Calculation of complex singular solutions to the 3D incompressible Euler equations

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Numerical Studies

• Axisymmetric flow with swirl and 2D Boussinesq convection

• High symmetry flows

• Antiparallel vortex tubes
  - Hou & Li (2006)

• Pauls et al (2006).: Study of complex space singularities for 2D Euler in short time asymptotic regime
Axisymmetric flow with swirl


\[ r^{-1} \partial_r (ru_r) + \partial_z u_z = 0 \]
\[ \partial_t u_z + \mathbf{u} \cdot \nabla u_z + \partial_z p = 0 \]
\[ \partial_t u_r + \mathbf{u} \cdot \nabla u_r - r^{-1} u_\theta^2 + \partial_r p = 0 \]
\[ \partial_t u_\theta + \mathbf{u} \cdot \nabla u_\theta + r^{-1} u_\theta u_r = 0. \]

- Annular geometry

\[ r_1 < r < r_2, \quad 0 < z < 2\pi \]

- Steady background flow

\[ \mathbf{u} = (0, \bar{u}_\theta, \bar{u}_z)(r) \]

chosen to satisfy Rayleigh’s criterion for instability and an unstable eigenmode

\[ \hat{\mathbf{u}}_1(r)e^{iz+\sigma t} \]
Background flow

• Background flow is smoothed vortex sheet at \( r_0 \)
  (motivated by Caflisch, Li, Shelley 1991)

\[
\overline{u_\theta} = \begin{cases} 
\frac{\Gamma_1}{2\pi r} & r_1 < r < r_0 \\
\frac{\Gamma_2}{2\pi r} & r_0 < r < r_2 
\end{cases}
\]

\[
\overline{u_z} = \begin{cases} 
w_1 & r_1 < r < r_0 \\
w_2 & r_0 < r < r_2 
\end{cases}
\]

\[
\overline{u_r} = 0.
\]

• Pure swirling flow is unstable if 
  \(|\Gamma_1| > |\Gamma_2|\) (Rayleigh criterion)
Traveling wave solution

- Construct complex, upper-analytic traveling wave solution
  Baker, Caflisch & Siegel (1993)

\[ u = \bar{u}(r) + u_+(r, z, t) \]

in which

\[ u_+ = \sum_{k=1}^{\infty} \hat{u}_k(r) e^{ik(z-i\sigma t)} \]

- Traveling wave with speed \( \sigma \) in \( \text{Im}(z) \) direction
- \( \hat{u}_1 \) is linearly unstable eigenmode with eigenvalue \( \sigma \)
- Traveling wave speed \( \sigma \) is thus determined from linear eigenvalue problem and is independent of the amplitude
Motivation for traveling wave form

• Construction of solution is greatly simplified
  - Degrees of freedom reduced
• One way coupling among wavenumbers so mode $k'$ depends only on $k < k'$
  - Computational errors minimized since no truncation or aliasing errors in restriction to finite number of Fourier components
• Equation for $\hat{u}_k$ has form
  $$L_k \hat{u}_k = F_k(\overline{u}, \hat{u}_1, \ldots, \hat{u}_{k-1})$$
• $L_k$ is second order ODE operator
Motivation (cont’d)

• Singularities at $z = z_r + i z_i$ travel with speed $\sigma$ in $\text{Im } z$ direction, reach real $z$ line in finite time (for $z_i \leq 0$)
• Singularities detected through asymptotics of Fourier coefficients $\hat{u}$
(Sulem, Sulem & Frisch 1983)
• Provide information on generic form of singularities
Perturbation construction of real singular solution

- Consider \( \mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}_+ + \mathbf{u}_- + \tilde{\mathbf{u}} \) where \( \mathbf{u}_- = \mathbf{u}_+^*(z^*) \)
- \( \overline{\mathbf{u}}, \overline{\mathbf{u}} + \mathbf{u}_+, \overline{\mathbf{u}} + \mathbf{u}_- \) are exact solutions of Euler equations
- \( \tilde{\mathbf{u}} \) satisfies system of equations in which forcing terms are quadratic, i.e.,
  \[ \mathbf{u}_+ \cdot \nabla \mathbf{u}_- + \mathbf{u}_- \cdot \nabla \mathbf{u}_+ \]
- We want \( \mathbf{u}_+, \mathbf{u}_- = O(\varepsilon) \Rightarrow \tilde{\mathbf{u}} = O(\varepsilon^2) \)
  \( \tilde{u}_{reg} \sim O(T), \tilde{u}_{sing} \sim O(T \varepsilon) + O(\varepsilon^2) \)
- Full construction requires analysis showing that singularity of \( \tilde{\mathbf{u}} \) is same or weaker than that of \( \mathbf{u}_+, \mathbf{u}_- \)
• Similar approach used in studies of singularity formation on vortex sheets
  - Siegel, Caflisch, Howison (2004)
  - Cordoba (2006)

• For vortex sheets, singularity formation is associated with ill-posedness

• For Euler equations, traveling wave solution comes from balance between instability and nonlinearity
• Numerical construction in Caflisch (1993) was for $\overline{u}_z = 0$

• Singularity position depends on $r$, i.e., $-\text{Im } z = \rho(r)$

• Result: $u_{r+} \sim c(z - i\sigma t - i r^2)^\alpha$ where $\alpha = -1/3$

• Amplitude of $u_{r+}$ (i.e., $c$) is $O(1)$
Vortex sheet analogue

- Vortex sheets in Boussinesq approximation
  Siegel (1992), (1995)
  \( \gamma \) - vortex sheet strength
  \( A \) - density difference
- Pure Boussinesq \((A = 1, \gamma = 0) \Rightarrow \) traveling waves of \( O(1) \) amplitude
- Pure vortex sheet \((A = 0, \gamma = 1) \Rightarrow \) no traveling waves due to conservation of vorticity on sheet
- For \( A \ll 1, \gamma = 1 \) small amplitude \( \varepsilon \) traveling waves \( \varepsilon \rightarrow 0 \) as \( A \rightarrow 0 \).
Numerical method

• Pseudospectral in \( z \), 4th order discretization (in \( r \)) for \( L_k \)
• Background velocities

\[
\bar{u}_z = \sin\left(\frac{\theta_\gamma \pi}{2}\right) \bar{u}_{z0}, \quad \bar{u}_\theta = \cos\left(\frac{\theta_\gamma \pi}{2}\right) \bar{u}_{\theta0}
\]

\( 0 < \theta_\gamma < 1 \)
• Numerical method is accurate but unstable
  - Instability controlled using high-precision arithmetic (10^{-100})
• Singularities detected through asymptotics of Fourier components
  (Sulem, Sulem, Frisch 1983)

\[
u_+ \approx c(z - (\mu - i \delta))^{\alpha - 1}
\]

\[
\hat{u}_k \approx c_1 k^{-\alpha} \exp(-k(\delta + i \mu))
\]
Caflisch & Siegel (2004)
Shift in time by $t_0 \equiv \text{mult. of } kth$

Fourier coeff. by $e^{\sigma k t_0} \equiv \text{shift in imag. component of sing. position by } \sigma t_0$

Adjustable parameters: $|\hat{u}_1|, \theta_\gamma$
Amplitude of $u$ and the singularity amplitude $|c|$ vs. the axial flow fraction $\theta_\gamma$.
• Square root singularity does not satisfy Beale, Kato, Majda theorem

Singularity formation at time $T \iff \int_T^\infty \sup_x |\omega(x,t)| \, dt = \infty$
3D traveling wave

- Control numerical instability
- Look for traveling wave solution, periodic in \((x, y, z)\)

\[
\mathbf{u} = \sum_{k>0} \hat{u}_k \exp(ik \cdot (x - i\sigma t))
\]

\(k = (k, l, m), \ \sigma = (\sigma_x, \sigma_y, \sigma_z)\)

- Simplify construction
  - Base flow \(\bar{\mathbf{u}} = 0\)
  - Instability driven by forcing term

\[
\mathbf{F}(\mathbf{x}) = \sum_{k<N} \hat{F}_k \exp(ik \cdot (x - i\sigma t))
\]

- Euler equations

\[
L_k \hat{u}_k = G_k (\hat{u}_{k_1}, \hat{u}_{k_2}, \ldots, \hat{u}_{k_n})
\]

\(k_j < k, \ j = 1, \ldots, n\)
\[
\begin{pmatrix}
\hat{U}_k \\
\hat{V}_k
\end{pmatrix} = \left\{(\sigma \cdot k)(k \cdot k)\right\}^{-1}
\begin{pmatrix}
(l^2 + m^2)\hat{M}_k^{(x)} - lk\hat{M}_k^{(y)} - km\hat{M}_k^{(z)} \\
-lk\hat{M}_k^{(x)} + (m^2 + k^2)\hat{M}_k^{(y)} - lm\hat{M}_k^{(z)}
\end{pmatrix}
\]

\[
\hat{w}_k = \left\{(\sigma \cdot k)k\right\}^{-1}(k\hat{M}_k^{(z)} - m\hat{M}_k^{(x)}) + mk^{-1}\hat{u}_k
\]

where \(\sigma \cdot k \neq 0, \ k \neq 0\)

\[
\hat{M}_k = \hat{F}_k + \hat{N}_k \quad \partial_k (\epsilon - \epsilon u \cdot \nabla u)
\]

- Small amplitude singularity by choice of forcing
- Introduce \(\epsilon\) into forcing; when \(\epsilon = 0\), solution \(u\) is entire.
- For small \(\epsilon\), singularity amplitude is \(O(\epsilon)\)
Numerical method

- Nonlinear terms $\hat{N}_k$ evaluated by pseudospectral method
- No truncation error in restriction to finite $k$
- Since $N$ is quadratic, padding with zeroes eliminates aliasing error from pseudospectral part of calculation
- Extreme numerical instability eliminated; very mild instability controlled by spectral filtering

We compute traveling wave $u_{+++}, u_{+++} + u_{---}$ is real
Fit of singularity parameters \( \sigma = (1, 0, 0), \epsilon = 1 \)

1D fit: \( u = \sum_{k=1}^{\infty} \hat{u}_k(y, z)e^{ikx} \)

\( u_{+++} \sim c \log(x - i|\sigma|t + \rho(y, z)) \)

- BKM satisfied
Fit of singularity parameters $\varepsilon = 0.1$

Graph showing the fit of singularity parameters $\alpha$, $c$, and $\delta$ as functions of $k$. The graph includes a legend indicating the parameters represented by different lines: 'delta', 'c', and 'alpha'.
Fit of singularity parameters $\varepsilon = 0.01$
Singularity amplitude

\[ \max |u_{+++}| \]

\[ |c| \]

\[ \mathcal{E} \]
Singular surface

\[-\text{Im } x = \rho(y, z) \quad \sigma = (1, 0, 0)\]

Geometry of singular surface is useful for analysis.
Contour plot of $100\delta(x,y)$ from data and quadratic fit: $L2(\text{err } \delta) = 1.5258e-005$

$x: \delta = -0.0087567 + 1.5084(x-y)^2 + 0.44261(x+y)^2$
Contour plot of $100\cdot\delta(y,z)$ from data and quadratic fit: $L_2(\text{err}\delta) = 1.3067e^{-005}$

$y: \delta = -0.0079482 + 1.483(x-y)^2 + 0.42761(x+y)^2$
Conclusion

• Introduced new method to compute singular solutions to 3D Euler equations with complex velocity

• Eliminated numerical instability observed in earlier calculations; introduced techniques to achieve small amplitude singularity

• Results suggest a traveling wave singularity to 3D complex Euler equations in which the velocity blows up; satisfies Beale, Kato, Majda theorem, smooth singular surface

• Easily generalized to other problems, e.g., 2D and 3D MHD, quasi-geostrophic equation, etc.