A Survey of Computational
High Frequency Wave Propagation II

Olof Runborg

NADA, KTH

High Frequency Wave Propagation
CSCAMM, September 19-22, 2005
Numerical methods

• Direct methods
  – Wave equation (time domain)
  – Integral equation methods (frequency domain)

• Asymptotic methods
  – Physical optics
  – Geometrical optics
  – (Gaussian beams)

• Hybrid methods
Direct numerical methods for the wave equation

Scalar wave equation

\[ u_{tt} - c(x)^2 \Delta u = 0, \quad (t, x) \in \mathbb{R}^+ \times \Omega, \quad \Omega \subset \mathbb{R}^d \]

+ boundary and initial data.

- Discretize \( \Omega \), time and \( u \).
- Many methods: FD (explicit, uniform staggered grids), FV, FEM (implicit or DG).
- Complexity \( O(\omega^{(1/p+1)(d+1)}) \) (including time).
Direct integral equation methods for Helmholtz

Scattering problem for Helmholtz equation: \( u = u_s + u_{\text{inc}}, \ c \equiv 1 \)

\[
\Delta u_s + \omega^2 u_s = 0, \quad x \in \mathbb{R}^d \setminus \Omega,
\]

\[
u_s = -u_{\text{inc}}, \quad x \in \partial \Omega \quad + \text{radiation condition}
\]

Rewrite as integral equation, e.g.

\[
u_{\text{inc}}(x) = -\oint_{\partial \Omega} G(\omega|x - x'|) \frac{\partial u(x')}{\partial n} \, dx', \quad \forall x \in \partial \Omega.
\]

Discretize \( \partial \Omega \) and \( \frac{\partial u}{\partial n} \Rightarrow O(\omega^{d-1}) \) unknowns.

Finite element/collocation methods, ”method of moments”.

Full matrix equation, direct solution, complexity \( O(\omega^{3(d-1)}) \).

Fast multipole methods, iterative solver, complexity \( \approx O(\omega^{(d-1)}) \).

Physical optics

Integral formulation of scattering problem

\[ u_s(x) = \oint_{\partial \Omega} G(\omega |x - x'|) \frac{\partial u(x')}{\partial n} \, dx', \quad x \in \mathbb{R}^d \setminus \Omega, \]

Approximate \( \frac{\partial u}{\partial n} \) by geometrical optics solution.
E.g. if \( u_{\text{inc}} = \exp(i\omega \alpha \cdot x) \) is a plane wave, \( \Omega \) convex, then

\[ \frac{\partial u}{\partial n} \approx \frac{\partial (u_{\text{inc}} + u_{s\text{GO}})}{\partial n} = \begin{cases} 2i\omega \alpha \cdot \hat{n}(x)e^{i\omega \alpha \cdot x}, & x \text{ illuminated,} \\ 0, & x \text{ in shadow.} \end{cases} \]

Cost of computing solution still depends on \( \omega \).

"Exact PO"

\[ \frac{\partial u}{\partial n} = A(x, \omega)e^{i\omega \alpha \cdot x}/\omega \]

then \( A(x, \omega) \) smooth, uniformly in \( \omega \), except at shadow boundaries.
Discretize and solve \( A(x, \omega) \) at cost independent of \( \omega \). [Bruno]
Geometrical optics models and numerical methods

\[ u_{tt} - c(\mathbf{x})^2 \Delta u = 0 \]

- Rays
  \[ \frac{d\mathbf{x}}{dt} = c^2 \mathbf{p}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\nabla c}{c} \]
  - Ray tracing

- Kinetic
  \[ f_t + c^2 \mathbf{p} \cdot \nabla_x f = 0 \]
  - Wavefront methods

- Eikonal
  \[ \phi_t + c |\nabla \phi| = 0 \]
  - Moment methods, Full phase space methods
  - Hamilton–Jacobi methods
Eikonal equation

- Time-dependent version.
  Wave equation plus ansatz $u(t, x) \approx A(t, x)e^{i\omega \phi(t,x)}$ give
  \[
  \phi_t + c(x)|\nabla \phi| = 0.
  \]
  Upwind, high-resolution (ENO, WENO) finite difference methods [Osher, Shu, et al]

- Stationary version.
  Helmholtz equation plus ansatz $u(x) \approx A(x)e^{i\omega \phi(x)}$ give
  \[
  |\nabla \phi| = c(x)^{-1}.
  \]
  Fast marching [Sethian] or fast sweeping methods [Zhao, Tsai, et al].

(Note, if IC and BC match, $\phi = \varphi - t$.)
Eikonal equation

- Ansatz only treats one wave. In general crossing waves
  \[ u(x) \approx A_1(x)e^{i\omega \varphi_1(x)} + A_2(x)e^{i\omega \varphi_2(x)} + \cdots \]

- Nonlinear equation, no superposition principle
- Viscosity solution, kinks
- First arrival property: \( \varphi_{\text{visc}}(x) = \min_n \varphi_n(x) \)
Example: Eikonal solution
Geometrical optics models and numerical methods

\[ u_{tt} - c(x)^2 \Delta u = 0 \]

- **Rays**
  \[ \frac{dx}{dt} = c^2 p, \quad \frac{dp}{dt} = -\frac{\nabla c}{c} \]
  - Ray tracing
  - Wavefront methods

- **Kinetic**
  \[ f_t + c^2 p \cdot \nabla_x f \]
  \[ -\frac{1}{c} \nabla c \cdot \nabla_p f = 0 \]
  - Moment methods, Full phase space methods

- **Eikonal**
  \[ \phi_t + c |\nabla \phi| = 0 \]
  - Hamilton–Jacobi methods
Ray tracing

Rays are the (bi)characteristics \((x(t), p(t))\) of the eikonal equation, given by ODEs

\[
\frac{dx}{dt} = c(x)^2 p, \quad \frac{dp}{dt} = -\frac{\nabla c(x)}{c(x)},
\]

Hamiltonian system with \(H = c(x)|p|\) and \(H \equiv 1\).

Solve with numerical ODE methods, e.g. Runge Kutta.

Note, if valid at \(t = 0\), then for all \(t > 0\):

- \(\varphi(x(t)) = t\), (phase \(\sim\) traveltime)
- \(\nabla \varphi(x(t)) = p(t)\), (local ray direction)
- \(|p(t)| = 1/c(x(t))\), (\(H = 1\) conserved, can reduce to \(p \in S^{d-1}\))

There are also ODEs for the amplitude along rays.

Issues: Diverging rays. Interpolation onto regular grid.
Example: Ray/Wavefront solutions
Ray tracing boundary value problem

Start and endpoint of ray given.

- Piecewise constant $c(x)$
  Rays piecewise straight lines. Find refraction/reflection points at interfaces by Newton’s method.

- Smoothly varying $c(x)$
  Ray tracing eq is a nonlinear elliptic boundary value problem

\[
\frac{d}{dt} \left( c(x)^{-2} \frac{dx}{dt} \right) = -\frac{\nabla c(x)}{c(x)},
\]

\[
x(0) = x_0,
\]

\[
x(t^*) = x_1.
\]

$t^*$ additional unknown.

Solve by shooting method or discretize PDE + Newton.

Multiple solutions difficult.
**Wavefront tracking**

Directly solve for wavefront given by $\varphi(x) = \text{const.}$

Suppose $\gamma(\alpha)$ is the initial wavefront, $\varphi(\gamma(\alpha)) = 0$.

Follow ensemble of rays

$$\frac{\partial x(t, \alpha)}{\partial t} = c^2 p,$$

$$x(0, \alpha) = \gamma(\alpha),$$

$$\frac{\partial p(t, \alpha)}{\partial t} = -\frac{\nabla c}{c},$$

$$p(0, \alpha) = \frac{\gamma'(\alpha) \perp}{c|\gamma'(\alpha)|}.$$

Note: Moving front in normal direction a possibility

$$x_t = c \frac{x^\perp_{\alpha}}{|x_{\alpha}|} \quad (\text{since } 0 = \partial_\alpha \varphi(x(t, \alpha)) = x_{\alpha} \cdot \nabla \phi = x_{\alpha} \cdot p)$$

But not good since wavefront non-smooth!
Phase space

Phase space \((x, p)\), where \(p \in \mathbb{S}^{d-1}\) is local ray direction

Observation: Wavefront is a smooth curve in phase space.

- 2D problems: 1D curve in 3D phase space \((x, y, \theta)\).
- 3D problems: 2D surface in 5D phase space \((x, y, z, \theta, \alpha)\).
Wavefront construction

Propagate Lagrangian markers on the wavefront in phase space.
Insert new markers adaptively by interpolation when front resolution deteriorates.
Interpolate traveltime/phase/amplitude onto regular grid.

[Vinje, Iversen, Gjøystdal, Lambaré, ...]
Geometrical optics models and numerical methods

\[ u_{tt} - c(x)^2 \Delta u = 0 \]

Rays

\[
\frac{dx}{dt} = c^2 p, \quad \frac{dp}{dt} = -\frac{\nabla c}{c}
\]

Ray tracing

Kinetic

\[
f_t + c^2 p \cdot \nabla_x f
\]

Wavefront methods

Moment methods, Full phase space methods

Eikonal

\[
\phi_t + c |\nabla \phi| = 0
\]

Hamilton–Jacobi methods
Kinetic formulation

Let $f(t, x, p)$ be the particle (photon) density in phase space. Bicharacteristic equations $\Rightarrow$

$$f_t + c^2 p \cdot \nabla_x f - \frac{\nabla c}{c} \cdot \nabla_p f = 0.$$ 

$f$ supported on $|p| = c(x)^{-1}$, $(H \equiv 1)$.

Can also be derived directly from wave eq. through e.g. Wigner measures [Tartar, Lions, Paul, Gerard, Mauser, Markowich, Poupaud, ...]

Relationship to wave equation solution:

$$u = Ae^{i\omega\phi} \sim f = A^2 \delta(p - \nabla \phi).$$

Note: Loss of phase information.
Moment equations

- Derived from transport equation in phase space + closure assumption for a system of equations representing the moments. (C.f. hydrodynamic limit from Boltzmann eq.)
- PDE description in the “small” \((t, x)\)-space.
- Arbitrary good superposition. \(N\) crossing waves allowed. (But larger \(N\) means a larger system of PDEs must be solved.)

[Brenier, Corrias, Engquist, OR] (wave equation),
[Gosse, Jin, Li, Markowich, Sparber] (Schrödinger)
Derivations, homogenous case \((c \equiv 1)\)

Starting point is

\[ f_t + \mathbf{p} \cdot \nabla_x f = 0. \]

Let \( \mathbf{p} = (p_1, p_2) \). Define the moments,

\[ m_{ij} = \int_{\mathbb{R}^2} p_1^i p_2^j f \, d\mathbf{p}. \]

From

\[ \int_{\mathbb{R}^2} p_1^i p_2^j (f_t + \mathbf{p} \cdot \nabla_x f) \, d\mathbf{p} = 0, \]

we get the infinite (valid \( \forall i, j \geq 0 \)) system of moment equations

\[ (m_{ij})_t + (m_{i+1,j})_x + (m_{i,j+1})_y = 0. \]
Derivations, homogenous case, cont.

Make the closure assumption

\[ f(\mathbf{x}, \mathbf{p}, t) = \sum_{k=1}^{N} A_k^2 \cdot \delta(|\mathbf{p}| - 1, \arg \mathbf{p} - \theta_k). \]

The moments take the form

\[ m_{i,j} = \sum_{k=1}^{N} A_k^2 \cos^i \theta_k \sin^j \theta_k. \]

Corresponds to a maximum of \( N \) waves at each point.

Choose equations for moments \( m_{2k-1,0} \) and \( m_{0,2k-1}, k = 1, \ldots, N \).

Gives closed system of \( 2N \) equations with \( 2N \) unknowns (the \( A_k \)'s and \( \theta_k \)'s).
Moment equations, examples

Ex. $N = 1$

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_t + \begin{pmatrix}
  \frac{u_1^2}{\sqrt{u_1^2 + u_2^2}} \\
  \frac{u_1 u_2}{\sqrt{u_1^2 + u_2^2}}
\end{pmatrix}_x + \begin{pmatrix}
  \frac{u_1 u_2}{\sqrt{u_1^2 + u_2^2}} \\
  \frac{u_2^2}{\sqrt{u_1^2 + u_2^2}}
\end{pmatrix}_y = 0.
\]

where $u_1 = m_{10} = A^2 \cos \theta$ and $u_2 = m_{01} = A^2 \sin \theta$.

For $N \geq 2$,

\[
F_0(u)_t + F_1(u)_x + F_2(u)_y = 0.
\]

where $F_0(u), F_1(u)$ and $F_2(u)$ are complicated non-linear functions.

- PDE = weakly hyperbolic system of conservation laws, (with source terms when $c$ varies)
- Flux functions in conservation law can be difficult to evaluate.
Wedge example

$N = 1$

$N = 2$

$N = 1$

$N = 2$
Hybrid methods

- Full Helmholtz or wave equation where variations in $c(x)$ and/or geometry on same scale as wavelength.
- GO elsewhere, typically for long range interactions.
  Ex. antenna + aircraft.

Coupling of models.
Other methods

- Hamilton–Jacobi methods
  [Vidale, van Trier, Symes, Engquist, Fatemi, Osher, Benamou, . . . ]

- Wavefront tracking using level sets in phase space
  [Osher, Tsai, Cheng, Liu, Jin, Qian, . . . ]

- Wavefront tracking using segment projection
  [Engquist, OR, Tornberg]

- Full phase space methods
  [Sethian, Fomel, Symes, Qian]