Optimal Interpolation and Nudging

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Thought Experiment

- You see two clocks, one says 9:30 and one says 9:40. What is your best estimate of the current time?
- For now, assume the clocks look identical and no other information is available.
- One possible answer: 9:35.
- Another possible answer: about 9:30 with probability 1/2, and about 9:40 with probability 1/2.
Thought Experiment

- If we perceive the clock that reads 9:40 to be more accurate (or less likely to have failed) than the other, we could consider 9:40 to be more probable than 9:30.
- If we make a nonprobabilistic estimate (focus of this lecture), it should be closer to 9:40 than 9:30.
- How much closer depends on quantifying the uncertainties in the clock readings and what we mean by “best estimate”.
- One notion of “best” is the minimum variance unbiased estimator (MVUE).
Unbiased Estimators

• Suppose we have two independent observations $y_1$ and $y_2$ of an unknown scalar quantity $x$, with different accuracies.

• More specifically, assume each $y_j$ is independently sampled from a distribution with (unknown) mean $x$ and (known) standard deviation $\sigma_j$.

• An estimator for $x$ is a scalar function $f(y_1, y_2)$.

• It is unbiased if the mean (over different samples) of $f(y_1, y_2)$ is $x$. 
MVUE

• For all $\lambda$, the statistic $\lambda y_1 + (1 - \lambda)y_2$ is an unbiased estimator of $x$.

• The variance of this estimator is $\lambda^2 \sigma_1^2 + (1 - \lambda)^2 \sigma_2^2$, which is minimized when

$$\lambda = \sigma_1^{-2}/(\sigma_1^{-2} + \sigma_2^{-2}).$$

• The MVUE for $x$ is

$$\frac{(\sigma_1^{-2} y_1 + \sigma_2^{-2} y_2)}{\sigma_1^{-2} + \sigma_2^{-2}}.$$

• This is also the maximum likelihood estimate for $x$ if (e.g.) the distributions are Gaussian and the “prior” is uniform.
Observation Bias

- On the previous slides, I assumed that the observations $y_j$ are unbiased: the mean of the error $y_j - x$ is 0.
- Real observations are likely to be biased.
- Conundrum:
  - If we know what the bias (the mean of $y_j - x$) is, we can subtract it from $y_j$ to get an unbiased observation.
  - If we don’t know what the bias is, then what is the relationship between $y_j$ and $x$?
Observation (Forward) Operator

- A possible answer to the previous question is to assume that $y_j$ is sampled from a distribution with mean $H_j(x)$; the function $H_j$ is called an observation operator or forward operator.
- If the same procedure is used to “observe” $x$ at many different times, one can try to adjust the function $H_j$ to improve the accuracy of this assumption.
- One may be tempted instead to assume that some function of $y_j$ has mean $x$, but this approach is less flexible.
Role of Data Assimilation

• Suppose $x$ is a time-varying vector, and the available observations at a given time form a vector $y$.

• Data assimilation is particularly useful in cases when $y$ does not contain enough information to uniquely determine $x$, perhaps b/c it is lower dimensional.

• On the other hand, if we are trying to predict future observations, it helps to have a model whose state $x$ has enough information to determine $y$. 
Scalar Optimal Interpolation

- Let’s return to the case when $x$ is a scalar, but plan ahead for the vector case.
- Assume each $H_j$ is linear: $y_j$ is sampled from a distribution with mean $H_j x$.
- The MVUE for $x$ is

$$
\frac{(\sigma_1/H_1)^{-2}y_1/H_1 + (\sigma_2/H_2)^{-2}y_2/H_2)}{(\sigma_1/H_1)^{-2} + (\sigma_2/H_2)^{-2}}
= \frac{H_1 \sigma_1^{-2} y_1 + H_2 \sigma_2^{-2} y_2}{H_1^2 \sigma_1^{-2} + H_2^2 \sigma_2^{-2}}.
$$
Scalar Optimal Interpolation

- Now assume $H_1 = 1$ and $y_1$ is a background estimate $x^b$ of $x$, which may be based in part on some previous observations, while $y_2$ is a new observation $y^o$.
- Replace $\sigma_1^2$ with $P^b$, $\sigma_2^2$ with $R^o$, and $H_2$ with $H$; the MVUE is then

$$
\frac{(P^b)^{-1}x^b + H(R^o)^{-1}y^o}{(P^b)^{-1} + H^2(R^o)^{-1}}
$$

$$
= x^b + \frac{H(R^o)^{-1}(y^o - Hx^b)}{(P^b)^{-1} + H^2(R^o)^{-1}}
$$
Vector Optimal Interpolation (OI)

- Next, assume $x$ and $y$ are vectors and $H$, $P^b$, and $R^o$ are matrices.

- The MVUE can then be written

$$x^a = x^b + G(y^o - Hx^b)$$

where

$$G = \left[ (P^b)^{-1} + H^T (R^o)^{-1} H \right]^{-1} H^T (R^o)^{-1}$$

$$= P^b H^T [HP^b H^T + R^0]^{-1}$$

- Here $x^a$ is the analysis ("after") estimate that takes into account the background ("before") $x^b$ and the new observations $y^o$. 
Consider now a sequence of observations and estimates indexed by $n$, representing times $t_n = n\Delta t$.

Let $M_n$ be a model such that the true state satisfies $x^t_{n+1} \approx M_n(x^t_n)$.

A data assimilation (DA) cycle is

$$x^a_n = x^b_n + G_n(y^o_n - Hx^b_n)$$

$$x^b_{n+1} = M_n(x^a_n)$$

An OI cycle, like 3DVar, typically uses a background covariance $P^b$, and hence a gain $G$, that is independent of $n$. 

Data assimilation cycle
• If $G$ is constant in time but formulated in a more ad hoc manner than the OI gain, I’ll call the resulting DA cycle discrete-time nudging; nudging is usually formulated in continuous time (limit as $\Delta t \to 0$).

• If $H = I$ and $G = I$, then $x^n_a = y^n_o$; this is direct insertion.

• Direct insertion also refers to the case when $H$ maps $x$ to a subset of its coordinates; then $x^n_a = y^n_o$ for observed coordinates and $x^n_a = x^n_b$ for unobserved coordinates.
Canonical OI Citations


OI vs. 3DVar

- The OI analysis computes the minimum of the 3DVar cost function in the case that the observation operator $h$ is linear ($H$).
- 3DVar often refers to a minimization implementation that allows nonlinear $h$ and may include e.g. preconditioning.
- OI is often presumed to include localization, where the estimate of $x$ at a given geographical location is done only with nearby observations.
- OI with localization was used at NCEP in 1980s [e.g., Derber, Parrish, Lord 1991].
Some Further OI Perspectives

http://www.ecmwf.int/newsevents/training/meteorological_presentations/pdf/DA/AssAlg_2.pdf

http://www.ecmwf.int/newsevents/training/rcourse_notes/DATA_ASSIMILATION/ASSIM_CONCEPTS/Assim_concepts8.html