High Frequency Scattering by Convex Polygons

Stephen Langdon
University of Reading, UK

Joint work with: Simon Chandler-Wilde, Steve Arden

Funded by: Leverhulme Trust

University of Maryland, September 2005
Contents

• The scattering problem
• The Burton & Miller integral equation
• Understanding solution behaviour
• Approximating the solution efficiently
  – Galerkin method
  – Collocation method
• Numerical Results
The Scattering Problem

\[ \Delta u + k^2 u = 0 \]

\[ u = 0 \]

\[ u^i, \text{ incident wave} \]

\[ u = 0 \]

\[ \Gamma \]

\[ D \]

obstacle

\[ x_1 \]

\[ x_2 \]
\[ \Delta u + k^2 u = 0 \]

\[ u = 0 \]

Green's representation theorem:

\[ u(x) = u^i(x) - \int_{\Gamma} \Phi(x, y) \frac{\partial u}{\partial n}(y) ds(y), \quad x \in D, \]

where \( \Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|) \).
\[ \Delta u + k^2 u = 0 \]

\[ u = 0 \quad \Gamma \]

\[ D \]

From Green's representation theorem (Burton & Miller 1971):

\[ \frac{1}{2} \frac{\partial u}{\partial n}(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} + i\eta \Phi(x, y) \right) \frac{\partial u}{\partial n}(y)ds(y) = f(x), \quad x \in \Gamma, \]

where

\[ f(x) := \frac{\partial u^i}{\partial n}(x) + i\eta u^i(x). \]
\[ \Delta u + k^2 u = 0 \]

\[ u = 0 \]

From Green's representation theorem:

\[
\frac{1}{2} \frac{\partial u}{\partial n}(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} + i\eta \Phi(x, y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.
\]

**Theorem** (follows from Burton & Miller 1971, Selepow 1969) If \( \eta \in \mathbb{R} \), \( \eta \neq 0 \), then this integral equation is uniquely solvable in \( L^2(\Gamma) \).
\[
\frac{1}{2} \frac{\partial u}{\partial n}(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} + i\eta \Phi(x, y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.
\]

**Conventional BEM:** Apply a Galerkin method, approximating \( \frac{\partial u}{\partial n} \) by a piecewise polynomial of degree \( P \), leading to a linear system to solve with \( N \) degrees of freedom. **Problem:** \( N \) of order of \( kL \), where \( L \) is linear dimension, so cost is \( O(N^2) \) to compute full matrix and apply iterative solver ... or close to \( O(N) \) if a fast multipole method (e.g. Amini & Profit 2003, Darve 2004) is used.

This is **fantastic** but still infeasible as \( kL \to \infty \).
\[ \frac{1}{2} \frac{\partial u}{\partial n}(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} + i\eta \Phi(x, y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma. \]

**Alternative:** Reduce $N$ by using new basis functions, e.g.

(i) approximate $\partial u/\partial n$ by taking a large number of plane waves and multiplying these by conventional piecewise polynomial basis functions (Perrey-Debain et al. 2003, 2004). **This is very successful (in 2D, 3D, for acoustic/elastic waves and Neumann/impedance b.c.s), reducing number of degrees of freedom per wavelength from e.g. 6-10 to close to 2.** However $N$ still increases proportional to $kL$. 
\[
\frac{1}{2} \frac{\partial u}{\partial n}(x) + \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial n(x)} + i \eta \Phi(x, y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.
\]

**Alternative:** Reduce \( N \) by using new basis functions, e.g.

(ii) for convex scatterers, remove some of the oscillation by factoring out the oscillation of the incident wave, i.e. writing

\[
\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y)
\]

and approximating \( F \) by a conventional BEM (e.g. Abboud et al. 1994, Darrigrand 2002, Bruno et al 2004).
Alternative: Reduce $N$ by using new basis functions, e.g.

(ii) for convex scatterers, remove some of the oscillation by factoring out the oscillation of the incident wave, i.e. writing

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y) \quad (*)$$

and approximating $F$ by a conventional BEM.

For smooth obstacles this works well: equation $(*)$ holds with $F(y) \approx 2$ on the illuminated side (physical optics) and $F(y) \approx 0$ in the shadow zone.
(ii) for convex scatterers, remove some of the oscillation by factoring out the oscillation of the incident wave, i.e. writing

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y) \quad (*)$$

and approximating $F$ by a conventional BEM. Not very effective for non-smooth scatterers.
Let

$$G(x, y) := \Phi(x, y) - \Phi(x, y')$$

be the Dirichlet Green function for the left half-plane $\Omega$. By Green’s representation theorem,

$$u(x) = u^i(x) + u^r(x) + \int_{\partial\Omega \setminus \Gamma} \frac{\partial G(x, y)}{\partial n(y)} u(y) ds(y), \quad x \in \Omega.$$
Understanding solution behaviour

In the left half-plane $\Omega$, 

$$ u(x) = u^i(x) + u^r(x) + \int_{\partial\Omega \setminus \Gamma} \frac{\partial G(x,y)}{\partial n(y)} u(y) ds(y) $$

$$ \Rightarrow \frac{\partial u}{\partial n}(x) = 2 \frac{\partial u^i}{\partial n}(x) + 2 \int_{\partial\Omega \setminus \Gamma} \frac{\partial^2 \Phi(x,y)}{\partial n(x) \partial n(y)} u(y) ds(y), \quad x \in \gamma = \partial\Omega \cap \Gamma. $$
Explicitly, where $s$ is distance along $\gamma$, and $\phi(s)$ and $\psi(s)$ are $k^{-1}\partial u/\partial n$ and $u$, at distance $s$ along $\gamma$,

$$\phi(s) = P.O. + \frac{1}{2} [e^{iks}v_+(s) + e^{-iks}v_-(s)]$$

where

$$v_+(s) := k \int_{-\infty}^{0} F(k(s - s_0))e^{-iks_0}\psi(s_0)ds_0.$$ 

and $F(z) := e^{-iz}H_1^{(1)}(z)/z$
\[
\phi(s) = P.O. + \frac{1}{2} \left[ e^{iks} v_+(s) + e^{-iks} v_-(s) \right]
\]

where
\[
v_+(s) := k \int_{-\infty}^{0} F(k(s-s_0)) e^{-iks_0} \psi(s_0) ds_0.
\]

Now \( F(z) := e^{-iz} H_1^{(1)}(z)/z \) which is non-oscillatory, in that
\[
F^{(n)}(z) = O(z^{-3/2-n}) \text{ as } z \to \infty.
\]
\[ \phi(s) = P.O. + \frac{i}{2} \left[ e^{ik_s} v_+(s) + e^{-ik_s} v_-(s) \right] \]

where

\[ v_+(s) := k \int_{-\infty}^{0} F(k(s - s_0)) e^{-ik_s_0} \psi(s_0) ds_0. \]

Now \( F(z) := e^{-iz} H^{(1)}_1(z)/z \) which is non-oscillatory, in that

\[ F^{(n)}(z) = O(z^{-3/2-n}) \text{ as } z \to \infty. \]

\[ \Rightarrow v_+^{(n)}(s) = O(k^n (k s)^{-1/2-n}) \text{ as } ks \to \infty. \]
\[ \phi(s) = P.O. + \frac{1}{2} \left[ e^{iks} v_+(s) + e^{-iks} v_-(s) \right] \]

where

\[ k^{-n} |v_+^{(n)}(s)| = O \left( (ks)^{-1/2-n} \right) \quad \text{as } ks \to \infty \]

and (by separation of variables local to the corner),

\[ k^{-n} |v_+^{(n)}(s)| = O \left( (ks)^{-\alpha-n} \right) \quad \text{as } ks \to 0, \]

where \( \alpha < 1/2 \) depends on the corner angle.
\[ \phi(s) = P.O. + \frac{i}{2} \left[ e^{iks} v_+(s) + e^{-iks} v_-(s) \right] \]

where

\[ k^{-n} |v_+^{(n)}(s)| = \begin{cases} 
O \left( (ks)^{-1/2-n} \right) & \text{as } ks \to \infty \\
O \left( (ks)^{-\alpha-n} \right) & \text{as } ks \to 0,
\end{cases} \]

where \( \alpha < 1/2 \) depends on the corner angle.

Thus approximate

\[ \phi(s) \approx P.O. + \frac{i}{2} \left[ e^{iks} V_+(s) + e^{-iks} V_-(s) \right], \]

where \( V_+ \) and \( V_- \) are piecewise polynomials on graded meshes.
Thus approximate

\[
\phi(s) \approx P.O. + \frac{i}{2} \left[ e^{iks} V_+(s) + e^{-iks} V_-(s) \right],
\]

where \( V_+ \) and \( V_- \) are piecewise polynomials on graded meshes.

Figure 1: Scattering by a square
Thus approximate

\[ \phi(s) \approx P.O. + \frac{1}{2} \left[ e^{iks} V_+(s) + e^{-iks} V_-(s) \right], \]

where \( V_+ \) and \( V_- \) are piecewise polynomials on graded meshes.

Figure 2: Scattering by a square
Thus approximate

\[
\phi(s) \approx P.O. + \frac{1}{2} \left[ e^{ik_s} V_+(s) + e^{-ik_s} V_-(s) \right],
\]

where \( V_+ \) and \( V_- \) are piecewise polynomials on graded meshes.

Figure 3: Scattering by a square
Approximation error

**Theorem:** If $V_+$ is the best $L_2$ approximation from the approximation space, then

$$k^{1/2} \| v_+ - V_+ \|_2 \leq C_p \frac{n^{1/2}(1 + \log^{1/2}(kL))}{Np+1},$$

where

- $N \propto$ degrees of freedom
- $p =$ polynomial degree
- $L =$ max side length
- $n =$ number of sides of polygon
Boundary integral equation method

Integral equation in parametric form

$$\varphi(s) + K\varphi(s) = F(s),$$

where

$$\varphi(s) := \frac{1}{k} \frac{\partial u}{\partial n}(x(s)) - P.O..$$

**Theorem.** The operator $(I + K) : L^2(\Gamma) \mapsto L^2(\Gamma)$ is bijective with bounded inverse

$$\| (I + K)^{-1} \|_2 \leq C,$$

so that the integral equation has exactly one solution.
Boundary integral equation method

Integral equation in parametric form

\[ \varphi(s) + K\varphi(s) = F(s), \]

where

\[ \varphi(s) := \frac{1}{k} \frac{\partial u}{\partial n}(x(s)) - P.O.. \]

**Difficulty 1** The operator \((I + K) : L^2(\Gamma) \mapsto L^2(\Gamma)\) is bijective with bounded inverse

\[ \|(I + K)^{-1}\|_2 \leq C(k), \]

where the dependence of \(C(k)\) on \(k\) is not clear.
Approximation space: seek

\[ \varphi_N(s) = \sum_{j=1}^{M} v_j \rho_j(s) \in V_N, \]

where

\[ \rho_j(s) := e^{\pm is} \times \text{piecewise polynomial supported on graded mesh}. \]

**Question:** how do we compute \( v_j \)?
Galerkin method

To solve

$$\varphi(s) + \mathcal{K}\varphi(s) = F(s),$$

seek $$\varphi_{NG} \in V_N$$ such that

$$(I + P_{NG}\mathcal{K})\varphi_{NG} = P_{NG}F,$$

where $$P_{NG}$$ is the orthogonal projection onto the approximation space. Equivalently

$$(\varphi_{NG}, \rho) + (\mathcal{K}\varphi_{NG}, \rho) = (F, \rho), \quad \forall \rho \in V_N,$$

$$\Rightarrow \sum_{j=1}^{M} v_j[(\rho_j, \rho_m) + (\mathcal{K}\rho_j, \rho_m)] = (F, \rho_m).$$

If $$\rho_j, \rho_m$$ supported on same side of polygon, integrals not oscillatory.
Theorem. For \( N \geq N^* \), the operator \((I + P_{NG} \mathcal{K}) : L^2(\Gamma) \rightarrow V_N\) is bijective with bounded inverse

\[
\| (I + P_{NG} \mathcal{K})^{-1} \|_2 \leq C_s.
\]
Galerkin method

**Difficulty 2.** For $N \geq N^*(k)$, the operator $(I + P_{NG}K) : L_2(\Gamma) \mapsto V_N$ is bijective with bounded inverse

$$\|(I + P_{NG}K)^{-1}\|_2 \leq C_s(k),$$

where the dependence of $N^*(k)$ and $C_s(k)$ on $k$ is not clear.
Collocation method

To solve
\[ \varphi(s) + \mathcal{K}\varphi(s) = F(s), \]
seek \( \varphi_{NC} \in V_N \) such that
\[ (I + P_{NC}\mathcal{K})\varphi_{NC} = P_{NC}F, \]
where \( P_{NC} \) is the interpolatory projection onto the approximation space. Equivalently
\[ \varphi_{NC}(s_m) + \mathcal{K}\varphi_{NC}(s_m) = F(s_m), \quad m = 1, \ldots, M, \]
\[ \Rightarrow \sum_{j=1}^{M} v_j [\rho_j(s_m) + \mathcal{K}\rho_j(s_m)] = F(s_m). \]
If \( \rho_j \) supported on same side of polygon as \( s_m \), integrals not oscillatory.
Collocation method

We have not shown that \((I + P_{NC} \mathcal{K}) : L_2(\Gamma) \leftrightarrow V_N\) is bijective with bounded inverse.
Galerkin vs. Collocation: error analysis

**Theorem** There exists a constant $C_p > 0$, independent of $k$, such that for $N \geq N^*$

$$k^{1/2} \| \varphi - \varphi_{NG} \|_2 \leq C_p C_s \sup_{x \in D} |u(x)| \frac{n^{1/2}(1 + \log^{1/2}(kL/n))}{N^{p+1}},$$

$$k^{1/2} |u(x) - u_{NG}(x)| \leq C_p C_s \sup_{x \in D} |u(x)| \frac{n^{1/2}(1 + \log^{1/2}(kL/n))}{N^{p+1}}.$$  

- Stability and convergence not proven for collocation scheme.
Galerkin vs. Collocation: conditioning

**Galerkin:** mass matrix $M_G := [(\rho_j, \rho_m)]$ has $\text{cond} M \leq (1 + \sigma)/(1 - \sigma)$, where

$$\sigma \leq \max \left\{ \frac{\min(y_j^+, y_m^-) - \max(y_{j-1}^+, y_{m-1}^-)}{\sqrt{(y_j^+ - y_{j-1}^+)(y_m^- - y_{m-1}^-)}} \right\} < 1,$$

and if side lengths and angles are equal we can prove

$$\sigma < \left( \frac{1}{kL} \right)^{1/2N \log k}.$$

**Collocation:** difficulty with choice of collocation points, $M_C := [\rho_j(s_m)]$ may be ill conditioned.
Galerkin vs. Collocation: implementation

Galerkin: need to evaluate numerically many integrals of form
\[
\int_{-a}^{d} \int_{c}^{d} \left[ H_{0}^{(1)}(k\sqrt{s^{2} + t^{2}}) + \frac{itH_{1}^{(1)}(k\sqrt{s^{2} + t^{2}})}{\sqrt{s^{2} + t^{2}}} \right] e^{ik(\sigma_{j}t - \sigma_{m}s)} dt ds.
\]

Collocation: need to evaluate numerically many integrals of form
\[
\int_{a}^{b} \left[ H_{0}^{(1)}(k\sqrt{s_{m}^{2} + t^{2}}) + \frac{itH_{1}^{(1)}(k\sqrt{s_{m}^{2} + t^{2}})}{\sqrt{s_{m}^{2} + t^{2}}} \right] e^{ik\sigma_{j}t} dt.
\]

- Collocation method easier to implement
Numerical results

scattering by a square, $k = 5$

scattering by a square, $k = 10$
Numerical results (scattering by a square)

Solution minus P.O. approximation;
Numerical results (scattering by a square)

Solution minus P.O. approximation;
Numerical results (scattering by a square)

Solution minus P.O. approximation;

![Graph showing numerical results](image-url)
Numerical results (scattering by a square)

Solution minus P.O. approximation;
Numerical results (scattering by a square)

Solution minus P.O. approximation;
Numerical results (scattering by a square)

Solution minus P.O. approximation;
Numerical results (scattering by a square)

Solution minus P.O. approximation;
Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 5$;
Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 10$;
Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 20$;
Numerical results (scattering by a square)

"Exact" solution minus P.O. approximation, $k = 40$;
<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>dof</th>
<th>$\frac{|\varphi - \varphi_{NC}|_2}{|\varphi|_2}$</th>
<th>$\frac{|\varphi - \varphi_{NC}|_2}{|\varphi|_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td>24</td>
<td>$1.1691 \times 10^0$</td>
<td>$7.5453 \times 10^{-1}$</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
<td></td>
<td>$4.3784 \times 10^{-1}$</td>
<td>$4.7335 \times 10^{-1}$</td>
</tr>
<tr>
<td>8</td>
<td>96</td>
<td></td>
<td>$2.2320 \times 10^{-1}$</td>
<td>$2.6980 \times 10^{-1}$</td>
</tr>
<tr>
<td>16</td>
<td>192</td>
<td></td>
<td>$1.2106 \times 10^{-1}$</td>
<td>$1.2670 \times 10^{-1}$</td>
</tr>
<tr>
<td>32</td>
<td>376</td>
<td></td>
<td>$1.1633 \times 10^{-1}$</td>
<td>$6.8440 \times 10^{-2}$</td>
</tr>
<tr>
<td>64</td>
<td>752</td>
<td></td>
<td>$2.8702 \times 10^{-2}$</td>
<td>$3.3034 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
Table 2: Relative errors, $k = 160$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>dof</th>
<th>$\frac{|\varphi - \varphi_{NG}|_2}{|\varphi|_2}$</th>
<th>$\frac{|\varphi - \varphi_{NC}|_2}{|\varphi|_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>160</td>
<td>2</td>
<td>32</td>
<td>$7.2765 \times 10^{-1}$</td>
<td>$6.8901 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>56</td>
<td>$4.2628 \times 10^{-1}$</td>
<td>$4.4455 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>112</td>
<td>$4.9060 \times 10^{-1}$</td>
<td>$4.6445 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>224</td>
<td>$1.2847 \times 10^{-1}$</td>
<td>$2.3456 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>456</td>
<td>$8.4578 \times 10^{-2}$</td>
<td>$9.3327 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>904</td>
<td>$3.4570 \times 10^{-2}$</td>
<td>$4.8153 \times 10^{-2}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$M_N$</td>
<td>$|\varphi - \varphi_N|_2$</td>
<td>$|\varphi - \varphi_N|_2/|\varphi|_2$</td>
<td>COND</td>
</tr>
<tr>
<td>-----</td>
<td>-------</td>
<td>-----------------</td>
<td>----------------</td>
<td>-------</td>
</tr>
<tr>
<td>5</td>
<td>360</td>
<td>$3.6171 \times 10^{-1}$</td>
<td>$6.8909 \times 10^{-2}$</td>
<td>$2.6 \times 10^{1}$</td>
</tr>
<tr>
<td>10</td>
<td>376</td>
<td>$8.5073 \times 10^{-1}$</td>
<td>$1.1633 \times 10^{-1}$</td>
<td>$1.8 \times 10^{2}$</td>
</tr>
<tr>
<td>20</td>
<td>392</td>
<td>$8.0941 \times 10^{-1}$</td>
<td>$7.9909 \times 10^{-2}$</td>
<td>$1.0 \times 10^{3}$</td>
</tr>
<tr>
<td>40</td>
<td>416</td>
<td>$1.1252 \times 10^{0}$</td>
<td>$8.0909 \times 10^{-2}$</td>
<td>$2.4 \times 10^{2}$</td>
</tr>
<tr>
<td>80</td>
<td>432</td>
<td>$1.6630 \times 10^{0}$</td>
<td>$8.7071 \times 10^{-2}$</td>
<td>$5.9 \times 10^{2}$</td>
</tr>
<tr>
<td>160</td>
<td>456</td>
<td>$2.1936 \times 10^{0}$</td>
<td>$8.4578 \times 10^{-2}$</td>
<td>$5.2 \times 10^{2}$</td>
</tr>
<tr>
<td>320</td>
<td>472</td>
<td>$3.5185 \times 10^{0}$</td>
<td>$1.0211 \times 10^{-1}$</td>
<td>$8.1 \times 10^{2}$</td>
</tr>
</tbody>
</table>

Table 3: Relative $L_2$ errors, various $k$, $N = 32$
| $k$ | $N$ | $\frac{|u_N - u_{256}|}{u_{256}} (-\pi, 3\pi)$ | $\frac{|u_N - u_{256}|}{u_{256}} (3\pi, 3\pi)$ | $\frac{|u_N - u_{256}|}{u_{256}} (3\pi, -\pi)$ |
|-----|-----|------------------------------------------|------------------------------------------|------------------------------------------|
| 5   | 4   | $1.9588 \times 10^{-2}$                  | $1.0071 \times 10^{-3}$                  | $1.5885 \times 10^{-2}$                  |
|     | 8   | $4.2631 \times 10^{-3}$                  | $2.8032 \times 10^{-3}$                  | $2.3213 \times 10^{-3}$                  |
|     | 16  | $3.6178 \times 10^{-4}$                  | $3.1438 \times 10^{-4}$                  | $1.3514 \times 10^{-3}$                  |
|     | 32  | $6.6463 \times 10^{-5}$                  | $2.9271 \times 10^{-5}$                  | $1.7115 \times 10^{-5}$                  |
|     | 64  | $1.1634 \times 10^{-5}$                  | $5.4525 \times 10^{-6}$                  | $3.8267 \times 10^{-6}$                  |

Table 4: Relative errors, for $u_N(x)$
| $k$ | $N$ | $|\frac{u_N - u_{256}}{u_{256}}(-\pi, 3\pi)|$ | $|\frac{u_N - u_{256}}{u_{256}}(3\pi, 3\pi)|$ | $|\frac{u_N - u_{256}}{u_{256}}(3\pi, -\pi)|$ |
|-----|-----|---------------------------------|---------------------------------|---------------------------------|
| 320 | 4   | $7.2339 \times 10^{-6}$        | $9.1702 \times 10^{-6}$        | $6.5155 \times 10^{-5}$        |
|     | 8   | $1.3617 \times 10^{-5}$        | $4.7357 \times 10^{-6}$        | $3.6329 \times 10^{-5}$        |
|     | 16  | $1.0694 \times 10^{-5}$        | $3.0122 \times 10^{-6}$        | $2.9284 \times 10^{-5}$        |
|     | 32  | $1.0691 \times 10^{-6}$        | $5.3066 \times 10^{-7}$        | $2.8225 \times 10^{-6}$        |
|     | 64  | $3.1606 \times 10^{-7}$        | $3.0148 \times 10^{-7}$        | $8.1702 \times 10^{-7}$        |

Table 5: Relative errors, for $u_N(x)$
What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

Figure 4: square, $k = 5$
What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

Figure 5: square, $k = 10$
What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

Figure 6: square, $k = 20$
What we actually are computing . . .

The difference between the exact solution and a leading order approximation;

Figure 7: square, $k = 40$
Summary and Conclusions

• Using Green’s representation theorem in a half-plane we can understand behaviour of the field on the boundary and its derivatives for scattering by a convex polygon (extends to convex polyhedron in 3D)

• For a convex polygon, design of an optimal graded mesh for piecewise polynomial approximation is then straightforward

• The number of degrees of freedom need only grow logarithmically with the wavenumber to maintain a fixed accuracy

• Ongoing considerations
  – Galerkin vs. Collocation - stability and convergence analysis
  – Better schemes for evaluating oscillatory integrals
  – $hp$ ideas