Vortex Method Applications

Peter S. Bernard
University of Maryland
Vortex Methods

Flow field is represented using gridfree vortex elements

Navier-Stokes equation governs the dynamics of the freely convecting vortex elements

Velocity is recovered from the vortices using the Biot-Savart law

For turbulent flow simulation vortex methods operate as a LES: not economical to resolve the smallest scales.
### Vortex Elements

**“Blobs”**

or: **filaments made up of tubes:**

<table>
<thead>
<tr>
<th></th>
<th>Viscous Diffusion</th>
<th>Vortex Stretching</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Blob methods</strong></td>
<td><strong>Accurate, if:</strong> blobs overlap occasional remeshing</td>
<td><strong>Unstable - depends on calculation of velocity derivatives.</strong></td>
</tr>
<tr>
<td><strong>Filament methods</strong></td>
<td><strong>Not straightforward.</strong></td>
<td><strong>Convect tube end points.</strong></td>
</tr>
</tbody>
</table>
Filament methods:

• preferred for turbulent flow simulation since vortex stretching is an essential and dominant flow process.

• viscous diffusion is not significant away from walls.

• viscous dissipation is "subgrid" and can be modeled via loop removal.

Blob methods:

• rarely applied to turbulence due to vortex stretching instability

• best suited to moderate Reynolds number flows where viscous diffusion is an essential aspect of the dynamics (e.g. laminar vortex rings, low speed jets).
Vortex Filament Scheme

- Computational elements: straight vortex tubes linked end-to-end forming filaments.
- Convect tubes via their endpoints: this models the convection and vortex stretching term in the equations of motion.
- Circulation $\Gamma$ on filaments is constant (Kelvin's Theorem)
- Tubes subdivide when stretched beyond length $h$. 

![Image of vortex filament scheme](image-url)
Not economical to resolve the dissipation scales.

Moreover, viscous diffusion would have to be modeled directly.

Loop removal offers a means around the impasse.
Vortex loops are removed as they form, thus providing spatially intermittent local dissipation: in fact, vortex loops contribute only to the local velocity field.

In principle, energy is removed at inertial range scales without the hardship of computing energy transfer to the smallest scales.

Loop removal is non-diffusive (unlike traditional sub-grid models).

Does not prevent or interfere with backscatter: (e.g. filaments combine to form larger scale structures).

Prevents runaway growth in the number of tubes.
Number of tubes vs. time

a: without loop removal, 
   h = 0.0375 (max tube length)

b: with loop removal, 
   h = 0.025

c: with loop removal 
   h = 0.0375
Near-Wall Considerations

Flow near solid walls contains strong vorticity gradients that determine viscous production of new vorticity.

In general, LES is not appropriate next to walls since all scales need to be resolved.

DNS resolution is necessary.
Near-wall vorticity gradients are not readily accounted for via gridfree vortex elements.

All viscous effects must be computed next to a wall.

Solid surfaces are often specified via triangularizations.

Sheet-like vortices on a thin prismatic wall mesh can be used to efficiently resolve the steep vorticity gradients.
With these constraints in mind:

a fine resolution finite volume scheme is employed to solve the full 3D, viscous vorticity equation on a thin region ($y^+ < 30-50$) next to boundaries.

Triangular prism mesh (usually 11 layers) is grown from surface triangles.

No-slip BC at wall; constant flux BC at top sheet layer.

\[ \Delta y^+ \approx 3 \]

\[ \Delta z^+ \approx 30 \]
New filaments are produced from the vorticity arriving at the top layer.

Orientation of new vortices is determined from local vorticity.

The circulation strength $\Gamma$ is determined by the condition that the prism and new tube have the identical far field velocity:

$$\Gamma |s| = |\Omega| V_T$$

$V_T = \text{prism volume}$ \quad $s = \text{axial vector on tube.}$
Velocity Computation

\[ U_{\text{total}} = U_{\text{tubes}} + U_{\text{sheets}} + U_{\text{potential}} \]

For N tubes: \[ U_{\text{tubes}} = -\frac{1}{4\pi} \sum_{i=1}^{N} \frac{r_i \times s_i}{|r_i|^3} \Gamma_i \phi(|r_i|/\sigma), \]

where \[ x_i = (x_{1,i} + x_{2,i})/2, \quad r_i = x - x_i, \quad r_i = |r_i|, \quad s_i = x_{2,i} - x_{1,i} \]

\[ \phi(|r_i|/\sigma) = 1 - \left(1 - \frac{3}{2}(|r_i|/\sigma)^3\right) e^{-(|r_i|/\sigma)^3}, \]

desingularizes the Biot-Savart kernel.

\[ U_{\text{sheets}} \rightarrow \text{sum over individual contributions: near field from exact formulas, far field as equivalent tubes.} \]

\[ U_{\text{potential}} \rightarrow \text{sum over surface sources to enforce non-penetration.} \]
Variations of the velocity over distances smaller than the tube length cannot be known accurately.

For example, $\sigma$ has some effect on the local velocity surrounding any vortex - but it is not physical.

Velocity on a line through a vortex.
Approximation to Biot-Savart law for a tube introduces local errors:

In this example, \( v(z) \) should be a constant, independent of \( z \), and it is once the tube size \( h < L \).

\( h = 1.25L \)

\( h = 2.5L \)

\( h = 0.87L \)

\( h = 0.63L \)

\( h, \Gamma \) and \( \Delta t \) should be kept as small as is practical.
Magnitude of $h$ has a significant visual effect on simulations:

$h = 0.005$

$h = 0.025$
Predictions converge to experimental values as h is reduced: 0.025 → 0.003125

- Bell/Mehta experiment
Use of Biot-Savart law means that in many applications vorticity outside the domain of interest may contribute to the velocity field.
Numerical Aspects of Velocity Computation

$O(N^2)$ cost reduced to $O(N)$ via use of an adaptive Fast Multipole Method (Greengard & Rohklin).

Parallel efficiency is excellent through 22 processors.

$O(N)$ scaling in the FMM

Parallel efficiency. (Dashed – ideal).
- 13M tubes
- 10M tubes + 8 periodic extensions
Summary of Numerical Parameters (Filament Calculations)

- $h$ – tube length
- $\Delta t$ - time step
- $\Gamma$ - circulation
- $\sigma$ - smoothing parameter in Biot-Savart Law
- $d$ - criterion for loop removal
Among the advantages of vortex methods:

The representation of vortices in terms of their end points and circulation represents a gain in efficiency over grid-based methods.

Vortices remain sharp (without dissipation) as they convect.

Opportunity to employ non-diffusive "subgrid" modeling.

Gridding requirements are easier to accommodate than grid-based schemes (e.g. number of prisms \( \sim \text{Re}^{3/2} \)).

Direct view of vortical structures provides a new way of exploring the physics of turbulence.
Some Applications of the Vortex Filament Scheme

1. "Isotropic Turbulence"

2. Spatially developing shear layer

3. Boundary layer

4. Automotive Flows

5. Rotorcraft Flows

6. Co-flowing round jet
"Isotropic" Turbulence
A more or less isotropic region of turbulence is created from a short duration pulse of a planar jet.

- Orifice has unit width.
- 20 layers of filaments, $h=0.005$.
- Incoming circulation corresponds to Poiseulle Flow at the orifice exit.
- 4 periodic extensions to either side used in computing velocities.
Turbulence statistics computed from spanwise velocity traces.
Two point longitudinal, f(r), and transverse, g(r), correlation functions

Symbols show consistency with the isotropy condition:

\[ g(r) = f(r) + \frac{r}{2} \frac{df}{dr}(r) \]
1D spectra and Kolmogorov law.

Assuming that the universal form: \( E(k) = 0.53 \, \varepsilon^{2/3} \, k^{-5/3} \) holds, the dissipation \( \varepsilon \) can be computed.
Structure functions $S_2(r)$, $S_3(r)$, $S_4(r)$ where $S_n(r) = |u(x+r)-u(x)|^n$.

Dashed lines have slopes, $2/3$, $1$, $4/3$, respectively.

$\varepsilon$ can also be computed from the universal form $S_n(r) = 2.13 \varepsilon^{2/3} r^{2/3}$.
Reynolds Number

Substituting $\varepsilon$ and $\lambda$ into the isotropic identity $\varepsilon = 15 \nu u'^2 / \lambda^2$ yields

$$R_e = \frac{UL}{\nu} = 15 u'^2 / (\lambda^2 \varepsilon)$$

and then

$$R_\lambda = \frac{u' \lambda}{\nu} = R_e u' \lambda.$$  

Furthermore, Kolmogorov length and time scales may be computed (non-dimensionalized with $U,L$):

$$\eta = R_e^{-3/4} \varepsilon^{-1/4} \quad t_d = (R_e \varepsilon)^{-1/2}$$

For blob flow with $\zeta = 0.01$:

$R_\lambda = 71 \quad R_e = 155354 \quad \lambda = 0.023 \quad \eta = 0.0017 \quad \varepsilon = 0.00008 \quad t_d = 0.23$
Mixing Layer
Spatial extent of one periodic image of the shear layer:

2000 θ X 1000 θ in the x and z directions, respectively,

where θ is the momentum thickness:

\[ \theta \equiv \int \frac{(U_h - \bar{U})(\bar{U} - U_l)dy}{(U_h - U_l)^2} \]
Quantitative predictions are consistent with Bell/Mehta experiment:

Correct linear growth in momentum thickness.

Mean velocity is self-similar and matches error function.

Reynolds stresses close to experiment.
Experimental observations have categorized 3 kinds of vortex structure in transition:

1. Roller/Rib

2. Vortex lattice (chain link fence)
   • Response to asymmetric forcing.

3. Oblique roller vortices with partial pairing.
   • Associated with upstream turbulence.
Vortex lattice/chain link fence transition
Oblique roller vortices with partial pairing
Close up view of roller/ribs

Close up view of lattice
Three transition modes seen in experiment:

- Roller/Rib (response to lateral perturbation)
- Vortex lattice
- Oblique roller vortices & partial pairing (response to upstream turbulence)
Boundary Layer
Boundary Layer Simulations

Test section: $-0.25 < z < 0.25$

- $U = 1$

- 62,272 surface triangles,
- 684,992 prisms,
- 22,000,000 vortex tubes
- Re = 50,000, 80,000

- 74,274 surface triangles,
- 817,014 prisms,
- 28,000,000 vortex tubes
- Re = 80,000
Mean Velocity Predictions

\[ \overline{U}^+ = y^+ \]

Computed \( \overline{U}^+ (R_\theta = 670) \) vs. Spalart DNS (\( R_\theta = 670 \)).

Blasius BL

Scaling with \( \delta \) and \( \theta \)
Reynolds Stresses
Transition is dominated by the appearance of vortex furrows - spaced approximately at the boundary layer thickness.
Vortex furrows override low speed streaks.
Transverse cuts through the furrows.
Vortex furrows erupting into mushroom-shaped filaments
Ejection of low speed fluid.
Counter-rotating motion is associated with the uplifted filaments in the furrows.
Initially the counter-rotating motion is produced by forward tilted filaments in the arches: there are no streamwise filaments.

True streamwise vorticity is prevalent in the lobes of the mushroom profile.

Furrow viewed from above. Tubes within $\pi/16$ of the streamwise direction are indicated in blue (+) and red (-).

Projection onto cross plane of vortex tubes in mushroom-shaped furrow.
When viewed as isocontours of rotational motion (2nd eigenvalue of $S^2+W^2$), the furrows have the appearance of hairpin vortices.

Isocontours (shown in red) representing the "legs of hairpin vortices mark the counter-rotating motion associated with the furrows."
Shear roll-up of spanwise vorticity is the apparent source of arch-type vortices that straddle the furrows.
Breakdown of furrows into turbulence
Ground Vehicle Flows
Ahmed body

12.5, 25, 30 degree base slant angle
$R_e = 500,000$
Inviscid ground plane
Front: $x=0$
Back: $x=1$
Ahmed Body with $30^\circ$ base slant angle

Side

Top

Rear during startup
Ahmed Body with 12.5° base slant angle
Ahmed Body with $30^\circ$ base slant angle
U on centerline.
U in wake.
U on window, x = 0.8678.
K on centerline and wake.
MIRA Vehicle

Re=500,000

Inviscid ground plane, or

Moving, viscous ground plane
Rotorcraft Simulations
Coflowing Round Jet
Co-flowing Round Jet

Coflows = 1/2, 1/3, 1/4, 1/5, 1/10.

5 vortex rings at unit diameter orifice.

Potential flow:
disk shaped source maintains unit velocity at inlet

(a) Remove rings when > L₁
(b) Remove rings if any part > L₂
Co-flowing Round Jet

<table>
<thead>
<tr>
<th>$U_c$</th>
<th>$h$</th>
<th>BC</th>
<th>L/d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>0.055</td>
<td>a</td>
<td>5.25</td>
</tr>
<tr>
<td>1/5</td>
<td>0.04</td>
<td>a</td>
<td>6.65</td>
</tr>
<tr>
<td>1/4</td>
<td>0.04</td>
<td>b</td>
<td>11.0</td>
</tr>
<tr>
<td>1/3</td>
<td>0.045</td>
<td>b</td>
<td>12.75</td>
</tr>
<tr>
<td>1/2</td>
<td>0.04</td>
<td>b</td>
<td>15.5</td>
</tr>
</tbody>
</table>

$$
\theta \equiv \sqrt{2\pi \int (\overline{U}(U-U_c))r dr / U_c}
$$

$$
\delta \equiv \sqrt{\int (\overline{U}-U_c)r^2 dr / \int (\overline{U}-U_c) dr}
$$

$$
\overline{U}/U_0 = e^{-r^2/b^2}
$$

$\theta \rightarrow$ momentum thickness

$b \rightarrow$ Gaussian scale
Velocity Excess in Coflowing Round Jet vs. Streamwise Position

\[ U_{\text{excess}} = \frac{(U_{\text{centerline}} - U_{\text{coflow}})}{U_{\text{coflow}}} \]

+ – experiments (Nickels and Perry)
\(x\) – experiments (Chu, Lee and Chu)

○ – 1/10
○ – 1/5
○ – 1/4
○ – 1/3
○ – 1/2
$\delta/\theta$ vs. streamwise position

+ – experiments (Nickels and Perry)
  o – 1/10
  o – 1/5
  o – 1/4
  o – 1/3
  o – 1/2
b/θ vs. streamwise position

+ – experiments
(Chu, Lee and Chu)

○ – 1/10
○ – 1/5
○ – 1/4
○ – 1/3
○ – 1/2
Self-similar Gaussian mean velocity in coflowing round jet

+ Chu, Lee, and Chu data
Concentration Statistics
The vortex filament approach appears to offer an attractive means for efficiently simulating a variety of complex turbulent flows.

Good resolution of the wall region flow is essential to accurate predictions. Further improvements to the numerical implementation (e.g. parallelism) will enable the treatment of higher Reynolds number flows.

The use of vortex filaments in directly representing vortical structure is seen to offer a view of the physics that has not been previously achieved with grid-based methods.