Well-Balanced Positivity Preserving Central-Upwind Scheme on Triangular Grids for the Saint-Venant System

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Modeling and Computations of Shallow-Water Coastal Flows
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Motivation

• Saint-Venant System of shallow water equations describes the fluid flow as a conservation law with an additional source term

• The general characteristic of shallow water flows is that vertical scales of motion are much smaller than the horizontal scales

• The shallow water equations are derived from the incompressible Navier-Stokes
Motivation

- This Saint-Venant System is widely used in many scientific and engineering applications related to
- Modeling of water flows in rivers, lakes and coastal areas
- The Development of robust and accurate numerical methods for Shallow Water Equations is an important and challenging problem
2-D Saint-Venant system of shallow water equations

\[
\begin{aligned}
&h_t + (hu)_x + (hv)_y = 0, \\
&(hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x + (huv)_y = -ghB_x, \\
&(hv)_t + (huv)_x + \left( hv^2 + \frac{1}{2}gh^2 \right)_y = -ghB_y,
\end{aligned}
\]  

- the function \( B(x, y) \) represents the bottom elevation
- \( h \) is the fluid depth above the bottom
- \((u, v)^T\) is the velocity vector
- \( g \) is the gravitational constant

One of the difficulties encountered:

- that system (1) admits nonsmooth solutions: shocks, rarefaction waves,
- the bottom topography function \( B \) can be discontinuous.
A good numerical method for Saint-Venant System should have at least two major properties, which are crucial for its stability:

(i) The method should be well-balanced, that is, it should exactly preserve the stationary steady-state solutions \( h + B \equiv \text{const}, \ u \equiv v \equiv 0 \) (lake at rest states).

This property diminishes the appearance of unphysical waves of magnitude proportional to the grid size (the so-called “numerical storm”), which are normally present when computing quasi steady-states;

(ii) The method should be positivity preserving, that is, the water depth \( h \) should be nonnegative at all times.

This property ensures a robust performance of the method on dry \((h = 0)\) and almost dry \((h \sim 0)\) states.
Semi-discrete central-upwind scheme

Central-Upwind schemes were developed for multidimensional hyperbolic systems of conservation laws in 2000 – 2007 by Kurganov, Lin, Noelle, Petrova, Tadmor, ...

- Central-Upwind schemes are Godunov-type finite-volume projection-evolution methods:
- At each time level a solution is globally approximated by a piecewise polynomial function,
- Which is then evolved to the new time level using the integral form of the conservation law system.
Key ideas of the scheme development for Saint-Venant system

- Change of conservative variables from \((h, hu, hv)^T\) to \((w := h + B, hu, hv)^T\)
- Replacement of the bottom topography function \(B\) with its continuous piecewise linear (or bilinear in the 2-D case) approximation
- Special positivity preserving correction of the piecewise linear reconstruction for the water surface \(w\)
- Development of a special finite-volume-type quadrature for the discretization of the cell averages of the geometric source term.
Description of the scheme

- We describe now, our new second-order semi-discrete central-upwind scheme for solving the Saint-Venant system of shallow water equations on triangular grids.

- We first denote the water surface by \( w := h + B \) and rewrite the original Saint-Venant system in terms of the vector \( U := (w, hu, hv)^T \):

\[
U_t + F(U, B)_x + G(U, B)_y = S(U, B)
\]

where the fluxes and the source terms are:

\[
F(U, B) = \left( hu, \frac{(hu)^2}{w - B} + \frac{1}{2} g(w - B)^2, \frac{(hu)(hv)}{w - B} \right)^T
\]

\[
G(U, B) = \left( hv, \frac{(hu)(hv)}{w - B}, \frac{(hv)^2}{w - B} + \frac{1}{2} g(w - B)^2 \right)^T
\]

\[
S(U, B) = \left( 0, -g(w - B)B_x, -g(w - B)B_y \right)^T.
\]
Description of the scheme: notations

- Triangulation $\mathcal{T} := \bigcup T_j$ of the computational domain: triangular cells $T_j$ of size $|T_j|$
- $\vec{n}_{jk} := (\cos(\theta_{jk}), \sin(\theta_{jk}))$ are the outer unit normals to the corresponding sides of $T_j$ of length $\ell_{jk}$, $k = 1, 2, 3$,
- $(x_j, y_j)$ are the coordinates of the center of mass for $T_j$ and $M_{jk} = (x_{jk}, y_{jk})$ is the midpoint of the $k$-th side of the triangle $T_j$, $k = 1, 2, 3$
- $T_{j1}, T_{j2}$ and $T_{j3}$ are the neighboring triangles that share a common side with $T_j$
Description of the central-upwind scheme on triangular grids

Denote $\bar{U}_j(t) \approx \frac{1}{|T_j|} \int_{T_j} U(x, y, t) \, dx \, dy$.

Second order central-upwind scheme on triangular grid for the Saint-Venant System:

$$
\frac{d\bar{U}_j}{dt} = 

\begin{align*}
&- \frac{1}{|T_j|} \sum_{k=1}^{3} \frac{\ell_{jk} \cos(\theta_{jk})}{a_{jk}^{in} + a_{jk}^{out}} \left[ a_{jk}^{in} F(U_{jk}(M_{jk}), B(M_{jk})) + a_{jk}^{out} F(U_j(M_{jk}), B(M_{jk})) \right] \\
&- \frac{1}{|T_j|} \sum_{k=1}^{3} \frac{\ell_{jk} \sin(\theta_{jk})}{a_{jk}^{in} + a_{jk}^{out}} \left[ a_{jk}^{in} G(U_{jk}(M_{jk}), B(M_{jk})) + a_{jk}^{out} G(U_j(M_{jk}), B(M_{jk})) \right] \\
&\quad + \frac{1}{|T_j|} \sum_{k=1}^{3} \frac{a_{jk}^{in} a_{jk}^{out}}{a_{jk}^{in} + a_{jk}^{out}} \left[ U_{jk}(M_{jk}) - U_j(M_{jk}) \right] + \bar{S}_j,
\end{align*}
$$

where $U_j(t)$ denotes the averaged state in tetrahedron $T_j$. The scheme is second order accurate in both space and time.
Description of the central-upwind scheme on triangular grids

- \( U_j(M_{jk}) \) and \( U_{jk}(M_{jk}) \) are the corresponding values at \( M_{jk} \) of the piecewise linear reconstruction

\[
\tilde{U}(x, y) := \bar{U}_j + (U_x)_j(x - x_j) + (U_y)_j(y - y_j), \quad (x, y) \in T_j
\]
of \( U \) at time \( t \)

- The quantity \( S_j \) in the scheme is an appropriate discretization of the cell averages of the source term

- The directional local speeds \( a_{jk}^{in} \) and \( a_{jk}^{out} \) are defined by

\[
a_{jk}^{in}(M_{jk}) = -\min\{\lambda_1[V_{jk}(U_j(M_{jk}))], \lambda_1[V_{jk}(U_{jk}(M_{jk}))], 0\}, \\
a_{jk}^{out}(M_{jk}) = \max\{\lambda_3[V_{jk}(U_j(M_{jk}))], \lambda_3[V_{jk}(U_{jk}(M_{jk}))], 0\},
\]

where \( \lambda_1[V_{jk}] \leq \lambda_2[V_{jk}] \leq \lambda_3[V_{jk}] \) are the eigenvalues of the matrix \( V_{jk} = \cos(\theta_{jk}) \frac{\partial F}{\partial U} + \sin(\theta_{jk}) \frac{\partial G}{\partial U} \).

- A fully discrete scheme is obtained by using a stable ODE solver of an appropriate order
Calculation of the numerical derivatives of the $i$th component of $U$

- Construct three linear interpolations $L_{12}^j(x, y)$, $L_{23}^j(x, y)$ and $L_{13}^j(x, y)$: conservative on $T_j$ and two of the neighboring triangles $(T_{j1}, T_{j2})$, $(T_{j2}, T_{j3})$ and $(T_{j1}, T_{j3})$

- Select the linear piece with the smallest magnitude of the gradient, say, $L_{km}^j(x, y)$, and set

\[
((U_{x}^{(i)})_j, (U_{y}^{(i)})_j)^T = \nabla L_{km}^j
\]

- Minimize the oscillations by checking the appearance of local extrema at the points $M_{jk}, 1, 2, 3$
**Piecewise linear approximation of the bottom**

- Replace the bottom topography function $B$ with its continuous piecewise linear approximation $\tilde{B}$, which over each cell $T_j$ is given by the formula:

$$
\begin{vmatrix}
    x - \tilde{x}_{j12} & y - \tilde{y}_{j12} & \tilde{B}(x, y) - B_{j12} \\
    \tilde{x}_{j23} - \tilde{x}_{j12} & \tilde{y}_{j23} - \tilde{y}_{j12} & B_{j23} - B_{j12} \\
    \tilde{x}_{j13} - \tilde{x}_{j12} & \tilde{y}_{j13} - \tilde{y}_{j12} & B_{j13} - B_{j12}
\end{vmatrix} = 0, \quad (x, y) \in T_j.
$$

- $B_{j\kappa}$ are the values of $\tilde{B}$ at the vertices $(\tilde{x}_{j\kappa}, \tilde{y}_{j\kappa})$, $\kappa = 12, 23, 13$, of the cell $T_j$

- $B_{j\kappa} := \frac{1}{2}(\max_{\xi^2+\eta^2=1} \lim_{h,\ell \to 0} B(\tilde{x}_{j\kappa} + h\xi, \tilde{y}_{j\kappa} + \ell\eta) + \min_{\xi^2+\eta^2=1} \lim_{h,\ell \to 0} B(\tilde{x}_{j\kappa} + h\xi, \tilde{y}_{j\kappa} + \ell\eta))$

- If the function $B$ is continuous at $(\tilde{x}_{j\kappa}, \tilde{y}_{j\kappa})$: $B_{j\kappa} = B(\tilde{x}_{j\kappa}, \tilde{y}_{j\kappa})$
**Positivity preserving reconstruction for \( w \)**

The idea of the algorithm that guarantees positivity of the reconstructed values of the water depth \( h_j(M_{jk}) := w_j(M_{jk}) - B_{jk}, \ k = 1, 2, 3, \) for all \( j \):

- The reconstruction \( \tilde{w} \) should be corrected only in those triangles, where \( \tilde{w}(\tilde{x}_{jk}, \tilde{y}_{jk}) < B_{jk} \) for some \( \kappa \), \( \kappa = 12, 23, 13 \)

- Since \( \bar{w}_j \geq B_j \), it is impossible to have \( \tilde{w}(\tilde{x}_{jk}, \tilde{y}_{jk}) < B_{jk} \) for all three values of \( \kappa \): at all three vertices of the triangle \( T_j \)

- Two cases in which a correction is needed are possible:
  - either there are two indices \( \kappa_1 \) and \( \kappa_2 \), for which \( \tilde{w}(\tilde{x}_{j\kappa_1}, \tilde{y}_{j\kappa_1}) < B_{j\kappa_1} \) and \( \tilde{w}(\tilde{x}_{j\kappa_2}, \tilde{y}_{j\kappa_2}) < B_{j\kappa_2} \),
  - or there is only one index \( \kappa_1 \), for which \( \tilde{w}(\tilde{x}_{j\kappa_1}, \tilde{y}_{j\kappa_1}) < B_{j\kappa_1} \)
Well-balanced discretization of the source term

- The well-balanced property of the scheme is guaranteed if the discretized cell average of the source term, $\bar{S}_j$, exactly balances the numerical fluxes.

- The desired quadrature for the source term that will preserve stationary steady states ($U_{jk}(M_{jk}) \equiv U_j(M_{jk}) \equiv (C, 0, 0)^T$, $\forall j, k$) is given by:

  $\bar{S}^{(2)}_j = \frac{g}{2|T_j|} \sum_{k=1}^{3} \ell_{jk}(w_j(M_{jk}) - B_{jk})^2 \cos(\theta_{jk}) - g(w_x)_j(\bar{w}_j - B_j)$

  $\bar{S}^{(3)}_j = \frac{g}{2|T_j|} \sum_{k=1}^{3} \ell_{jk}(w_j(M_{jk}) - B_{jk})^2 \sin(\theta_{jk}) - g(w_y)_j(\bar{w}_j - B_j)$
Main theorem: positivity property of the new scheme

Theorem 1 Consider the Saint-Venant system in the new variables \( U := (w, hu, hv)^T \) and the central-upwind semi-discrete scheme (with well-balanced quadrature for the source \( S \), positivity preserving reconstruction for \( w \))

- Assume that the system of ODEs for the fully discrete scheme is solved by the forward Euler method and that for all \( j \), \( \bar{w}_j^n - B_j \geq 0 \) at time \( t = t^n \)

- Then, for all \( j \), \( \bar{w}_j^{n+1} - B_j \geq 0 \) at time \( t = t^{n+1} = t^n + dt \), provided that \( dt \leq \frac{1}{6a} \min_{j,k} \{ r_{jk} \} \), where \( a := \max_{j,k} \{ a_{jk}^{\text{out}}, a_{jk}^{\text{in}} \} \) and \( r_{jk}, k = 1, 2, 3 \), are the altitudes of triangle \( T_j \).

Remark. Theorem 1 is still valid if one uses a higher-order SSP ODE solver (either the Runge-Kutta or the multistep one), because such solvers can be written as a convex combination of several forward Euler steps.
**Accuracy test**

The scheme is applied to the Saint-Venant system subject to the following initial data and the bottom topography:

\[ w(x, y, 0) = 1, \quad u(x, y, 0) = 0.3, \]
\[ B(x, y) = 0.5 \exp(-25(x - 1)^2 - 50(y - 0.5)^2). \]

- For a reference solution, we solve this problem with our method on a $2 \times 400 \times 400$ triangular grid.
- By $t = 0.07$ the solution converges to the steady state.
Accuracy test

- $w$ component of the reference solution of the IVP on a $2 \times 400 \times 400$ grid: the 3-D view (left) and the contour plot (right).

- $L^1$- and $L^\infty$-errors and numerical orders of accuracy.

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<th>Order</th>
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Small perturbation of a stationary steady-state solution

• Solve the initial value problem (IVP) proposed by R. Leveque.

• The computational domain is \([0, 2] \times [0, 1]\) and the bottom consists of an elliptical shaped hump:

\[
B(x, y) = 0.8 \exp(-5(x - 0.9)^2 - 50(y - 0.5)^2).
\]

• Initially, the water is at rest and its surface is flat everywhere except for \(0.05 < x < 0.15\):

\[
w(x, y, 0) = \begin{cases} 
1 + \varepsilon, & 0.05 < x < 0.15, \\
1, & \text{otherwise,}
\end{cases}
\]

\[u(x, y, 0) \equiv v(x, y, 0) \equiv 0,\]

where the perturbation height is \(\varepsilon = 10^{-4}\).
Perturbation of a stationary steady-state: well-balanced scheme (left) and non well-balanced (right)
Perturbation of a stationary steady-state: well-balanced scheme (left) and non well-balanced (right)
Saint-Venant System with friction and discontinuous bottom

- More realistic shallow water models include additional friction and/or viscosity terms
- Presence of friction and viscosity terms guarantees uniqueness of the steady state solution
- We consider the simplest model in which only friction terms, $-\kappa(h)u$ and $-\kappa(h)v$, are added to the rhs of the second and third equations of the Saint-Venant System

\[
\begin{align*}
    h_t + (hu)_x + (hv)_y &= 0, \\
    (hu)_t + \left( hu^2 + \frac{1}{2}gh^2 \right)_x + (huv)_y &= -ghB_x - \kappa(h)u, \\
    (hv)_t + (huv)_x + \left( hv^2 + \frac{1}{2}gh^2 \right)_y &= -ghB_y - \kappa(h)v.
\end{align*}
\]
Saint-Venant System with friction and discontinuous bottom

- We numerically solve the shallow water model with friction term on the domain $[-0.25, 1.75] \times [-0.5, 0.5]$

- We assume that the friction coefficient is

$$\kappa(h) = 0.001(1 + 10h)^{-1}$$

- The bottom topography function has a discontinuity along the vertical line $x = 1$ and it mimics a mountain river valley
Saint-Venant System with friction and discontinuous bottom: description of the initial and boundary conditions

- We implement reflecting (solid wall) boundary conditions at all boundaries.

- Our initial data correspond to the situation when the second of the three dams, initially located at the vertical lines $x = -0.25$ (the left boundary of the computational domain), $x = 0$, and $x = 1.75$ (the right boundary of the computational domain), breaks down at time $t = 0$, and the water propagates into the initially dry area $x > 0$, and a “lake at rest” steady state is achieved after a certain period of time.
• We plot 1-D slices of the numerical solution along the $y = 0$ line

• Plots clearly show the dynamics of the fluid flow as it moves from the region $x < 0$ into the initially dry area $x > 0$ and gradually settles down into a “lake at rest” steady state

• This state includes dry areas and therefore its computation requires a method that is both well-balanced and positivity preserving on the entire computational domain
Flow in converging-diverging channel

• The exact geometry of each channel is determined by its breadth, which is equal to $2y_b(x)$, where

$$y_b(x) = \begin{cases} 
0.5 - 0.5(1 - d) \cos^2(\pi(x - 1.5)), & |x - 1.5| \leq 0.5, \\
0.5, & \text{otherwise}, 
\end{cases}$$

• $d = 0.6$ is the minimum channel breadth
Flow in converging-diverging channel

- The initial conditions:
  \[ w(x, y, 0) = \max\left\{ 1, B(x, y) \right\}, \quad u(x, y, 0) = 2, \quad v(x, y, 0) = 0. \]

- The upper and lower \( y \)-boundaries are reflecting (solid wall), the left \( x \)-boundary is an inflow boundary with \( u = 2 \) and the right \( x \)-boundary is a zero-order outflow boundary.

- The bottom topography is given by
  \[ B(x, y) = \left( e^{-10(x-1.9)^2-50(y-0.2)^2} + e^{-20(x-2.2)^2-50(y+0.2)^2} \right), \]
Flow in converging-diverging channel: \( w \)

Steady-state solution \((w)\) for \((d, B_{\text{max}}) = (0.6, 1)\) on \(2 \times 200 \times 200\) (left) and \(2 \times 400 \times 400\) (right) grids.
Conclusions/Difficulties

- We developed a simple central-upwind scheme for the Saint-Venant system on triangular grids.
- We proved that the scheme both preserves stationary steady states (lake at rest) and guarantees the positivity of the computed fluid depth.
- It can be applied to models with discontinuous bottom topography and irregular channel widths.
- Method is sensitive to the accuracy of the boundary representation.