Driven Motion of Interfaces

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Central theme

- Nonequilibrium (driven)
- Macroscopic evolution (pattern formation, instabilities)
- Fluctuations

My talk: 2D droplet growth

Much simpler
unstable fluid

stable fluid

interface

- bulk nucleation is on a longer time scale

2D thin film 1D interface

//shape fluctuations//

- experiment on turbulent liquid crystal, Takeuchi, Sano 2010

- theory universal probability density functions
1. Experiment by Takeuchi and Sano, Tokyo Univ.

- thin film of turbulent liquid crystal
  - $25^\circ C$, voltage 26V at 250 Hz
  - $\Rightarrow$ cell size 16mm x 16mm x 12 $\mu$m

- in-plane isotropic

- two phase coexistence
  - DSM 1 unstable grey
  - DSM 2 stable black

- $\Rightarrow$ point seed of DSM 2
  - “droplet”

- $\Rightarrow$ line seed of DSM 2
  - “flat”
Statistics of shape fluctuations (height) 1100 repeats

$h(t)$ height (radius) along fixed direction

- nonuniversal properties (KPZ theory)

\[ h(t) = v_\infty t + c_2 t^{1/3} \]

random amplitude

1. asymptotic growth velocity $v_\infty$
2. coupling strength (nonlinearity) $\alpha = v_\infty$ isotropic
3. stationary height-height correlations at small distances

\[ \langle (h(x,t) - h(0,t))^2 \rangle \]

NO adjustable parameters
Probability densities are known from random matrix theory.

**Droplet** \( \text{GUE} \ \beta = 2 \)

A is \( N \times N \) Hermitian matrix

\[
\frac{1}{Z_N} e^{-\frac{1}{2N} \text{tr} A^2}
\]

Eigenvalue spacing \( 1 \)

\( \lambda_N \approx 2N + N^{1/3} \chi_2 \)

**Flat** \( \text{GOE} \ \beta = 1 \)

A is \( N \times N \) real symmetric

\[
\lambda_N = 2N + N^{1/3} \chi_1
\]
TW
-- GUE
-- GOE

circular

flat

\[ \rho(\chi) \]

rescaled height \( \chi \)
2. theory of Kardar, Parisi, Zhang 1986

- top part only, height function $h(x, t)$

$$\frac{2}{\partial t} h = \frac{1}{2} \lambda \left( \frac{\partial h}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2}{\partial x^2} h + W$$

nonlinearity $\lambda > 0$

Gaussian white noise $W(x, t)$

$$< W(x, t) W(x', t') > = \delta(x - x') \delta(t - t')$$

- sharp wedge initial conditions

$$h(x, 0) = -\frac{1}{a} \text{1x1}$$

$a \to 0$

shape fluctuations $t^{1/3}$
equivalently

- Noisy Burgers equation

\[ \frac{\partial}{\partial x} \mathcal{H} = u \]

\[ \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \left[ -\lambda u^2 - \frac{1}{2} \frac{\partial^2}{\partial x^2} u - W \right] = 0 \]

- Cole-Hopf, directed polymer

\[ Z(x,t) = e^{\mathcal{H}(x,t)} \]

\[ \frac{\partial}{\partial t} \mathbb{E} \left[ e^{\int_0^t \mathcal{W}(b(s),s)} \delta(b(t) - x) \right] \]

\[ \mathbb{E} \left[ e^{\int_0^t \mathcal{W}(b(s),s)} \delta(b(t) - x) \right] = Z(x,t) \]

Construction of solution

\[ \lambda = 1 \]

\[ Z(x,0) = \delta(x) \]

Sharp wedge

Brownian motion

Random potential

Point-to-point

Random

Elastic string
3. generating function

\[ \langle \exp \left[ -e^{-s} + \left( \mathcal{H}(x,t) + t + \frac{x^2}{2t} \right) \right] \rangle = \det \left( 1 - \mathcal{K}_s \right) \]

white noise

kernel

\[ \mathcal{K}_s(x,y) = \frac{e^{t^{1/3}x - s}}{1 + e^{t^{1/3}x - s}} \mathcal{K}_{Ai}(x,y) \]

Airy kernel

\[ \mathcal{K}_{Ai}(x,y) = \int_0^\infty dw \, \mathcal{A}_i(x+w) \mathcal{A}_i(y+w) \]

\[ t \to \infty \quad \mathcal{H}(x,t) = -t - \frac{x^2}{2t} + t^{1/3} \mathcal{A}_2 \]

replace \( s \to at^{1/3} \)

\[ \mathcal{P}(\mathcal{A}_2 \leq a) = \det \left( 1 - P a \mathcal{K}_{Ai} P a \right) \]

\[ \{ \text{Tracy-Widom, GUE, } \beta = 2 \}

projects onto \([a, \infty)\)
Probability densities

- fixed $t$

$$h(x, t) = t + \frac{x^2}{2t} + t^{1/3} \frac{3}{t}$$

$$\lim_{t \to \infty} \frac{3}{t} = \chi_2$$

$\frac{3}{t}$ has probability density $p_t$

$$p_t(s) = p_{GU} \ast q_t(s)$$

- Gumbel density

$$t^{1/3} e^{t^{1/3}} \times e^{-e^{t^{1/3}}}$$

- $q_t$ is difference of two Fredholm determinants

$\uparrow$

computable by $100 \times 100$ approximation
puzzle of finite time correction

relative to TW

slowest mode: mean with decay \( c_0 t^{-1/3} \)

sign of \( c_0 \)?

KPZ equation \( c_0 \) negative

experiment \( c_0 \) positive

KPZ holds for weak asymmetry

strong asymmetry

single step growth model

\[ h \]

\[ x \]
4. Method/Generalizations

- Approximation through weakly asymmetric single step growth
  Sasamoto, U.S. 2010
  Independently Amir, Corwin, Quastel 2010
  Yields density $p_t(s)$ based on Tracy, Widom 2009

- Replica method, Kardar 1987
  Moments $< Z(x,t)^n > = \text{attractive } \delta \text{-Bose gas on the line}$
  Divergent series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{n^3}$
  Independently Calabrese, Le Doussal, Rosso 2010
  Dotsenko 2010
  Yields generating function
two-point function

sharp wedge

generating function + shift

\[
\langle \exp[-e^{-s_1 + h(x_1, t)}] - e^{-s_2 + h(x_2, t)} \rangle \quad \text{if same time}
\]

stationary, depends only on \( x = x_2 - x_1 \)

\[
= \text{det}(1 - K_{s_1, s_2, x}) \sim L^2(\mathbb{R})
\]

\[
H = -\frac{d^2}{du^2} + u
\]

\[
K_{s_1, s_2, x}(u, v) = \frac{e^{-t^{1/3} u - s_1} + e^{-t^{1/3} v - s_2}}{1 + e^{t^{1/3} u - s_1} + e^{t^{1/3} v - s_2}}(e^{-t^{-2/3} 1 \times 1} H)(u, v) \int_0^\infty \int_0^{2/3} 1 \times 1 W \text{Ai}(u+w) \text{Ai}(v+w)
\]

Airy process, Dyson's Brownian motion
\[ C'_2(u) \equiv C_2(l) / \left( A^2 \lambda t/2 \right)^{2/3} \]

\[ u \equiv (Al/2)(A^2 \lambda t/2)^{-2/3} \]

\[ g_2 \equiv g_{\text{int}}(t) - C_2(t) \]

Slope: -1/3
Same as facet edge fluctuations

experiment T. Einstein, Williams et al. 2006

(1,1,1) facet of Pb on Ru

facet

Ru

top

as droplet

\[ R \approx t \]
5. Summary / Outlook

- droplet growth
- fluid / fluid interface
  *driven* (stable/unstable)
- experiment: turbulent liquid crystal
- exact solution of 1D KPZ equation

future

flat initial conditions \( h(x,0) = 0 \)

exact solution?