j+1} = H v_{j+1} + \xi_{j+1} \quad \text{for} \quad \xi_{j+1} \text{iid } N(0, \Gamma).$$
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We estimate using an **ensemble** of particles $\{u^{(k)}\}_{k=1}^{K}$. Each particle is a statistical **representative** of the **posterior**.

For each particle, we have an **artificial observation**

$$y_{j+1}^{(k)} = y_{j+1} + \xi_{j+1}^{(k)}, \quad \xi_{j+1}^{(k)} \text{ iid } N(0, \Gamma).$$

We update each particle using the **Kalman update**

$$u_{j+1}^{(k)} = \Psi_h(u_j^{(k)}) + G(u_j) (y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)})),$$

where $G(u_j)$ is the **Kalman gain** computed using the **forecasted ensemble covariance**

$$\hat{C}_{j+1} = \frac{1}{K} \sum_{k=1}^{K} (\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)})^T (\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)}).$$
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Filter divergence

It has been observed (⋆) that the ensemble can blow-up (ie. reach machine-infinity) in finite time, even when the model has nice bounded solutions.

This is known as catastrophic filter divergence.

It is suggested in (⋆) that this is caused by numerically integrating a stiff-system. Our aim is to “prove” this.

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Discrete time results

We make a “dissipativity” assumption on $F$. Namely that

$$F(\cdot) = A \cdot + B(\cdot, \cdot) \quad (\dagger)$$

with $A$ linear elliptic and $B$ bilinear, satisfying certain estimates and symmetries.

Eg. 2d-Navier-Stokes, Lorenz-63, Lorenz-96.

Theorem (AS,DK)

If $H = I$ and $\Gamma = \gamma^2 I$, then there exists constant $\beta, K$ such that

$$E|u_j^{(k)}|^2 \leq e^{2\beta jh} E|u_0^{(k)}|^2 + 2K\gamma^2 \left( \frac{e^{2\beta jh} - 1}{e^{2\beta jh} - 1} \right)$$

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The EnKF equations look like a discretization

Recall the ensemble update equation

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\begin{align*}
    u_{j+1}^{(k)} &= \Psi_h(u_j^{(k)}) + G(u_j) \left( y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)}) \right) \\
    &= \Psi_h(u_j^{(k)}) + \hat{C}_{j+1} H^T (H^T \hat{C}_{j+1} H + \Gamma)^{-1} \left( y_{j+1}^{(k)} - H\Psi_h(u_j^{(k)}) \right)
\end{align*}
\]

Subtract \( u_j^{(k)} \) from both sides and divide by \( h \)

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\begin{align*}
    \frac{u_{j+1}^{(k)} - u_j^{(k)}}{h} &= \frac{\Psi_h(u_j^{(k)}) - u_j^{(k)}}{h} \\
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Clearly we need to rescale the noise (ie. \( \Gamma \)).
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**Continuous-time limit**

If we set $\Gamma = h^{-1}\Gamma_0$ and substitute $y_{j+1}^{(k)}$, we obtain

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\begin{align*}
\frac{u_{j+1}^{(k)} - u_j^{(k)}}{h} &= \frac{\Psi_h(u_{j}^{(k)}) - u_j^{(k)}}{h} + \hat{C}_{j+1} H^T (hH^T \hat{C}_{j+1} H + \Gamma_0)^{-1} \\
&\left( H \nu + h^{-1/2} \Gamma_0^{1/2} \xi_{j+1} + h^{-1/2} \Gamma_0^{1/2} \xi_{j+1}^{(k)} - H \Psi_h(u_j^{(k)}) \right)
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But we know that

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\Psi_h(u_j^{(k)}) = u_j^{(k)} + O(h)
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and

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\hat{C}_{j+1} = \frac{1}{K} \sum_{k=1}^{K} (\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)})^T (\Psi_h(u_j^{(k)}) - \overline{\Psi_h(u_j)}) = \frac{1}{K} \sum_{k=1}^{K} (u_j^{(k)} - \overline{u_j})^T (u_j^{(k)} - \overline{u_j}) + O(h) = C(u_j) + O(h)
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We end up with

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\frac{u_j^{(k)} - u_j^{(k)}}{h} = \frac{\Psi_h(u_j^{(k)}) - u_j^{(k)}}{h} - C(u_j)H^T \Gamma_0^{-1} H(u_j^{(k)} - \nu_j) + C(u_j)H^T \Gamma_0^{-1} \left( h^{-1/2} \xi_j + h^{-1/2} \xi_j^{(k)} \right) + O(h)
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This looks like a numerical scheme for

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\]

Rmk. The extra dissipation term only sees differences in observed space and only dissipates in the space spanned by ensemble.
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\frac{u_{j+1}^{(k)} - u_j^{(k)}}{h} = \frac{\Psi_h(u_j^{(k)}) - u_j^{(k)}}{h} - C(u_j)H^T\Gamma_0^{-1}H(u_j^{(k)} - v_j) \\
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Theorem (AS,DK)

Suppose the model $v$ satisfies (†) and $\{u^{(k)}\}_{k=1}^K$ satisfy (●). Let

$$e^{(k)} = u^{(k)} - v.$$ 

If $H = I$ and $\Gamma = \gamma^2 I$, then there exists constant $\beta, K$ such that

$$\mathbb{E} \sum_{k=1}^K |e^{(k)}(t)|^2 \leq \mathbb{E} \sum_{k=1}^K |e^{(k)}(0)|^2 \exp(\beta t).$$
Summary + Future Work

(1) Writing down an SDE/SPDE allows us to see the important quantities in the algorithm.

(2) Does not “prove” that filter divergence is a numerical phenomenon, but is a decent starting point.

(1) Improve the condition on $H$.

(2) If we can measure the important quantities, then we can test the performance during the algorithm.

(3) Suggests new EnKF-like algorithms, for instance by discretising the stochastic PDE in a more numerically stable way.
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