Network Constrained Coalitional Dynamic Games and Evolution of Network Topologies

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# Taxonomy of Networked Systems

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Collaborative Robotic Swarms
Autonomous Swarms – Networked Control
Biological Networks

- Systematic approaches to study large numbers of proteins, metabolites, and their modification have revealed complex molecular networks
- Significantly different from random networks and often exhibit ubiquitous properties in terms of their structure and organization
- They are actually dynamic, interacting, weighted hypergraphs. Weights exist at nodes and links. Weights can be numerical, logical, ODEs, rules, etc. (various annotations).
- Analyzing these networks provides novel insights in understanding basic mechanisms controlling normal cellular processes and disease pathologies
- Indispensable component of Systems Biology
Networks and Networked Systems

Internet backbone
(Lumeta Corp.)

Physical

Vehicle, robot networks

Logical

Trust
(J Golbeck - Science, 2008)

Internet: North American cities
(Chris Harrison)
Outline

• Multiple interacting dynamic hypergraphs – three challenges
• Networks and Collaboration
  Constrained Coalitional Games
• Trust and Networks
• Topology Matters
• Conclusions and Future Directions
Multiple Interacting Dynamic Hypergraphs

- **Multiple Interacting Graphs**
  - *Nodes*: agents, individuals, groups, organizations
  - Directed graphs
  - *Links*: ties, relationships
  - Weights on links: value (strength, significance) of tie
  - Weights on nodes: importance of node (agent)
- **Value directed graphs with weighted nodes**
- **Real-life problems**: Dynamic, time varying graphs, relations, weights, policies

Networked System architecture & operation
Three Fundamental Challenges

• Multiple interacting dynamic hypergraphs involved
  – Collaboration hypergraph: who has to collaborate with whom and when.
  – Communication hypergraph: who has to communicate with whom and when.

• Effects of connectivity topologies:
  Find graph topologies with favorable tradeoff between performance improvement (benefit) of collaborative behaviors vs cost of collaboration
  – Small word graphs achieve such tradeoff
  – Two level algorithm to provide efficient communication

• Need for different probability models – the classical Kolmogorov model is not correct
  – Probability models over logics and timed structures
  – Logic of projections in Hilbert spaces – not the Boolean of subsets
Outline

• Multiple interacting dynamic hypergraphs – three challenges
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What is a Network ...?

• In several fields or contexts:
  – social
  – economic
  – communication
  – sensor
  – biological
  – physics and materials
A Network is ...

- A collection of nodes, agents, ... that **collaborate** to accomplish actions, gains, ...
  that cannot be accomplished with out such collaboration

- Most significant concept for **dynamic autonomic networks**
The Fundamental Trade-off

• The nodes **gain** from collaborating
• But collaboration has **costs** (e.g. communications)
• **Trade-off:** gain from collaboration vs cost of collaboration

Vector metrics involved typically

**Constrained Coalitional Games**

• **Example 1:** Network Formation -- Effects on Topology
• **Example 2:** Collaborative robotics, communications
• **Example 3:** Web-based social networks and services
• **Example 4:** Groups of cancer tumor or virus cells
Example: Autonomic Networks

• Autonomic: self-organized, distributed, unattended
  – Sensor networks
  – Mobile ad hoc networks
  – Ubiquitous computing

• Autonomic networks depend on collaboration between their nodes for all their functions
  – The nodes gain from collaboration: e.g. multihop routing
  – Collaboration introduces cost: e.g. energy consumption for packet forwarding
Example: Social Webs

• In August 2007, there were totally 330,000,000 unique visits to social web sites. (Source: Nielsen Online)
  – 9 sites with over 10,000,000 unique visits
  – MySpace, Facebook, Windows Live Spaces, Flickr, Classmates Online, Orkut, Yahoo! Groups, MSN Groups

• Main types of social networking services
  – directories of some categories: e.g. former classmates
  – means to connect with friends: usually with self-description pages
  – recommender systems linked to trust/reputation
Game Theoretic Approach

- The conflict between the benefit from collaboration and the required cost naturally leads to game-theoretic studies.
  - Nodes strategically decide the degree to which they volunteer their resources for the common good of the network.
  - Nodes attempt to maximize an objective function that takes the form of a payoff, which depends on the pattern of collaboration.

- We study collaboration based on the notion of coalitions.
  - In coalitions, users connect to (join) each other, and are able to acquire access to each other.
  - The notion of coalitions can be well captured by coalitional game theory (aka cooperative game theory).
Coalitional Games

• The central concept is that of coalition formation: subsets of users that join their forces and decide to act together.
  – Players form coalitions to obtain the optimal payoffs
  – Players can negotiate collectively
  – The coalitional game model fits better to the practical scenarios, where agents naturally form coalitions, such as soldiers in the same group.

• Coalitional Games in characteristic function form
  – The coalitional game $G = \{N, \nu\}$, where $N = \{1, 2, \ldots, n\}$ is the set of all nodes
  – Characteristic function $\nu : 2^N \rightarrow \mathbb{R}$, on all subsets $S$ (coalitions) of $N$, represents the total payoff of a coalition
Network Model

- The communication structure of the network is represented as an **undirected** graph $G$.
  - Undirected links: the willingness of both nodes is necessary to establish and maintain a link.
  - In wireless networks, reliable transmissions require that two nodes interact to avoid collisions and interference.

- If $i$ and $j$ agree to collaborate with each other, the link $ij \in G$.
  - Add link $ij$ to the existing graph $g$: $G + ij$;
  - Sever link $ij$ from $g$: $G - ij$.

- A **coalition** of $G$ is a subgraph $G' \subseteq G$, where $\forall i \in G'$ and $j \in G'$
  - there is a path in $G'$ connecting $i$ and $j$;
  - $ij \in G$ implies $ij \in G'$.
Gain

- Users gain by joining a coalition.
  - **Wireless networks**
    - The benefit of nodes in wireless networks can be the rate of data flow they receive, which is a function of the received power
      \[
      B_{ij} = f(P_j l(d_{ij}))
      \]
      \(P_j\) is the power to generate the transmission and \(l(d_{ij}) < 1\) is the loss factor
      e.g: \(B_{ij} = \log(1 + (P_j l(d_{ij}) / N_0))\)
  - **Social connection** model (Jackson & Wolinsky 1996)
    \[
    B_{ij} = \sum_{j \in g} V \delta^{r_{ij}} \quad \text{or} \quad w_i(G)
    \]
    - \(r_{ij}\) is # of hops in the shortest path between \(i\) and \(j\)
    - \(0 \leq \delta \leq 1\) is the connection gain depreciation rate
Cost

• Activating links is costly.  \[ c_i(G) = \sum_{j \in N^i_i} C_{ij} \]
  
  – Wireless networks
    • Energy consumption for sending data:  \[ C_{ij} = R S d_{ij}^\alpha \]
      RS depends on transmitter/receiver antenna gains and system loss not related to propagation
      \( \alpha \) : path loss exponent
    • Data loss during transmission
      \( \nu_i \) is the environment noise and \( I_{ij} \) is the interference
      \[ C_{ij} = h(\nu_i, I_{ij}) > 0 \]
  
  – Social connection model
    • The more a node is trusted, the lower the cost to establish link
      e.g. suppose that the trust \( i \) has on \( j \) is \( s_{ij} \) (between 0 and 1),
      we can define the cost as the inverse of the trust values
      \[ C_{ij} = \frac{1}{s_{ij}} \]
Pairwise Game and Convergence

- **Payoff** of node $i$ from the network $G$ is defined as
  \[ v_i(G) = \text{gain} - \text{cost} = w_i(G) - c_i(G) \]

- **Iterated process**
  - Node pair $ij$ is selected with probability $p_{ij}$
  - If link $ij$ is already in the network, the decision is whether to sever it, and otherwise the decision is whether to activate the link
  - The nodes act **myopically**, activating the link if it makes each at least as well off and one strictly better off, and deleting the link if it makes either player better off
  - **End**: if after some time, no additional links are formed or severed
  - **With random mutations**, the game converges to a unique Pareto equilibrium (underlying Markov chain states)
Pairwise Game

- Pairwise game is modeled as an **iterated process**
  - Individual nodes activate and delete links based on the improvement that the resulting network offers them relative to the current network.

- A **strategy** of node $i$ is a vector defined as
  $$\gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,i-1}, \gamma_{i,i+1}, \ldots, \gamma_{i,n}).$$
  - $\gamma_{i,j} = 1$ (or 0): node $i$ wants (or does not) to form a link with node $j$.
  - A link $ij$ is formed only if $\gamma_{i,j} = 1$ and $\gamma_{j,i} = 1$.

- A **strategy profile** $\gamma^{(t)} = (\gamma_1^{(t)}, \ldots, \gamma_n^{(t)})$ at time period $t$ corresponds to the network $G^{(t)}$ at time $t$.

$$\gamma_1 = \{0, 1\}$$
$$\gamma_2 = \{0, 1\}$$
$$\gamma_3 = \{1, 1\}$$
Convergence of the Iterated Pairwise Game

- **Pairwise stability**
  - No more link is added and no existing link is deleted
- **Lemma**: the iterated pairwise game converges to a pairwise stable network or a cycle of networks.
  - The converging pairwise stable network may be inefficient

Random mutations are introduced, the game converges to a unique Pareto equilibrium (Markov chain states strategy profiles $\gamma$). Intent of players is carried out with probability $1 - \epsilon$. 

Cost between neighboring nodes is 1

$V = 0.9, \delta = 0.3$
Stochastic Stability

- Dynamic process is now a finite state, aperiodic, irreducible Markov chain (graph process)—steady-state distribution, $\Pi(g, \varepsilon)$.
- A network $g$ is **stochastically stable** if $\Pi(g, \varepsilon)$ is bounded below as the error rate, $\varepsilon$, tends to zero; $\Pi(g, \varepsilon) \to a > 0$, as $\varepsilon \to 0$.
  - Stochastically stable networks must be pairwise stable networks or networks of closed cycles
  - Stochastic stability identifies the most “robust” or easy to reach networks in a particular sense (the most mutations needed to get “unstuck”).
  - The above example converges to a Pareto efficient pairwise stable network by considering all the possible dynamic paths between the left and right networks.
Network Formation Dynamics

Parameter Values:
\[ \delta = 0.2 \]
\[ \alpha = 2 \]
\[ V = 1 \]
\[ P = 2 \]
Coalition Formation at the Stable State

- The cost depends on the physical locations of nodes
  - Random network where nodes are placed according to a uniform Poisson point process on the [0,1] x [0,1] square.

- **Theorem**: The coalition formation at the stable state for $n \to \infty$
  - Given $\delta = 0$, $V = P\left(\frac{\ln n}{n}\right)^{\frac{\alpha}{2}}$ is a sharp threshold for establishing the grand coalition (number of coalitions = 1).
  - For $0 < \delta \leq 1$, the threshold is less than $P\left(\frac{\ln n}{n}\right)^{\frac{\alpha}{2}}$. 

![Graph](image)
Topologies Formed

(a) $P = 0.5$ (low cost): complete graph

(b) $P = 2$ (middle cost): small world topology

(c) $P = 4$ (high cost): partitioned network
Stability of Coalitions

- **Core stability**
  - A network $G$ is core stable if there is no subset of nodes $S$ who prefer another network $\hat{G}$ to $G$ and who can change the network from $G$ to $\hat{G}$ without the cooperation from the rest of the set of nodes $N \setminus S$.

  $x_i(\hat{G}) \geq x_i(G)$ for all $i \in S$ and there is at least one strict inequality

  If $ij \in \hat{G}$ but $ij \notin G$, then $i, j \in S$

  If $ij \notin \hat{G}$ but $ij \in G$, then $i \in S$ and/or $j \in S$

  - Core stability allows that a node is able to interact and coordinate with any other nodes in the same coalition.

- Core stability is stronger than pairwise stability.
Formation Topology

• Conditions under which the formation game converges to a network with small-world properties

• Network model:
  – All nodes are equally placed on a circle
  – Benefit: \( B_{ij} = \sum_{j \in g} V \delta^{r_{ij} - 1} \)
  – Cost: \( C_{dr} \) (cost of establishing a link between two nodes that are \( r \) hops away)

• Formation process
  – Initial network where nodes only connecting to their immediately neighbors, i.e.,
    \[
    C_{d_1} < B < \frac{C_{d_2}}{1 - \delta^{\lfloor \frac{n}{2} \rfloor - 1}}
    \]
  – Direct connections between nodes that are at least \( r \) hops away on the circle if
    \[
    B > \frac{C_{d_r}}{1 - \delta^{\lfloor \frac{n}{2} \rfloor - r + 1}}
    \]
We investigate the effect of shortcuts following the perturbation approach to small worlds proposed by Higham (Higham, 2003)

- $\varepsilon$ represents the probability that a shortcut is added to the initial network.
- Assume the shortcuts are added to nodes that are at most $r$ hops away on the circle

**Proposition:** Let $r\varepsilon = K / n^\beta$, where $K > 0$ and $\beta \geq 0$. For $\beta > 2$, the effect of shortcuts on convergence rate is negligible. $\beta = 2$ is the threshold. For $\beta < 2$, the shortcuts are dominantly decreasing the SLEM, thus the small-world topology appears.

Given that $B > \frac{C_d \sqrt{\frac{n}{2} - r}}{1 - \delta^{\frac{n-r}{2}}}$, small-world topology appears if more than $K/n$ shortcuts are added.
Time-dependent Game

• The game is time-dependent
  – The payoff players receive varies over time.
  – The dynamics of the game can be separated in rounds of successive coalition expansions (or contractions).

• The dynamic coalition formation process is described as an iterated game
  – $x_i^t$: the action $i$ chooses at time $t$.
  – $v_i(x^t)$: the payoff of user $i$ at time $t$.
  – $q^t(x)$: players’ probability of playing action $x$ at time $t$.
  – $C_i^t$: the set of users that form the coalition user $i$ belongs to at time $t$.
  – user $i$ and user $j$ decide to activate link $ij$ at time $t$:

$$C_i^t = C_j^t = C_{i}^{t-1} \cup C_{j}^{t-1}$$
Value Function

• Value function for coalition $C$ (component-wise additive value function)

\[ v(C) = \sum_{i \in C} v_i(g) \]

• Value function depends on topology

Same coalition $C=\{1,2,3\}$ with different topology

$v(\{12,13\}) \neq v(\{12,23,13\})$
• **Nash equilibrium**
  – Player action probability $q$ is a Nash equilibrium if no player $i$ can deviate from $q$ and achieve a higher payoff

\[
\forall i, \sum_{x_{-i} \in X_{-i}} v_i(x'_i, x_{-i}) \prod_{j \neq i} q_j(x_j) \leq \sum_{x \in X} v_i(x) \prod_{j} q_j(x_j).
\]

• **Core stability**
  – A network $g$ is **core stable** if there is no subset of nodes $S$ who prefer another network $g'$ to $g$ and who can change the network from $g$ to **without the cooperation** from the rest of the set of nodes $M \setminus S$. 

Stability of Game Dynamics
Learning via Regret

• **Question:**
  – Are there simple strategies that lead our coalition formation game to equilibrium?

• **Solution:**
  – Players stochastically adjust their strategies by a reinforcement learning rule guided by “regret”.

• **Learning strategy:**
  – At each period, a player may either choose to continue playing the same action as in the previous period, or switch to other actions with probabilities that are proportional to how much higher his accumulated payoff would have been if he had always made that change in the past.
Regret Matching Strategy

• Average payoff through time $t$ for user $i$

$$\bar{v}_i^t = \frac{1}{t} \sum_{1 \leq s \leq t} v_i(x_i^s, x_{-i}^s).$$

• Average regret from not having played $x_i'$

$$\bar{r}_{i,x_i'}^t = \frac{1}{t} \sum_{1 \leq s \leq t} v_i(x_i', x_{-i}^s) - \bar{v}_i^t.$$

• Regret matching strategy
  - at each time period $t + 1$, the player $i$ plays either action activate or not activate with a probability proportional to the nonnegative part of his regret up to time $t$

$$q_i^{t+1}(x_i) = \frac{(\bar{r}_{i,x_i}^t)^+}{\sum_{x_i'} (\bar{r}_{i,x_i'}^t)^+}.$$
Results on Stability

• Nash equilibrium
  – \( \phi^t(x) \): the proportion of time up to \( t \) that each action-tuple \( x \) was played.
  – \( \phi \): the empirical distribution.
  – Results: Given that all players use the regret matching strategy, the empirical distribution converges almost surely to the set of Nash equilibria.

• Core existence
  – If \( \forall i, j, v_{ij} + v_{ji} \geq 0 \), the core of the formation game is nonempty
Simple Case Study

- **Gain**: users benefit from connecting to as many other users as possible, directly (one-hop) or indirectly (multi-hop, through other users)
  \[ b_i(g) = |C^t(i) - 1| \]
- **Cost**: random variable with an exponential probability distribution with parameter \( \lambda \).
- **Result 1**: All coalitions formed at the Nash equilibrium are trees.
- **Result 2**: The probability that the game has nonempty core is greater than or equal to
  \[ (1 - e^{-4\lambda})^{N(N-1)}. \]
Coalitions, Networks and Constraints

• **Cooperative Game** in characteristic function form \( \Gamma = \{\mathcal{N}, v\} \), \( \mathcal{N} = \{1, 2, \ldots, N\} \), \( v : 2^N \rightarrow \mathbb{R} \), on all subsets \( S \) (coalitions) of \( \mathcal{N} \)

• All coalitions cannot be formed

• To collaborate agents need to communicate

• Communication Network \((N, L)\)
  – Edges – links between payers
  – \( i \) and \( j \) directly connected
  – \( i \) and \( j \) path connected

• Cooperation components

• Links between players in \( S \) , \( L(S) \)

• Network \((S, L(S))\) induces a partition of \( S \)
Constrained Coalitional Games

- Network-restricted cooperation game or constrained coalition \( \{\mathcal{N}, v^L\} \)
- \( \{\mathcal{N}, v, L\} \) communication situation

- Characteristic function

\[
v^L(S) = \sum_{C \in S / L} v(C) \quad \text{for each} \quad S \subseteq \mathcal{N}
\]

- Myerson value: Shapley value of \( \{\mathcal{N}, v^L\} \)
Network Formation

• Form links pairwise
• Iterative game
• Better understanding of topologies – dynamics – topology control
• **Network formation with costs** for establishing links
• \( \{N, v, L, c\} \) \( \{N, v^{L,c}\} \)

\[
v^{L,c}(S) = \sum_{C \in S/L} v(C) - c |L(S)| \quad \text{for each} \quad S \subseteq N
\]

• **Stability** vs **efficiency** of the resulting network
• Small world graphs, expander graphs …
Outline

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  Constrained Coalitional Games
  • Trust and Networks
• Topology Matters
• Conclusions and Future Directions
Networks and Trust

• Trust and reputation critical for collaboration
• Characteristics of trust relations:
  – Integrative (Parsons1937) – main source of social order
  – Reduction of complexity – without it bureaucracy and transaction complexity increases (Luhmann 1988)
  – Trust as a lubricant for cooperation (Arrow 1974) – rational choice theory
• Social Webs, Economic Webs
  – MySpace, Facebook, Windows Live Spaces, Flickr, Classmates Online, Orkut, Yahoo! Groups, MSN Groups
  – e-commerce, e-XYZ, services and service composition
  – Reputation and recommender systems
Ising and Spin Glass Models

- **Statistical Physics models for magnetization**
  - Orientation of each particle’s spin depends on its neighbors
  - *Ising Model*: behavior of simple magnets
  - *Spin Glass Model*: complex materials

- **Interpretation:**
  - $s = \{s_1, s_2, ..., s_n\}$ is a configuration of $n$ particle spins -- $s_j = 1$ or -1 (up or down)

  Energy for configuration $s$

  $$H(s) = -\frac{1}{T} \sum_{i \in V} \sum_{j \in N_i} J_{ij} s_i s_j - \frac{mH}{T} \sum_i s_i$$

  - **Ising Model**: $J_{ij} = J$ for all $i, j$
  - **Spin Glass Model**: $J_{ij}$ depend on $i, j$ and can be random
Ising/SG Models and Games

- Ising/SG models can be interpreted as dynamic (repeated) games:
  - The value of $s_i$ represents whether node $i$ is willing to cooperate or not
  - each particle selects spin to maximize its own payoff
    $$\pi_i = (\sum_{j \in N_i} J_{ij} s_i s_j) / T$$
    - Ising model ($J_{ij} = J > 0$): align its spin with the majority of neighbors spin
      - High $T$, conservative agents, not willing to change, small payoffs
      - Low $T$, aggressive agents, larger payoffs
    - Collection of local decisions reduces the total energy of the interacting particles

- Inspires an approach where trust is an incentive for cooperation
  - $J_{ij}$ can be interpreted as the worth of player $j$ to player $i$
  - decide to cooperate or not based on benefit from cooperation and trust values of neighbors
Spin Glass Cooperative Game

• Spin glass model as a cooperative game (spin glass game)
  – \( S \subseteq N = \{1, 2, \ldots, n\} \) is a coalition, in which all nodes cooperate
  – Interaction topology (\( J_{ij} \)’s) moderates effects pos. and neg. feedback
  – \( v(S) \): value of the characteristic function of the game, \( v: 2^N \rightarrow \mathbb{R} \), which is the maximum payoff \( S \) can get without cooperation from other nodes \( N / S \).

\[
v(S) = \sum_{i \in S} \pi_i = \sum_{i,j \in S} J_{ij} - \sum_{i \in S, j \notin S} J_{ij}
\]

– The cooperative game is denoted as \( \Gamma = (N, v) \)

• **Object**: to find what form or policy for \( J_{ij} \) can induce all (or most) nodes to cooperate: maximize the coalition
Spin Glass Cooperative Game Properties

- Spin Glass game is a convex and superadditive game iff (net pos. effects)
  \[ \forall i, j, J_{ij} + J_{ji} \geq 0 \]

- **Shapley value**: \[ \varphi(v)_i = \sum_{j \in \mathcal{N}} J_{ij} \] in the core

- Not well understood in the regime of both negative and positive net effects

- Effects of interaction matrix structure (sparsity, neighborhood structure, range of interactions, strength of interactions) not well understood; Topology effects in network analog

- Oriented Spin Glass Game \( \Gamma(\mathcal{N}, \nu) \) where \( \nu \) now depends on both an interaction matrix \( J \) and a preference vector \( L \); a pair of char. fcns

  \[
  \nu_{\pm}(S) = \sum_{i,j \in S} J_{ij} - \sum_{i \in S, j \notin S} J_{ij} \pm \sum_{i \in S} L_i
  \]

- **Replica method** can be used to analyze various problems under various models and constraints on \( J \) and \( L \)
Cooperative Games with Negotiation

- **Theorem**: \( \Gamma = (\mathcal{N}, v) \) has a nonempty core if \( J_{ij} + J_{ji} \geq 0, \forall i, j \). The payoff allocation to node \( i \), \( x_i = \sum_{j \in N_i} x_{ij} \), where \( x_{ij} \) is computed as follows:

\[
x_{ij} = \begin{cases} 
J_{ij}, & \text{if } J_{ij} \geq 0, J_{ji} \geq 0 \\
J_{ij} + \lambda_{ij}J_{ji}, & \text{if } J_{ij} \leq 0, J_{ji} > 0 \\
(1 - \lambda_{ij})J_{ij}, & \text{if } J_{ij} > 0, J_{ji} \leq 0
\end{cases}
\]

with \( 0 \leq \lambda_{ij}, \lambda_{ji} \leq 1 \) is a solution in the core.

- This payoff allocation indicates a way to encourage cooperation.
- **Players with positive gain can negotiate with their neighbors by sacrificing certain gain** (offering their partial gain \( \lambda_{ij}x_{ij} \) )
Trust as Mechanism to Induce Collaboration

- **Trust is an incentive** for collaboration
  - Nodes who refrain from cooperation get lower trust values
  - Eventually penalized because other nodes tend to only cooperate with highly trusted ones.

- For node $i$ **loss for not cooperating** with node $j$ is a nondecreasing function of $J_{ji}$, $f(J_{ji})$,

- New characteristic function is

$$
\nu(S) = \sum_{i,j \in S} J_{ij} - \sum_{i \in S, j \notin S} f(J_{ij})
$$

- **Theorem**: if $\forall i, j, J_{ij} + f(J_{ji}) \geq 0$, the core is nonempty and $x_i = \sum_{j \in N_i} J_{ij}$ is a feasible payoff allocation in the core.

By introducing a trust mechanism, all nodes are induced to collaborate without any negotiation.
Dynamic Coalition Formation

Two linked dynamics
- Trust propagation and Game evolution

\[ \gamma_i(t + 1) = f_i(x_i(t), \gamma_i(t), \gamma_j(t), t_{ij}(t)) \]
\[ t_{ik}(t) = g_i(t_{ij}(t), v_{jk}(t)) \quad \forall k \in N \]
\[ x_i(t) = h_i(\gamma_i(t), \gamma_j(t)) \]
\[ v_{ij}(t) = p_i(\gamma_j(t), t_{ji}(t)) \]

An example of constrained coalitional games

Stability of dynamic coalition Nash equilibrium
Game Evolution

- **Strategy** of node $i$: $s_{ij} \in \{-1, 1\}, \forall j \in N_i$
  - $s_{ij} = 1 (= -1)$ represents that $i$ cooperates (does not cooperate) with its neighbor $j$

- **Payoff** for node $i$ when interacting with $j$: $x_{ij} = J_{ij} s_{ij} s_{ji}$
  - $x_{ij} > 0 (< 0)$ positive link (negative link)
  - **Node selfishness** → cooperate with neighbors on positive links

- **Strategy updates**: node $i$ chooses $s_{ij}=1$ only if all of the following are satisfied:
  - Neighbor $j$ is trusted
  - $x_{ij} > 0$, or the cumulative payoff of $i$ is less than the case when it unconditionally conducts $s_{ij}=1$.

- **Trust evaluation**:
  - The deterministic voting rule
  - **Reestablishing period** $\tau$: once a node is not trusted, in order to reestablish trust it has to cooperate for $\tau$ consecutive time steps
Results of Game Evolution

- **Theorem**: \( \forall i \in N_i \) and \( x_i = \sum_{j \in N_i} J_{ij} \), there exists \( \tau_0 \), such that for a reestablishing period \( \tau > \tau_0 \)
  - Iterated game converges to Nash equilibrium;
  - In the Nash equilibrium, all nodes cooperate with all their neighbors.

- **Compare games with (without) trust mechanism, strategy update:**

![Graph 1: Percentage of cooperating pairs vs negative links](image1)
![Graph 2: Average payoffs vs negative links](image2)
Next Generation Trust Analytics

• Trust evaluation, trust and mistrust dynamics
  – Spin glasses (from statistical physics), phase transitions

\[ s_i(k+1) = f \left( \hat{J}_{ji}, s_j(k) \mid j \in N_i \right) \]

• **Indirect** trust; reputations, profiles; Trust computation via ‘linear’ iterations in ordered semirings

\[ a \otimes b \leq a, b \]

\[ a \oplus b \geq a, b \]

2007 IEEE Leonard Abraham prize

*New Book* “Path Problems in Networks” 2010

• **Direct trust**: Iterated pairwise games on graphs with players of many types
Constrained Coalitional Games: Trust and Collaboration

Two linked dynamics
- Trust / Reputation propagation and Game evolution

- Integrating network utility maximization (NUM) with constraint based reasoning and coalitional games
- Beyond linear algebra and weights, semirings of constraints, constraint programming, soft constraints semirings, policies, agents
- Learning on graphs and network dynamic games: behavior, adversaries
- Adversarial models, attacks, constrained shortest paths, …
Outline

• Multiple interacting dynamic hypergraphs – three challenges
• Networks and Collaboration
  Constrained Coalitional Games
• Trust and Networks
• Topology Matters
• Conclusions and Future Directions
Networks: Different Linked Views

Networks:

– as distributed, asynchronous, feedback (many loops), hybrid automata (dynamical systems)
– as distributed asynchronous active databases and knowledge bases
– as distributed asynchronous computers
Distributed algorithms are essential
  – Group of agents with certain abilities
  – Agents communicate with neighbors, share/process information
  – Agents perform local actions
  – Emergence of global behaviors

Effectiveness of distributed algorithms
  – The speed of convergence
  – Robustness to agent/connection failures
  – Energy/ communication efficiency

Group topology affects group performance

Design problem:
Find graph topologies with favorable tradeoff between performance improvement (benefit) vs cost of collaboration

Example: Small Word graphs in consensus problems
The Importance of Being Well-Connected

- **Local majority voting** (Peleg ’96)
  - Each of \( n \) citizens has an opinion about voting **Yes** or **No**
  - **Rule**: Each citizen’s vote is based on the **majority** of its neighbors, including itself
  - **What is the minimum number of No-voters that can guarantee a No result?**
  - A few number of well connected nodes can determine the outcome of the process!
The Importance of Being Well-Connected (cont.)

White circles: **NO** voters

Black circles: **YES** voters

Order of voting matters!

**Iterative polling**: Oscillation or

If **NO** voters do not follow the protocol, then 2 **NO** voters, are sufficient to change the other n-2 **YES** voters’ opinion.

Even if **NO** voters follow the protocol a **small minority** of $2\sqrt{n}$ can result in one step convergence to **NO**
Consensus problems

- A Simple model:

\[
\theta_i(t + 1) = f_{ii}(t)\theta_i(t) + \sum_{j \in N(i)} f_{ij}(t)\theta_j(t)
\]

\[
\forall i \in \{1, \ldots, n\} : \sum_j f_{ij} = 1
\]

\[
\forall i, j \in \{1, 2, \ldots, n\} : f_{ij} \geq 0
\]

\[
\forall i \in \{1, \ldots, n\} : f_{ii} \geq \alpha > 0
\]
Vicsek’s model
(Vicsek et al., Jadbabaie et al.)

- A flock of $n$ agents moving at the same speed $s$, but with different headings
- Each agent updates its heading angle as an average of its neighbors including itself
- $D$ is diagonal matrix of nodes’ degrees
- $A$ is adjacency matrix

\[
\theta_i(t+1) = \langle \theta_i(t) \rangle = \frac{1}{1 + n_i(t)} [\theta_i(t) + \sum_{j \in N_i(t)} \theta_j(t)]
\]

\[
\theta(t+1) = F_{\sigma(t)} \theta(t)
\]

\[
F_p = (I + D_p)^{-1}(A_p + I)
\]

$G = (V, E)$

$\mathcal{G} = \{G_0, \ldots, G_{M-1}\}$

$\mathcal{M} = \{0, \ldots, M - 1\}$

$\sigma : \mathbb{N} \cup \{0\} \to \mathcal{M}$
Design of information flow

\[ x(k + 1) = F(k)x(k) \]

\[ F(k) = (I + D(k))^{-1}(A(k) + I) \]

\[ F(k) = I - hL(k) \]

Symmetric communication

- **Fixed graphs**: Geometric convergence with rate equal to Second Largest Eigenvalue Modulus (SLEM)
- How does graph topology affect location of eigenvalues?
- How can we design graph topologies which result in good convergence speed?
Small World Graphs

Simple Lattice $C(n,k)$

Small world: Slight variation adding $nk \Phi$

Adding a **small portion** of well-chosen links $\rightarrow$ **significant increase** in convergence rate
Mean Field Explanation and Perturbation Approach

Initial graph

Adjacency/ $F$ matrix

Final graph

Perturbed
• Random graph approach
  (e.g. Durrett 2007, Tahbaz and Jadababaie 2007)

• **Perturbation** approach (Higham 2003)
  – Start from lattice structure $G_0 = C(n,k) \iff F_0$
  – Perturb zero elements in the positive direction by $\varepsilon = \frac{K}{n^\alpha}$
    for fixed $K > 0$ and $\alpha > 1$.
  – Perturb the formerly nonzero elements equally, such that the stochastic structure of the $F$ matrix is preserved $F_\varepsilon$
  – Analyze the SLEM as a function of the perturbation as $\alpha$ varies
• Refer to the perturbations as $\varepsilon$-shortcuts
• In the limit of large $n$:
  – For $\alpha > 3$ the effect of $\varepsilon$-shortcuts on convergence rate is negligible
  – For $\alpha = 3$ the effect of $\varepsilon$-shortcuts on convergence rate starts (spectral gap gain perturbation of same order)
  – For $\alpha = 2$ the shortcuts dominantly decrease SLEM
  – For $\alpha = 1$ SLEM is very small
• $\varepsilon$-shortcuts are loosely analogous to the shortcuts in Small World networks
• $a = 3$ can be considered as the onset of small world effect with small world effect happening at $\alpha = 2$
Analysis of W-S model

• A graph is small-worldizable if \( \frac{SLEM(F)}{1 - SLEM(F)} \gg \frac{1}{n\varepsilon} \).

• For the ring type structure, in the limit of large \( n \):
  – For \( \alpha > 3 \) the effect of \( \varepsilon \)-shortcuts is negligible
  – For \( \alpha = 3 \) the effect of \( \varepsilon \)-shortcuts starts (spectral gap gain perturbation of same order)
  – For \( \alpha = 2 \) the shortcuts dominantly decrease SLEM

• \( \alpha = 3 \) onset of small world effect; small world effect happening at \( \alpha = 2 \).

<table>
<thead>
<tr>
<th>( 1 - \mu(F_0) )</th>
<th>Onset of SW</th>
<th>SW dominant</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O(n^{-1}) )</td>
<td>( \varepsilon = O(n^{-2}) )</td>
<td>( \varepsilon = O(\frac{1}{n \log n}) )</td>
</tr>
<tr>
<td>( O((\log n)^{-1}) )</td>
<td>( \varepsilon = O(\frac{1}{n \log n}) )</td>
<td>( \varepsilon = O(\frac{1}{n \log \log n}) )</td>
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</tbody>
</table>
Distributed exploration of the graph structure

- Self-organization for better performance and resiliency
- Hierarchical scheme to design a network structure capable of running distributed algorithms with high convergence speed
- A two stage algorithm:
  1- Find the most effective choice of local leaders
  2- Provide nodes with information about their location with respect to other nodes and leaders and the choice of groups to form
- Divide $N$ agents into $K$ groups with $M$ members each

\[ N = K \times M, \quad K \leq M \times N \]

select ‘leaders’
Distributed self-organization

Determine leaders → Find influence → Remove unnecessary links

Semi-decentralized → Decentralized: Dirichlet problem on the graph

**Goal:** design a scheme that gives each node a vector of compact global information
Social degrees and leaders

- **Social degree** of order 2: $SD^{(2)}(v) =$ number of neighbors of node $v$
- Social degree of order $k > 2$: $SD^{(k)}(v) =$ number of cycles of length $k$ passing through node $v$
- Social degrees of order 2 and 3 can be determined by a simple query
- A node is called a leader of order $k$ if its social degree of order $k$ is greater than that of its neighbors
Influence vector as a metric for well-connectedness

- $K$ local leaders, $N - K$ nodes regular nodes
- Regular nodes need to determine how well they are located with respect to local leaders and how they are influenced by them
- Distance to leaders does not include information on how “well-connected” a regular node is to leaders
- Consider a random walk on the graph starting from regular node $i$, with leader nodes as absorbing states, the influence of leader $k$ on regular node $i$, is the probability that the random walk hits $k$ before other leaders
Two stage semi-decentralized algorithm

• **Stage 1: Determining $K$ leaders**
  – Each node determines its social degree via local query
  – Dominant nodes in each neighborhood send their degrees to the central authority
  – Central authority computes their social scores

\[
SC(k) = \alpha SD^{(2)}(k) + (1 - \alpha) SD^{(3)}(k)
\]

Choice of $\alpha$ determines whether leaders in star-like neighborhoods are preferred

– The central authority selects the $K$ nodes with highest scores as social leaders and gives them an arbitrary order
• **Stage 2: Determining the influence vectors**
  - Based on its order each leader takes its influence vector to be the fixed vector $e_i$
  - Regular nodes update their influence vector entries:
    $$x_i^k(t + 1) = \frac{1}{n_i + 1} \left[ x_i^k(t) + \sum_{j \in N_i(t)} x_j^k(t) \right]$$

• For connected graphs, for $t$ large enough, $x_i^k(t)$ converges to the influence of leader $k$ on node $i$

• Upon calculation of influence vectors, each regular node determines its local leader and stops its communication with neighbors who have other leaders

• **Graph decomposes into two level hierarchy with efficient communication pattern**
Reliability and Spanning Trees

- End to end applications
- Spanning tree as a minimally connected graph
- \( T(G) \) as a measure of robustness to losses

**References:** Kelmans, Colbourn

**Graph** \( \mathcal{G}(\mathcal{V}, \mathcal{E}) \)

\( \mathcal{V} = \{1, 2, \ldots, n\} \)

\( \mathcal{E} = \{l_1, l_2, \ldots, l_e\} \)

\( p \): Constant link loss probability

\( N_i \): # of connected components with \( i \) edges

\( \tau(\mathcal{G}) \): Number of spanning trees

\[
\text{Rel}(\mathcal{G}, p) = \sum_{i=n-1}^{e} N_i (1 - p)^i p^{e-i}
\]

For sufficiently large \( p \):

\[
\tau(\mathcal{G})(1 - p)^{n-1} p^{e-n+1} \leq \text{Rel}(\mathcal{G}, p) \leq \tau(\mathcal{G})(1 - p)^{n-1}
\]
Graph Theory for Robust Network Design

- **Goal:** Given a base topology add \( k \) edges from a set of \( m \) candidates such that results in maximum number of spanning trees.

- **Number of spanning trees**
  \[
  \tau(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i(L) = \frac{1}{n} \det(L + \frac{11^T}{n})
  \]

- **Incidence vector of an edge** shows between which nodes it is

\[
\begin{align*}
  f_i : \text{incidence vector} \\
  f_i &= e_\alpha - e_\beta \\
  f_{(1,5)} &= e_1 - e_5 = \\
  [1 \ 0 \ 0 \ 0 \ -1]^T
\end{align*}
\]

- **Graph Laplacian**
  
  \[
  L = D - A = F_{nm} F_{mn}^T = \sum_{i=1}^{m} f_i f_i^T
  \]
Counting Spanning Trees

- **Normalized Laplacian** \( \mathcal{L} = D^{-1/2} LD^{1/2} \)
  - If \( G=(V_n,E) \) is connected: \( \lambda_n = 0 < \lambda_{n-1} \leq \lambda_{n-2} \leq \ldots \leq \lambda_1 < 2 \)
- **Random walk matrix** \( P = D^{-1} A = I - D^{-1} L \)
  \[ \lambda_i(P) = 1 - \lambda_{n+1-i}(\mathcal{L}) \]
- **Matrix-Tree theorem (Kirchhoff)**

\[
\text{Adj}(L) = \tau(\mathcal{G})L \\
\tau(\mathcal{G}) = \frac{1}{n} \prod_{j=2}^{n} \lambda_j(L) = \frac{1}{n} \det(L + \frac{1}{n} J) = \\
\prod_{j=2}^{n} \lambda_j(\mathcal{L}) \frac{\prod_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_i} = \det(Q_k) \prod_{i \neq k} d_i, \quad k = 1,\ldots,n.
\]

\( J = \frac{1}{n} 1_n^T \)

\( Q_i \) is the \( i \)th principal submatrix of \( I - P \)
Problem Statement

• **Goal**: Given a base topology add $k$ edges from a set of $m$ candidates such that results in maximum number of spanning trees

\[ \tau(\mathcal{G}) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i(L) = \frac{1}{n} \det(L + \frac{11^T}{n}) \]

• Dynamic graph process resulting from adding edges

Maximize $\tau(G(t + k))$

Subject to:

\[
\begin{align*}
G(t + 1) &= \text{Add}(G(t), u(t)), & t = 0, 1, \ldots, k - 1 \\
u(t) &= e(t + 1), & e(t + 1) \in S \subseteq E(\overline{G}(t)) \\
G(t) &= \mathcal{G}_0
\end{align*}
\]
Formulation and Relaxation

- **Goal**: Given a base topology add $k$ edges from a set of $m$ candidates such that results in maximum number of spanning trees (Approach similar to Ghosh and Boyd 06)

Maximize \( \tau \left( L_0 + \sum_{i=1}^{m} x_i f_i f_i^T \right) \) or equivalently

\[
\log \det \left( L_0 + \frac{1}{n} J + \sum_{i=1}^{m} x_i f_i f_i^T \right)
\]

Subject to:

- \( 1^T x = k \)
- \( x \in \{0,1\}^m \)

- Relax to \( x \geq 0 \)

\( x_i^* > 0 \Rightarrow \frac{\partial \tau(x^*)}{\partial x_i} \geq \frac{\partial \tau(x^*)}{\partial x_j}, \forall j \)

- At maximum \( \tau(x) \) has equal derivatives for positive \( x_i \)s
Robust Network Design

• Derivative:

\[ f_i^T \left( L_0 + \frac{1}{n} J + \sum_{i=1}^{m} x_i f_i f_i^T \right)^{-1} f_i = \lambda, \ \forall i \in \text{Chosen edge set} \ (x_i > 0) \]

\[ R_{\text{eff}} (i) = f_i^T \left( L + \frac{1}{n} J \right)^{-1} f_i \]

• If feasible, add edges such that the effective resistance distance of all selected edges become equal and greater than the effective resistance distance between non-selected candidates

\[ R_{\text{eff}} (\alpha, \beta) = V_{\alpha\beta} \]
• Adding 1 edge to a general graph
  – In a given graph which shortcut will result in more spanning trees?
  – If the edge is between nodes $\alpha$ and $\beta$:
    \[
    \tau(G(1)) = \left(1 + R_{\text{eff}}(\alpha, \beta)\right)\tau(G_0)
    \]
    – Select the edge corresponding to the maximal resistance distance
    – Example: Adding a shortcut to a ring
Special Cases (cont’d)

• Adding 2 edges \((\alpha, \beta)\) and \((\gamma, \delta)\)

\[
\tau(G(2)) = \left[ (1 + R_{eff}(\alpha, \beta))(1 + R_{eff}(\gamma, \delta)) - \right. \\
\left. - \left( (z_{\gamma\alpha} - z_{\gamma\beta}) - (z_{\delta\alpha} - z_{\delta\beta}) \right)^2 \right] \tau(G_0),
\]

\[
Z = [z_{ij}] = \left( L + \frac{1}{n} J \right)^{-1}
\]

Maximized by adding edge between high resistance distance nodes

Maximized by adding edge to symmetrize the graph

• Adding 3 or more edges similar: more complex terms due to compromising between symmetrizing the graph and joining nodes with the highest resistance distance
Small World Phenomenon

- Small world phenomenon as the trade-off in adding $k$ shortcuts to a base graph such that the number of spanning trees is maximized
- Perturbation based method to model Watts-Strogatz small world networks (based on Higham 03, Baras-Hovareshti 08)
  - Performance measure: $\tau(\mathcal{G})$

Capture performance measure of $G_0$, as property of $L(G_0)$

Perturb zero elements of $L_0$ by $\epsilon = Kn^{-\alpha}$, $\rightarrow L_\epsilon$

Do spectral analysis of $G_0, L_0$

Analyze $\frac{\tau(\mathcal{G}_\epsilon)}{\tau(\mathcal{G})}$ for large $n$ as a function of $\epsilon$, as $\epsilon$ varies

Interpret the result as structural perturbation
Small World (cont’d)

- Consider the ratio of the increase in the number of spanning trees as the result of adding $\varepsilon$ weights: $\varepsilon = Kn^{-\alpha}$.

$$r = \frac{\tau(G_\varepsilon)}{\tau(G_0)}$$

- Starting from a ring structure, in the limit of large $n$:
  - For $\alpha > 3$ the effect of $\varepsilon$-shortcuts is negligible
  - For $\alpha = 3$ the effect of $\varepsilon$-shortcuts starts (spectral gap gain perturbation of same order)
  - For $1 < \alpha \leq 3$ the shortcuts dominantly increase the number of spanning trees, i.e. $\lim_{n \to \infty} r = \infty$
Expander Graphs

- Fast synchronization of a network of oscillators
- Network where any node is “nearby” any other
- Fast ‘diffusion’ of information in a network
- Fast convergence of consensus
- Decide connectivity with smallest memory
- Random walks converge rapidly
- Easy to construct, even in a distributed way (ZigZag graph product)

Graph $G$, **Cheeger constant $h(G)$**
- All partitions of $G$ to $S$ and $S^c$,
  
  $h(G) = \min \left( \frac{\text{#edges connecting } S \text{ and } S^c}{\text{#nodes in smallest of } S \text{ and } S^c} \right)$

- $(k, N, \varepsilon)$ **expander** : $h(G) > \varepsilon$; sparse but locally well connected $(1 - SLEM(G) \text{ increases as } h(G)^2)$
Expander Graphs – Ramanujan Graphs
Constructing Expander Graphs

• Possible methods:
  – Form a random expander as a $2d$-regular multi-graph in which the set of edges consists of $d$ separate Hamiltonian cycles on APs (Law and Siu 2003)
  – Form a union of two spanning trees chosen independently from the uniform distribution over all spanning trees of a complete graph, implementable by a random walk method (Goyal et al. 2009)
Outline

• Multiple interacting dynamic hypergraphs – three challenges
• Networks and Collaboration
  Constrained Coalitional Games
• Trust and Networks
• Topology Matters
  • Conclusions and Future Directions
Conclusions

• Fundamental tradeoff between the benefit from collaboration and the required cost for collaboration
• Game theoretic studies for such conflict
• Two-phase coalitional games
• The convergence of the iterated pairwise games
• Phase transition of the coalition formation
• Stability of the formed coalitions
• Trust as a catalyst for collaborations
• Effects of topology on distributed algorithm performance
• Performance vs. efficiency – small world graphs – expander graphs
How Biology Does IT?
Control vs Communication

• Many graphs as abstractions

• **Collaboration graph** – or a model of what the system does (**behavior**)

• **Communication graph** – or a model of what the system consist of (**structure**)

• Nodes with **attributes** – several graphs

• **Key question 1**: Given behavior, what structure (subject to constraints) gives best performance?

• **Key question 2**: Given structure (and constraints) how well behavior can be executed?
Lessons Learned --
Future Directions

• Constrained coalitional games – unifying concept
• Generalized networks, flows - potentials, duality and network optimization (monotropic optimization)
• Time varying graphs – mixing – statistical physics
• Understand autonomy – better to have self-organized topology capable of supporting (scalable, fast) a rich set of distributed algorithms (small world graphs, expander graphs) than optimized topology
• Given a set of distributed computations is there a small set of simple rules that when given to the nodes they can self-generate such topologies?
Thank you!

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Questions?