Unfolding Complex Singularities for the Euler Equations

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Outline

• Complex singularities: PDE examples
• Numerical construction of complex singularities for Euler
• Unfolding complex singularities
Methods for Construction of Possible Euler Singularities

• **Numerical construction**

• **Similarity solutions**
  – Childress et al. (1989), …

• **Complex variables**
  – Bardos, Benachour & Zerner (1976), REC (1993)

• **Unfolding / catastrophe theory**
  – Ercolani, Steele & REC (1996)
Canonical Example 1: Cauchy-Riemann Equations

- Laplace equation in $x, t$
  \[ u_{tt} + u_{xx} = 0 \]

- Complex traveling wave solution
  \[ u(x, t) = w(x + it) + w^*(x - it) \]

- Singularities in initial data move toward the real axis.
Canonical Example 2: Burgers Equations

- **Burgers equation**
  - Initial value problem
    \[ u_t + uu_x = 0 \]
    \[ u(0, x) = u_0(x) \]
  - Characteristic form
    \[ \partial_t u = 0 \quad \text{on} \quad \partial_x x = u = u_0 \]
  - Invert initial data
    \[ x_0(u) : u_0(x_0(u)) = u \]
  - Implicit solution
    \[ x = x_0(u) + tu \]

- **Singularities**
  \[ u_x = \infty \iff x_u = 0 \]
  i.e. \[ 0 = \partial_u x_0(u) + t \]
Canonical Example 2: Burgers Equations (Cont)

- Burgers equation singularity condition
  \[ 0 = \partial_u x_0(u) + t \]

- Example
  - Initial data
    \[ u_0(x_0) = -x_0^{1/3} \]
    \[ x_0(u) = -u^3 \]
  - Singularity condition
    \[ 0 = -3u^2 + t \]
  - Real singularities for \( t > 0 \)
    \[ u = \pm \sqrt{t/3} \]
  - Complex singularities for \( t < 0 \)
    \[ u = \pm i \sqrt{|t|/3} \]

- Complex singularities collide forming shock
Kelvin-Helmholtz Instability

• Moore (1979) constructed singularities through asymptotics, as traveling waves in complex plane
  - \( z = x + iy \approx \gamma + (1+i) \varepsilon (\sin \gamma)^{3/2} \)
  - \( \gamma \) = circulation variable
  - Curvature singularity in sheet

• REC and Orellana (1989) constructed solutions, including solutions with singularities and ill-posedness, starting from analytic initial data.

• Wu (2005) showed that any solution, satisfying some mild regularity conditions, is analytic for \( t > 0 \).
Vortex Sheet Singularity for Kelvin-Helmholtz

- Moore (1979)
  - \( z = x + iy \approx \gamma + (1+i)\varepsilon (\sin \gamma)^{3/2} \)
  - \( \gamma \) = circulation variable
  - Curvature in shape of sheet
  - Cusp in sheet strength \( (z_\gamma)^{-1} \)

Sheet position at various times (Krasny)
Moore’s Construction
(REC & Orellana interpretation)

Birkhoff-Rott Equation
\[ \partial_t z^*(\gamma, t) = BR(z) = \frac{1}{2\pi i} PV \int (z(\gamma, t) - z(\gamma', t))^{-1} d\gamma' \]

- Look for \( z = \gamma + z_+ + z_- \)
  - upper analytic \( z_+ \)
  - lower analytic \( z_- \)

- Ignore interactions between \( z_+ \) and \( z_- \) (Moore’s approx)
  - Evaluate BR for lower analytic functions \( z_-, z_+^* \) by contour integration

\[ \partial_t z_+^*(\gamma, t) = BR(z_-) = \frac{1}{1 + \partial_\gamma z_-} \]
\[ \partial_t z_-^*(\gamma, t) = BR(z_+) = \frac{-1}{1 + \partial_\gamma z_+^*} \]

- Nonlinearization of CR eqtns, complex characteristics construction of solutions with singularities
Generalizations of Moore’s Construction

• Rayleigh-Taylor
  – Siegel, Baker, REC (1993)

• Muskat problem (2-sided Hele-Shaw, porous media)
  – Cordoba
Complex Euler Singularities: Numerical Construction

- **Axisymmetric flow with swirl**
  - REC (1993)
- **2D Euler**
  - Pauls, Matsumoto, Frisch & Bec (2006)
- **3D Euler (Pelz and related initial data) – talk by Siegel**
  - Siegel & REC (2006)
- **Singularity detection via asymptotics of fourier components**
Singularity Analysis

- Fit to asymptotic form of fourier components in 1D

\[ \hat{u}_k \approx c k^{-\alpha} e^{-ikz_*} \quad \Rightarrow \quad u \approx c (z - z_*)^{\alpha - 1} \]

- Apply 3-point fit, to get singularity parameters \( c, \alpha, z_* \) as function of \( k \)
  - Successful fit has \( c, \alpha, z_* \) nearly independent of \( k \)
Axisymmetric flow with swirl

REC (1993)

- **Solution method**
  - Moore’s approximation: \( u = u_+ + u_- \)
  - \( u_+ \) upper analytic in \( z \), \( u_- = u_+^* \) lower analytic, no interaction between them
  - Traveling wave ansatz (Siegel’s thesis 1989 for Rayleigh-Taylor)
    \[ u_+ (r, z, t) = u_+ (r, z - i\sigma t) \]
  - Ultra-high precision,
    - needed to control amplification of round-off error

- **Singularity type**
  - \( u \approx x^{-1/3} \)
  - \( \omega \approx x^{-4/3} \)

- **Real singularity? No**
  - Violates Deng-Hou-Yu (DHY) criterion, restricted directionality
Complex upper analytic solution: pure swirling flow

- Flow in periodic anulus,
  - \( r_1 < r < r_2 \) (no normal flow BCs)
  - \( 0 < z < 2\pi \) (periodic BCs)
2D Euler

Pauls, Matsumoto, Frisch & Bec (2006)

• Solution method
  – Small time asymptotics, spectral computation
  – Ultra-high precision,
    • needed singularity detection, since singularities are far from reals

• Singularity type
  – $\omega \approx x^{-\beta}$ with $5/6 \leq \beta \leq 1$

• Real singularity? No
  – Vorticity does not grow in 2D $\rightarrow$ no singularities
  – $u=(u,v,0)$ $\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u = 0$
  – $\omega=(0,0,\zeta)$

Fig. 3. Local prefactor exponent $\alpha_{\text{loc}}(k)$ versus wavenumber for two values of the slope.

Unfolding Singularities

- General method
Unfolding Singularities

- **General method**
  - Unfolding variable $\eta$
  - Mapping $q(x,t, \eta)=0$ defines relation between $(x, t)$ and $\eta$
  - Rewrite PDE in terms of $(x, t, \eta)$
    - $u = u(x, t, \eta)$
    - $\partial_t = \partial_t - q_{\eta}^{-1} \eta_t \partial_\eta$

- **Special method**
  - Include $\sqrt{\xi}$ in solution
  - $\xi = \xi(x,t)$ a smooth function
  - Works only for a single sqrt singularity
Boussinesq and Unfolding

Boussinesq eqtns

\[
\begin{align*}
(\partial_t + u \cdot \nabla) \rho &= f \\
(\partial_t + u \cdot \nabla) \zeta &= -\partial_z \rho + g \\
u &= (u, \nu) = \nabla^\perp \psi = (-\partial_z \psi, \partial_r \psi) \\
\zeta &= \nabla^2 \psi = -\partial_z u + \partial_r \nu.
\end{align*}
\]

Unfolding ansatz

\[
\begin{align*}
u &= u_0 + \xi^{\frac{1}{2}} u_1 \\
\rho &= \rho_0 + \xi^{\frac{1}{2}} \rho_1 \\
\zeta &= \zeta_0 + \xi^{-\frac{1}{2}} \zeta_1 \\
\psi &= \psi_0 + \xi^{\frac{3}{2}} \psi_1
\end{align*}
\]

\[u_i, \rho_i, \psi_i, \zeta_i, \xi \text{ smooth functions}\]
Unfolded eqtns

• Div

\[ \mathbf{u}_0 = \nabla^\perp \psi_0 \]
\[ \mathbf{u}_1 = \frac{3}{2} \psi_1 \nabla^\perp \xi + \xi \nabla^\perp \psi_1 \]

• \( \zeta \) definition

\[ \zeta_0 = \nabla \times \mathbf{u}_0 \]
\[ \zeta_1 = \frac{1}{2} \nabla \xi \times \mathbf{u}_1 + \xi \nabla \times \mathbf{u}_1 \]

• desingularization

\[ \xi_t + \mathbf{u}_0 \cdot \nabla \xi = 0. \]

• \( \rho \) eqtn

\[ (\partial_t + \mathbf{u}_0 \cdot \nabla) \rho_0 + \frac{1}{2} \alpha \xi \rho_1 + \xi \mathbf{u}_1 \cdot \nabla \rho_1 = f \]
\[ (\partial_t + \mathbf{u}_0 \cdot \nabla) \rho_1 + \mathbf{u}_1 \cdot \nabla \rho_0 = 0 \]

• \( \zeta \) eqtn

\[ (\partial_t + \mathbf{u}_0 \cdot \nabla) \zeta_1 + \xi \mathbf{u}_1 \cdot \nabla \zeta_0 = -\xi \partial_z \rho_1 - \frac{1}{2} \rho_1 \partial_z \xi \]
\[ (\partial_t + \mathbf{u}_0 \cdot \nabla) \zeta_0 + \mathbf{u}_1 \cdot \nabla \zeta_1 - \frac{1}{2} \xi \alpha \xi = -\partial_z \rho_0 + g. \]
Unfolded eqtns

- Div

\[ u_0 = \nabla^\perp \psi_0 \]
\[ u_1 = \frac{3}{2} \psi_1 \nabla^\perp \xi + \xi \nabla^\perp \psi_1 \]

- \( \xi \) definition

\[ \xi_0 = \nabla \times u_0 \]
\[ \xi_1 = \frac{1}{2} \nabla \xi \times u_1 + \xi \nabla \times u_1 \]

- \( \xi \) eqtn

\[ (\partial_t + u_0 \cdot \nabla) \xi_1 + \xi u_1 \cdot \nabla \xi_0 = -\xi \partial_x \rho_1 - \frac{1}{2} \rho_1 \partial_x \xi \]
\[ (\partial_t + u_0 \cdot \nabla) \xi_0 + u_1 \cdot \nabla \xi_1 - \frac{1}{2} \xi_1 \alpha \xi = -\partial_x \rho_0 + g. \]

Unfolded eqtns

• Div
  
  \[ \mathbf{u}_0 = \nabla^\perp \psi_0 \]
  
  \[ \mathbf{u}_1 = \frac{3}{2} \psi_1 \nabla^\perp \xi + \xi \nabla^\perp \psi_1 \]

• \( \xi \) definition
  
  \[ \xi_0 = \nabla \times \mathbf{u}_0 \]
  
  \[ \xi_1 = \frac{1}{2} \nabla \xi \times \mathbf{u}_1 + \xi \nabla \times \mathbf{u}_1 \]

• desingularization
  
  \[ \xi_t + \mathbf{u}_0 \cdot \nabla \xi = 0. \]

• \( \rho \) eqtn
  
  \[(\partial_t + \mathbf{u}_0 \cdot \nabla) \rho + \frac{1}{2} \alpha \xi \rho + \xi \mathbf{u}_1 \cdot \nabla \rho = f \]
  
  \[(\partial_t + \mathbf{u}_0 \cdot \nabla) \rho + \mathbf{u}_1 \cdot \nabla \rho = 0 \]

• \( \xi \) eqtn
  
  \[(\partial_t + \mathbf{u}_0 \cdot \nabla) \xi_1 = -\frac{1}{2} \rho \partial_z \xi \quad \mathbf{u}_1 \cdot \nabla \xi_0 = -\partial_z \rho_1 \]
  
  \[(\partial_t + \mathbf{u}_0 \cdot \nabla) \xi_0 + \mathbf{u}_1 \cdot \nabla \xi_1 - \frac{1}{2} \xi_1 \alpha \xi = -\partial_z \rho_0 + g. \]

• This system is well-posed but nonstandard.
  • Unfolding through mapping \( q(x, t, \eta) = 0 \) leads to a well-posed system that is more complicated but more standard.

Conclusions

- Inviscid singularities may play a role in viscous turbulence.
- Complex variables approach successful for interface problems, including singularity formation and global existence.
- Complex singular solutions for Euler constructed by special methods.
- Unfolding of weak complex singularities and their dynamics.
- Attempting to turn this into a real singular solution for Euler.
Equations for $u^+$

$$r^{-1} \partial_r (r u_r^+) + \partial_z u_z^+ = 0$$

$$(\bar{u}_z - i\sigma) \partial_z u_z^+ + u_r^+ \partial_r \bar{u}_z + \partial_z p^+ = a$$

$$(\bar{u}_z - i\sigma) \partial_z u_r^+ - 2r^{-1} u_\theta u_r^+ + \partial_r p^+ = b$$

$$(\bar{u}_z - i\sigma) \partial_z u_\theta^+ + u_r^+ \partial_r \bar{u}_\theta + r^{-1} \bar{u}_\theta u_r^+ = c$$

$$a = -u^+ \cdot \nabla u_z^+$$

$$b = -u^+ \cdot \nabla u_r^+ + r^{-1} u_\theta^2$$

$$c = -u^+ \cdot \nabla u_\theta^+ - r^{-1} u_\theta^+ u_r^+$$

$$u_r^+ \bar{\omega}_z = r^{-1} \partial_r (r \bar{u}_\theta) u_r^+$$
Simplified eqtn for $u^+_r$

$$\partial_r (r^{-1} \partial_r (ru^+_r)) + \partial_z^2 u^+_r - \eta u^+_r = d$$

$$\eta = (\bar{u}_z - i\sigma)^{-1} \{ \partial_r^2 \bar{u}_z - r^{-1} \partial_r \bar{u}_z - 2r^{-1} (\bar{u}_z - i\sigma)^{-1} \bar{u}_\theta \bar{\omega}_z \}$$

$$d = (\bar{u}_z - i\sigma)^{-1} \{-\partial_r a + \partial_z b + 2r^{-1} \bar{u}_\theta (\bar{u}_z - i\sigma)^{-1} c\}$$
Instability of $u_k$ equations

- Solution of $k$ eqtn depends on $k'$ with $k'<k$
- Roundoff error grows as $k$ increases
- Controlled through use of ultra high precision
  - MPFUN by David Bailey
- Limitation on size of computation
Hele-Shaw

• Flow through porous media with a free boundary
  – Darcy’s law and incompressibility
    \[ \mathbf{u} = V\mathbf{j} - k\nabla p \quad \nabla \cdot \mathbf{u} = 0 \]
  – Boundary conditions
    \[ p = 0 \quad \mathbf{u} \cdot \mathbf{n} = V_n \]

• Exact solution with cusp singularities in the boundary
Hele-Shaw

- The zero surface tension limit $\gamma \to 0$ is singular. Singularities in the complex plane move toward the real boundary, but they can be preceded by daughter singularities (Tanveer, Siegel, ...).
Muskat Problem

- **Two sided Hele-Shaw**
  - Darcy’s law and incompressibility ($i=1,2$)
    \[ \mathbf{u}_i = V_j - k_i \nabla p_i \quad \nabla \cdot \mathbf{u}_i = 0 \]
  - Boundary conditions
    \[ p_1 = p_2, \quad \mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n} = V_n \]

- **Singularities**
  - No exact solutions

- **Analysis by Siegel, Howison & REC**
  - Global existence in stable case (more viscous fluid moving into less viscous)
    - Initial data in Sobolev space, then becomes analytic for $t>0$
  - Analytic construction of singularities in unstable case
    - Curvature singularities, cusps not analyzed

Derivation of Singularity Requirements 
for Inviscid Energy Dissipation

- For singularity set S of codimension \( \kappa \), singularity order \( \alpha \)
  \[
dx = r^{\kappa-1} dr dx_S \\
u \approx r^\alpha
  \]

- Time derivative of energy \( u_t + u \cdot \nabla u + \nabla p = 0 \)
  \[
  (d/dt) \int |u|^2 \ dx = \int u \cdot (u \cdot \nabla) u + u \cdot \nabla p dx \\
  = \int_S \int r^{3\alpha-1} r^{\kappa-1} dr dx_S
  \]

- The convective integral is nonzero, only if it isn’t absolutely integrable; i.e.
  \[
  3\alpha - 1 + \kappa - 1 < -1 \\
  3\alpha + \kappa < 1
  \]

Upper analytic solutions

- Look for upper analytic solution \((k \geq 0)\)

\[
\begin{align*}
\mathbf{u} &= \mathbf{\bar{u}} + \mathbf{u}^+ \\
\mathbf{\bar{u}} &= (0, \overline{u}_z, \overline{u}_\theta)(r) \\
\mathbf{u}^+ &= (u_r^+, u_z^+, u_\theta^+)(r, z)
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}^+(r, z) &= \sum_{k \geq 1} \hat{u}_k(r) e^{ikz} \\
\mathbf{u}^+(r, z, t) &= \sum_{k \geq 1} \hat{u}_k(r) e^{ikz + \sigma kt}
\end{align*}
\]

- Because wavenumbers add, the coupling is one way (Siegel)

\[
\begin{align*}
M_k \hat{u}_k &= A_k(\sigma, \hat{u}_0, \ldots, \hat{u}_{k-1}) \\
M_k &= M_k(\sigma, \hat{u}_0)
\end{align*}
\]
3D Euler (Pelz initial data)

Siegel & REC (2006)

• Solution method
  – Moore’s approximation: \( u = u_+ + u_- \)
    • \( u_+ \) upper analytic in \( x,y,z \), \( u_- = u_+^* \) lower analytic, no interaction between them
  – Traveling wave ansatz
    \( u_+(x, y, z, t) = u_+(x, y, z - i\sigma t) \)
  – No need for ultra-high precision
  – Highly symmetric (Kida)

• Singularity type
  – \( u_+ \approx \epsilon x^{-1/2} \)
  – \( \omega_+ \approx \epsilon x^{-3/2} \)

• Real singularity? ?
  – Satisfies known singularity criteria
  – Attempting to construct real singular solution as \( u = u_+ + u_- + \epsilon^2 u_c \)