ANALYTICITY OF THE SUBCRITICAL AND CRITICAL QUASI-GEOSTROPHIC EQUATIONS IN BESOV SPACES

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Abstract. In this paper, we establish analyticity of the subcritical and critical quasi-geostrophic equations in critical Besov spaces. The main method is so-called Gevrey estimates, which is motivated by the work of Foias and Temam [21]. We show that mild solutions are Gevrey regular, i.e. they satisfy the estimate \( \sup_{t>0} ||e^{\alpha t^{\gamma}} \theta(t)||_{\mathcal{L}} < \infty \), where \( \mathcal{L} \) is a scaling invariant Besov space, and \( \alpha = 1 \) for the subcritical case and \( \alpha = \frac{1}{4} \) for the critical case.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

It is well-known that regular solutions of many dissipative equations, such as the Navier-Stokes equations, the Kuramoto-Sivashinsky equation and the Smoluchowski equation are in fact analytic, in both space and time variables ([5, 20, 32, 37]). In fluid-dynamics, the space analyticity radius has an important physical interpretation: at this length scale the viscous effects and the (nonlinear) inertial effects are roughly comparable. Below this length scale the Fourier spectrum decays exponentially ([14, 19, 23, 24]). In other words, the space analyticity radius yields a Kolmogorov type length scale encountered in turbulence theory. At a more practical level, this fact can be used to show that the finite dimensional Galerkin approximations converge exponentially fast in these cases ([13]). Other

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applications of analyticity radius occur in establishing sharp temporal decay rates of solutions in higher Sobolev norms ([33]), establishing geometric regularity criteria for the Navier-Stokes equations, and in measuring the spatial complexity of fluid flow (see [22, 28, 29]).

In this paper, we study analyticity properties of the dissipative quasi-geostrophic equations in two dimensions. These equations are derived from the more general quasi-geostrophic approximation for a rapidly rotating fluid flow with small Rossby and Ekman numbers. The system of equations is given by

\begin{align}
\theta_t + v \cdot \nabla \theta + \kappa \Lambda^\gamma \theta &= 0, \\
v &= (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta),
\end{align}

where the scalar function \( \theta \) is the potential temperature, \( v \) is the fluid velocity, \( \kappa \) is the viscosity coefficient which for simplicity we set as \( \kappa = 1 \), and \( \mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2) \) are the Riesz transforms whose symbols are given by \( \frac{i\xi_l}{|\xi|} \) for \( l = 1, 2 \). \( \Lambda^\gamma \) is a differential operator whose symbol is given by \( |\xi|^\gamma \), and the cases \( \gamma > 1 \), \( \gamma = 1 \), and \( \gamma < 1 \) are called respectively subcritical, critical and supercritical. In this paper, we only consider the subcritical and critical cases.

Formally, we can express a solution \( \theta \) in the integral form:

\[
\theta(t) = e^{-t \Lambda^\gamma} \theta_0 - \int_0^t \left[ e^{-(t-s) \Lambda^\gamma} (v \cdot \nabla \theta)(s) \right] ds.
\]

Any solution satisfying this integral equation is called a mild solution. We can find it by using a fixed point argument for the function \( \theta \mapsto F(\theta) \),

\[
F(\theta)(t) = e^{-t \Lambda^\gamma} \theta_0 - \int_0^t \left[ e^{-(t-s) \Lambda^\gamma} (v \cdot \nabla \theta)(s) \right] ds.
\]

The invariant space for solving this integral equation corresponds to a scaling invariance property of the equation. Assume that \((\theta, v)\) solves (1.1). Then, the same is true for rescaled functions:

\[
\theta_\lambda(t, x) = \lambda^{\gamma-1} \theta(\lambda^\gamma t, \lambda x), \quad v_\lambda(t, x) = \lambda^{\gamma-1} v(\lambda^\gamma t, \lambda x) \quad \forall \lambda > 0.
\]

Under these scalings, \( L^{\frac{2}{\gamma-1}} \) (\( \gamma \geq 1 \)), \( H^{2-\gamma} \), and \( B^{\frac{2}{p}+1-\gamma}_{p,q} \) are critical spaces for initial data \((t = 0)\), i.e., the corresponding norms are invariants under these scaling. One can find various well-posedness results for small initial data in these critical spaces in [2, 10, 25, 26, 34, 40, 41]. There are also many well-posedness results for large data; [9] in the Lebesgue space \( L^{\frac{2}{\gamma-1}} \) for \( \gamma > 1 \), [15] in the energy space \( H^1 \), and [1, 17, 38] in Besov spaces.

The goal of this paper is to show analyticity of mild solutions by establishing Gevrey regularity. The use of Gevrey regularity was pioneered by Foias and Temam ([21]) for estimating space analyticity radius for the Navier-Stokes equations and was subsequently used by many authors (see [4, 6, 8, 18], and the references there in). In this paper, we will show that mild solutions of (1.1) are Gevrey regular, i.e. they satisfy the estimate \( \sup_{t > 0} \| e^{\alpha t \Lambda^\gamma} \theta \|_{\mathcal{L}^1} \) where \( \mathcal{L} \) is a critical Besov space, and \( \alpha = 1 \) for \( \gamma > 1 \) and \( \alpha = \frac{1}{4} \) for \( \gamma = 1 \). We emphasize that here the exponential operator \( e^{\alpha t \Lambda^\gamma} \) is quantified by \( \Lambda^\gamma \) whose symbol is given by the \( l^1 \) norm \( |\xi| = |\xi_1| + |\xi_2| \) rather than the usual \( \Lambda = \sqrt{-\Delta} \). This approach enables one to avoid cumbersome recursive estimation of higher order derivatives.
In order to explain the main idea, we consider the subcritical case ($\gamma > 1$). The same approach will be used for the critical case. To show Gevrey regularity, we define $\Theta$ and $V$ by

$$\Theta = e^{t^{\frac{1}{2}}A_1}\theta, \quad V = e^{t^{\frac{1}{2}}A_1}v.$$  

Then, $\Theta$ and $V$ satisfy the following equation

$$\Theta(t) = e^{t^{\frac{1}{2}}A_1-tA_{\gamma}}\theta_0 - \int_0^t \left[ e^{t^{\frac{1}{2}}A_1-(t-s)A_{\gamma}}(e^{-s^{\frac{1}{2}}A_1}V \cdot \nabla e^{-s^{\frac{1}{2}}A_1}\Theta)\right] ds.$$  

Since $e^{t^{\frac{1}{2}|\xi|}}$ is dominated by $e^{-t|\xi|^\gamma}$ for $|\xi| \gg 1$, the linear term, $e^{t^{\frac{1}{2}}A_1-tA_{\gamma}}\theta_0$, closely resembles that of $\theta$. The estimates of the nonlinear term are similar to those of $\theta$ due to the nice boundedness property of the following bilinear operator $B_l$:

$$B_s(f,g) = e^{s^{\frac{1}{2}}A_1} \left( e^{-s^{\frac{1}{2}}A_1}f e^{-s^{\frac{1}{2}}A_1}g \right).$$  

As noticed from the above argument, the existence result of $\theta(t)$ is crucial to establish Gevrey regularity. Thus, in section 3 (for $\gamma > 1$) and section 4 (for $\gamma = 1$), we will first show the existence of a mild solution and then proceed to explain how to modify the existence proof to obtain Gevrey regularity.

We now present our existence/analyticity results for $\gamma > 1$ and $\gamma = 1$ separately. For notational simplicity, we will suppress the dependence on $p$, $q$ and other relevant indices when we define norms below.

### 1.1. Subcritical Case: $\gamma > 1$.  

#### 1.1.1. Existence.  

We begin with the existence result in critical spaces. Let us take initial data $\theta_0$ in $B_{p,q}^{2+1-\gamma} \cap L^p$, with $1 \leq q \leq \infty$ and $p_1 \geq \frac{2}{\gamma-1}$. Since the dissipation rate $\gamma > 1$ dominates the derivative in the advection term, we can follow Weissler’s idea ([39]) to define a function space $E_{\gamma,T} = K_{\gamma,T} \cap G_{\gamma,T}$, where

$$\|\theta\|_{K_{\gamma,T}} = \sup_{0 < t \leq T} \left[ \|\theta(t)\|_{\dot{B}_{p,q}^{\frac{2}{\gamma}+1-\gamma} + t^{\frac{q}{2}}\|\theta(t)\|_{\dot{B}_{p,q}^{\frac{2}{\gamma}+1-\gamma+\alpha}}} \right]$$  

and

$$\|\theta\|_{G_{\gamma,T}} = \sup_{0 < t < T} \left[ \|\theta(t)\|_{L^p} + t^\beta \|\theta(t)\|_{L^r} \right], \quad \beta = 1 - \frac{1}{\gamma} - \frac{2}{\gamma r}.$$  

We note that the time weights $t^{\frac{q}{2}}$ and $t^\beta$ indicate the gain of regularity and integrability of the solution through the dissipation $\Lambda^\gamma$. To prove the existence result stated below, we need some conditions for the indices $(\alpha, \beta, p, r)$:

$$\begin{align*}  
  (i) \quad &0 < \frac{2}{r} < \gamma - 1, \quad (ii) \quad \frac{2}{p} + 1 - \gamma - \frac{2}{r} > 0, \quad (iii) \quad \beta + \frac{\alpha}{\gamma} < 1. 
\end{align*}$$  

The first condition implies that $\beta > 0$. The other two conditions will be used to estimate the quadratic term $v\theta$ in the high frequency part (which will be defined in Section 2). $p$ is not necessarily equal to $p_1$. But, as proved in [9], the condition $p_1 \geq \frac{2}{\gamma-1}$ is necessary to show the existence of the solution in $G_{\gamma,T}$, which will be used to show the existence in $E_{\gamma,T}$. The first result of this paper is the following. We note that the size of initial data is arbitrary.
Theorem 1.1 (Existence). For any initial data $\theta_0 \in B_{p,q}^{\frac{2}{p}+1-\gamma} \cap L^p$, there exists a global-in-time solution $\theta \in E_{\gamma,T}$ such that
\[
\|\theta\|_{E_{\gamma,T}} \lesssim C(T)\|\theta_0\|_{B_{p,q}^{\frac{2}{p}+1-\gamma}}, \quad C(T) \sim 2^T.
\]

1.1.2. Analyticity. Analyticity of solutions in Theorem 1.1 can be obtained by establishing Gevrey regularity. To this end, we investigate the equation (1.5) of $\Theta$ and $V$ defined in (1.4). As we use the existence of $\theta$ in $G_{\gamma,T}$ to show the existence proof of $\theta$ in $E_{\gamma,T}$, the proof consists of two parts: the existence of $\Theta$ in $G_{\gamma,T}$ (Theorem 1.2) and the existence of $\Theta$ in $E_{\gamma,T}$ (Theorem 1.3).

Theorem 1.2 (Existence of $\Theta$). For any initial data $\theta_0 \in L_\gamma^{\frac{2}{p}+1-\gamma} \cap L^p$, there exists a global-in-time solution $\Theta(t) \in G_{\gamma,T}$. Moreover, for any time interval $[t, t + \tau]$, \[
\lim_{\tau \to 0} \left[ \tau^\beta \|\Theta(t + \tau)\|_{L^q} \right] = 0
\] uniformly in $t$.

Remark 1.1. We note that Theorem 1.2 provides a new proof of the decay estimates in [16]. Namely, we can obtain the decay rate of the $L^p$ norm of the solution:
\[
\|\Lambda^\xi \theta(t)\|_{L^p} = \left\| \Lambda^\xi e^{-t^\frac{1}{\gamma}A_1} e^{\frac{1}{t^{\gamma}}A_1} \theta(t) \right\|_{L^p} \leq C_\xi t^{-\frac{\gamma}{2}} \|\Theta(t)\|_{L^p} \lesssim \frac{C_\xi t^{-\frac{\gamma}{2}}}{},
\]
where $C_\xi = \|\Lambda^\xi e^{-A_1}\|_{L^1}$ and $t^{-\frac{\gamma}{2}}$ can be achieved by the scaling: $\xi \mapsto t^{\frac{1}{\gamma}}\xi$. Our approach enables one to avoid cumbersome recursive estimation of higher order derivatives and the proof is reduced significantly.

Using the existence of $\Theta$ in $G_{\gamma,T}$, we can prove that solutions of Theorem 1.1 are, in fact, analytic in the following sense.

Theorem 1.3 (Analyticity). For any initial data $\theta_0 \in B_{p,q}^{\frac{2}{p}+1-\gamma} \cap L^p$, there exists a global-in-time solution $\Theta \in E_{\gamma,T}$ such that
\[
\|\Theta\|_{E_{\gamma,T}} \lesssim C(T)\|\theta_0\|_{B_{p,q}^{\frac{2}{p}+1-\gamma}}, \quad C(T) \sim 2^T.
\]

1.2. Critical case: $\gamma = 1$.

1.2.1. Existence. We now consider the case $\gamma = 1$ with initial data $\theta_0$ in critical spaces. Compared to previous results [1, 7, 15, 17, 27, 38] where the global well-posedness is established for large data, the results we provide below are rather restricted due to the mild solution setting. Namely, the dissipation term $e^{-tA}$ is not strong enough to overcome the derivative in the advection term. Therefore, we need some smallness conditions for initial data. The following result was proved in [3] with initial data $\theta_0 \in L^\infty \cap \hat{H}^1 \cap B_{p,q}^{\frac{2}{p}}$. In the proof of [3], we need $v \in L^\infty$ to estimate $\theta$ in $B_{p,q}^{\frac{2}{p}}$. Since the Riesz transforms $\mathcal{R}$ defining $v$ do not map $L^\infty$ to $L^\infty$, we thus assume $v_0 \in L^\infty$. The main idea of proving the result in [3] consists of two parts. First, we show the existence of solutions in $L^\infty_t (L^\infty \cap \hat{H}^1) \cap L^\frac{2}{p}_t \hat{H}^{\frac{2}{p}}$ with small initial data in $L^\infty \cap \hat{H}^1$. Using this smallness condition, we next prove the existence of solutions in $E_1$ with the following norm,
\[
\|\theta\|_{E_1} = \|\theta\|_{L^\infty_t B_{p,q}^{\frac{2}{p}}} + \|\theta\|_{L^\frac{1}{2}_t B_{p,q}^{\frac{2}{p}}}. \tag{1.10}
\]
This space indicates a gain of one derivatives from the maximal regularity of the Poisson kernel.
**Theorem 1.4** (Existence [3]). Let $2 \leq p \leq \infty$ and $1 \leq q < 2$, or $1 \leq p < 2$ and $q \geq 2$. There exists a constant $\epsilon_0 > 0$ such that for all initial data $\theta_0 \in L^\infty \cap \dot{H}^1 \cap \dot{B}_{p,q}^{\frac{3}{2}}$ and $v_0 \in L^\infty$, with
\[
\|\theta_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|\theta_0\|_{\dot{H}^1} < \epsilon_0,
\]
there exists a global-in-time solution $\theta \in L_t^\infty(L^\infty \cap \dot{H}^1) \cap L_t^2\dot{H}^\frac{3}{2} \cap E_1$.

**Remark 1.2.** (1) To estimate $v$ in $L^\infty$, we will use the Oseen kernel representation. The definition of the Oseen kernel and its property will be presented in section 4.

(2) We take different ranges of $p$ and $q$ to avoid embedding $\dot{H}^1 \subset \dot{B}_{p,q}^{\frac{3}{2}}$ and $\dot{B}_{p,q}^{\frac{3}{2}} \subset \dot{H}^1$.

1.2.2. **Analyticity.** To show that solutions in Theorem 1.4 are analytic using Gevrey regularity, we need to define the Gevrey operator more carefully. Let
\[
\Theta(t) = e^{\frac{4}{2}tA_1} \theta(t), \quad V(t) = e^{\frac{4}{2}tA_1} v(t).
\]
Then, $\Theta$ and $V$ satisfy the following equation;

\[
\Theta(t) = e^{\frac{4}{2}tA_1-tA_0} - \int_0^t \left[ e^{\frac{4}{2}(t-s)} A \left( e^{-\frac{1}{2} s A_1} V \cdot \nabla e^{-\frac{1}{2} s A_1} \Theta \right) (s) \right] ds.
\]

As we will see in Section 4, the fact $\frac{1}{4} |\xi|_1 < \frac{1}{2} |\xi|$ allows that (1.11) is equivalent to the integral equation of $\theta$. Therefore, we can prove the following result along the lines of the proof of Theorem 1.3 and Theorem 1.4.

**Theorem 1.5** (Analyticity). Let $2 \leq p \leq \infty$ and $1 \leq q < 2$, or $1 \leq p < 2$ and $q \geq 2$. There exists a constant $\epsilon_0 > 0$ such that for all initial data $\theta_0 \in L^\infty \cap \dot{H}^1 \cap \dot{B}_{p,q}^{\frac{3}{2}}$ and $v_0 \in L^\infty$, with
\[
\|\theta_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|\theta_0\|_{\dot{H}^1} < \epsilon_0,
\]
there exists a global-in-time solution $\Theta \in L_t^\infty(L^\infty \cap \dot{H}^1) \cap L_t^2\dot{H}^\frac{3}{2} \cap E_1$.

**Remark 1.3.** In [12], they proved that for initial data $\theta_0 \in \dot{H}^2$ with the smallness condition to $\|\theta_0\|_{L^\infty}$, there exists $t_0 > 0$ such that the solutions of the critical quasi-geostrophic equations are analytic for $t > t_0$. By contrast, our result indicates that the solution of Theorem 1.5 is analytic immediately.

2. **Notations: the Littlewood-Paley decomposition and paraproduct**

We begin with some notation.

(i) $L^p(0,T;X)$ denotes the Banach set of Bochner measurable functions $f$ from $(0,T)$ to $X$ endowed with either the norm \( \left( \int_0^T \|f(\cdot,t)\|_X^p \, dt \right)^{\frac{1}{p}} \) for $1 \leq p < \infty$ or \( \sup_{0 \leq t \leq T} \|f(\cdot,t)\|_X \) for $p = \infty$. For $T = \infty$, we use $L^p_t X$ instead of $L^p(0,\infty;X)$.

(ii) For a sequence \( \{a_j\}_{j \in \mathbb{Z}} \), \( \{a_j\}_B := \left( \sum_{j \in \mathbb{Z}} |a_j|^p \right)^{\frac{1}{p}} \), with the usual change for $q = \infty$.

(iii) $A \lesssim B$ means there is a constant $C$ such that $A \leq CB$. 
We next provide notation and definitions in the Littlewood-Paley theory. We take a couple of smooth functions \((\chi, \varphi)\) supported on \(\{\xi; |\xi| \leq 1\}\) with values in \([0, 1]\) such that for all \(\xi \in \mathbb{R}^d\),

\[
\chi(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j} \xi) = 1, \quad \psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi)
\]

and we denote \(\psi(2^{-j} \xi)\) by \(\psi_j(\xi)\). The homogeneous dyadic blocks and lower frequency cut-off functions are defined by

\[
\triangle_j u = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) u(x - y) dy, \quad S_j u = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^j y) u(x - y) dy,
\]

with \(h = \mathcal{F}^{-1} \psi\) and \(\tilde{h} = \mathcal{F}^{-1} \chi\). Then, we can define the homogeneous Littlewood-Paley decomposition by

\[
\triangle_j u = \psi_j \text{in} \mathcal{S}', \quad u = \sum_{j \in \mathbb{Z}} \triangle_j u \text{in} \mathcal{S}',
\]

where \(\mathcal{S}'\) is the space of tempered distributions \(u\) such that \(\lim_{j \to -\infty} S_j u = 0\) in \(\mathcal{S}'\). Using this decomposition, we define stationary/ time dependent homogeneous Besov spaces as follows:

\[
\dot{B}^s_{p,q} = \left\{ f \in \mathcal{S}' ; \|f\|_{\dot{B}^s_{p,q}} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty \right\},
\]

\[
L^r(0,T; \dot{B}^s_{p,q}) = \left\{ f \in \mathcal{S}' ; \|f\|_{L^r(0,T; \dot{B}^s_{p,q})} := \left( \int_0^T \|\sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty \right\},
\]

\[
\dot{L}^r(0,T; \dot{B}^s_{p,q}) = \left\{ f \in \mathcal{S}' ; \|f\|_{\dot{L}^r(0,T; \dot{B}^s_{p,q})} := \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f\|_{L^p(0,T; L^r)}^q \right)^{\frac{1}{q}} < \infty \right\}
\]

with the usual change if \(q = \infty\). According to the Minkowski inequality, we have

\[
\|f\|_{\dot{L}^r(0,T; \dot{B}^s_{p,q})} \leq \|f\|_{L^r(0,T; \dot{B}^s_{p,q})} \text{ if } r \leq q,
\]

\[
\|f\|_{\dot{L}^r(0,T; \dot{B}^s_{p,q})} \geq \|f\|_{L^r(0,T; \dot{B}^s_{p,q})} \text{ if } r \geq q.
\]

The concept of paraproduct enables to deal with the interaction of two functions in terms of low or high frequency parts, \([11]\). For two tempered distributions \(f\) and \(g\),

\[
fg = Tfg + Tg f + R(f, g),
\]

\[
Tfg = \sum_{i \leq j \leq 2} \triangle_i f \triangle_j g = \sum_{j \in \mathbb{Z}} S_{j-1} f \triangle_j g, \quad R(f, g) = \sum_{|j - j'| \leq 1} \triangle_j f \triangle_{j'} g.
\]

Then, up to finitely many terms,

\[
\triangle_j (Tfg) = S_{j-1} f \triangle_j g, \quad \triangle_j R(f, g) = \sum_{k \geq j - 2} \triangle_k f \triangle_k g.
\]

We finally recall a few inequalities which will be used in the sequel.
Bernstein’s inequality \([11]\). For \(1 \leq p \leq q \leq \infty\) and \(k \in \mathbb{N}\),
\[
\sup_{|\alpha|=k} \| \partial^\alpha \Delta_j f \|_{L^p} \simeq 2^j k \| \Delta_j f \|_{L^p}, \quad \| \Delta_j f \|_{L^q} \lesssim 2^j k \| \Delta_j f \|_{L^p}.
\]
\[(2.7)\]

Localization of the fractional heat kernel \([25]\).
\[
\| e^{-t \Lambda^\gamma} \Delta_j f \|_{L^p} \lesssim e^{-\frac{2}{\gamma} j} \| \Delta_j f \|_{L^p}.
\]
\[(2.8)\]

3. Subcritical case: Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

In this section, we prove the existence and analyticity of the subcritical quasi-geostrophic equations with large initial data in \(\dot{B}^{\frac{2}{p}+\frac{1}{q}+1-\gamma}_{p,q} \cap L^{p_1}\). As we already mentioned in the introduction, we need the following existence result in \(G_{\gamma,T}\) to show the existence of a solution in \(E_{\gamma,T}\). We recall the definition of the norm of \(G_{\gamma,T}\):
\[
\| \theta \|_{G_{\gamma,T}} = \sup_{0 < t < T} \left[ \| \theta(t) \|_{L^{p_1}} + t^\beta \| \theta(t) \|_{L^r} \right], \quad \beta = 1 - \frac{1}{\gamma} - \frac{2}{r^\gamma} > 0.
\]
\[(3.1)\]

**Theorem 3.1** \([9]\). Let \(1 < \gamma < 2\) and \(\theta_0 \in L^{p_1}\) and \(\frac{2}{\gamma} < p_1 < r < \infty\). Then, there exists a global-in-time solution \(\theta\) in \(G_{\gamma,\infty}\). Moreover, for any time interval \([t, t+\tau]\),
\[
\lim_{\tau \to 0} \left[ t^\beta \| \theta(t+\tau) \|_{L^r} \right] = 0
\]
uniformly in \(t\).

We will use Theorem 3.1 to show the global existence of a solution in \(K_{\gamma,T}\). Let us recall the definition of the norm of \(K_{\gamma,T}\):
\[
\| \theta \|_{K_{\gamma,T}} = \sup_{0 < t < T} \left[ \| \theta(t) \|_{B^{\frac{2}{p}+1-\gamma}_{p,q}} + t^\alpha \| \theta(t) \|_{B^{\frac{2}{r}+1-\gamma+a}_{p,q}} \right]
\]
\[(3.2)\]

We note that the time weights appearing in (3.2) is introduced by the bounds of the fractional Laplacian in the Fourier space:
\[
|\xi|^a e^{-t |\xi|^\gamma} \lesssim t^{-\frac{a}{\gamma}}.
\]
\[(3.3)\]
In addition, we will use the following lemma repeatedly in the proof of Theorem 1.1.

**Lemma 3.1.** For any \(0 < a < 1\) and \(0 < b < 1\),
\[
\int_0^t [(t-s)^{-a}s^{-b}] ds \lesssim t^{1-a-b}.
\]

**Proof.** We can prove this easily by decomposing the time integral into two parts.
\[
\int_0^t [(t-s)^{-a}s^{-b}] ds = \int_0^{\frac{t}{2}} [(t-s)^{-a}s^{-b}] ds + \int_{\frac{t}{2}}^t [(t-s)^{-a}s^{-b}] ds
\]
\[
\lesssim t^{-a} \int_0^{\frac{t}{2}} s^{-b} ds + t^{-b} \int_{\frac{t}{2}}^t (t-s)^{-a} ds.
\]
Integrating in time, we complete the proof. \(\square\)
3.1. Existence: Proof of Theorem 1.1. We first show that the following a priori estimate

\[ \| \theta \|_{K_{\gamma,T}} \lesssim \| \theta_0 \|_{B^{\gamma+1}_{p,q}} + \sup_{0 < \tau \leq T} \left[ \tau^\beta \| \theta(\tau) \|_{L^p} \right] \| \theta \|_{K_{\gamma,T}} \]  

implies (1.7), the proof comes as follows. We take \( T_1 \) such that \( \tau^\beta \| \theta(\tau) \|_{L^p} \leq \frac{1}{2} \) on \([0, T_1]\). Then, \( \| \theta \|_{K_{\gamma,T}} \leq 2\| \theta_0 \|_{B^{\gamma+1}_{p,q}} \) on \([0, T_1]\). In particular, \( \| \theta(T_1) \|_{B^{\gamma+1}_{p,q}} \leq 2\| \theta_0 \|_{B^{\gamma+1}_{p,q}} \). By Theorem 3.1, we can take the next step by taking \( T_2 = 2T_1 \). Then, \( \| \theta(t) \|_{B^{\gamma+1}_{p,q}} \leq 4\| \theta_0 \|_{B^{\gamma+1}_{p,q}} \) on \([T_1, 2T_1]\). Inductively, we can obtain that for \( t \in [nT_1, (n + 1)T_1] \),

\[ \| \theta \|_{K_{\gamma,T}} \lesssim 2^{(n+1)}\| \theta_0 \|_{B^{\gamma+1}_{p,q}}, \]

which implies (1.7).

To obtain (3.4), we express a solution \( \theta \) in the integral form:

\[ \theta(t) = e^{-t\Lambda^\gamma} \theta_0 - \int_0^t \left[ \nabla e^{-(t-s)\Lambda^\gamma} \cdot (v\theta)(s) \right] ds = e^{-t\Lambda^\gamma} \theta_0 - B(v, \theta). \]

It is enough to estimate \( B(v, \theta) \) in \( K_{\gamma,T} \). Let us decompose \( \theta \) as a paraproduct: \( v\theta = T_\gamma \theta + T_\delta \theta + R(v, \theta) \). Then,

\[ B(v, \theta) = \int_0^t \left[ \nabla e^{-(t-s)\Lambda^\gamma} \cdot (v\theta)(s) \right] ds \]

\[ = \int_0^t \left[ \nabla e^{-(t-s)\Lambda^\gamma} \cdot (T_\gamma \theta + T_\delta \theta + R(v, \theta)) \right] ds := B_1(v, \theta) + B_2(v, \theta) + B_3(v, \theta). \]

In the sequel, we will treat \( v \) as \( \theta \) in the estimations of \( B_1(v, \theta) \) s in \( K_{\gamma,T} \) because \( v \) is the image of \( \theta \) under the Riesz transforms and these transforms are bounded in \( L^p \) for \( 1 < s < \infty \).

3.1.1. Estimation of \( B_1(v, \theta) \) and \( B_2(v, \theta) \). We take \( \triangle \) to \( B_1(v, \theta) \) and take the \( L^p \) norm. By (2.6)-(2.8), we have

\[ \| \triangle_j B_1(v, \theta)(t) \|_{L^p} \lesssim \int_0^t \left[ 2^j e^{-(t-s)2^{2j}} \| S_{j-1} \theta(s) \|_{L^\infty} \| \triangle_j \theta(s) \|_{L^p} \right] ds. \]

We now estimate \( \| S_{j-1} \theta \|_{L^\infty} \):

\[ \| S_{j-1} \theta(s) \|_{L^\infty} \lesssim \sum_{k = -\infty}^{j-1} 2^{2k} \| \triangle_j \theta(s) \|_{L^\infty} \lesssim 2^{2j} s^{-\beta} \| \theta \|_{L^p} \]

from which we have

\[ \| \triangle_j B_1(v, \theta)(t) \|_{L^p} \lesssim \int_0^t \left[ 2^j e^{-(t-s)2^{2j}} \| S_{j-1} \theta(s) \|_{L^\infty} \| \triangle_j \theta(s) \|_{L^p} \right] ds \]

\[ \lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \| \theta(\tau) \|_{L^p} \right] \int_0^t \left[ 2^{j(1+\frac{\beta}{2})} e^{-(t-s)2^{2j}} s^{-\beta} \| \triangle_j \theta(s) \|_{L^p} \right] ds. \]

From (3.3), we have \( 2^{j(1+\frac{\beta}{2})} e^{-(t-s)2^{2j}} \lesssim (t-s)^{-\frac{1}{2}(1+\frac{\beta}{2})} \). Therefore,

\[ \| \triangle_j B_1(v, \theta)(t) \|_{L^p} \lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \| \theta(\tau) \|_{L^p} \right] \int_0^t \left[ (t-s)^{-\frac{1}{2}(1+\frac{\beta}{2})} s^{-\beta} \| \triangle_j \theta(s) \|_{L^p} \right] ds. \]
We multiply (3.9) by $2^{j(\frac{3}{4}+1-\gamma)}$ and take the $l^q$ norm. Then,
\[
\|B_1(v,\theta)(t)\|_{B^\gamma_{p,q}} \lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \sup_{0 < \tau \leq t} \left[ \|\theta(\tau)\|_{B^{\gamma+1}_{p,q}} \right] \int_0^t \left[ (t-s)^{-\frac{1}{2}(1+\frac{3}{4})} s^{-\beta} \right] ds
\]
(3.10)
\[
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,t}} \int_0^t \left[ (t-s)^{-\frac{1}{2}(1+\frac{3}{4})} s^{-\beta} \right] ds
\]
\[
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,t}},
\]
where we use $\beta = 1 - \frac{1}{\gamma} - \frac{2}{\gamma^2}$ and Lemma 3.1 to bound the time integration by a constant.

Next, we estimate $\|B_1(v,\theta)\|_{B^\gamma_{p,q}}$. From (3.9),
\[
2^{j(\frac{3}{4}+1-\gamma+\alpha)} \|\triangle_j B_1(v,\theta)(t)\|_{L^p}
\]
(3.11)
\[
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \int_0^t \left[ (t-s)^{-\frac{1}{2}(1+\frac{3}{4})} s^{-\beta} 2^{j(\frac{3}{4}+1-\gamma+\alpha)} \|\triangle_j \theta(s)\|_{L^p} \right] ds
\]
\[
= \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \int_0^t \left[ (t-s)^{-\frac{1}{2}(1+\frac{3}{4})} s^{-\beta} s^{-\frac{\alpha}{2}} 2^{j(\frac{3}{4}+1-\gamma+\alpha)} \|\triangle_j \theta(s)\|_{L^p} \right] ds.
\]
By taking the $l^q$ norm, we have
\[
\|B_1(v,\theta)(t)\|_{B^{\gamma+1}_{p,q}}
\]
(3.12)
\[
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \sup_{0 < \tau \leq t} \left[ \tau^\gamma \|\theta(\tau)\|_{B^{\gamma+1}_{p,q}} \right] \int_0^t \left[ (t-s)^{-\frac{1}{2}(1+\frac{3}{4})} s^{-\beta} s^{-\frac{\alpha}{2}} \right] ds
\]
\[
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,t}} \int_0^t \left[ (t-s)^{-\frac{1}{2}(1+\frac{3}{4})} s^{-\beta} s^{-\frac{\alpha}{2}} \right] ds
\]
\[
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,t}} \cdot t^{-\frac{\alpha}{2}},
\]
where we use $\beta + \frac{\alpha}{2} < 1$ to apply Lemma 3.1 to bound the time integration by $t^{-\frac{\alpha}{2}}$. By (3.10) and (3.12), for any time interval $[0,T]$ we have
\[
\|B_1(v,\theta)\|_{K_{\gamma,T}} \lesssim \sup_{0 \leq \tau \leq T} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,T}}.
\]
(3.13)
Since $B_2(v,\theta)$ has the same structure as $B_1$, we also obtain that
\[
\|B_2(v,\theta)\|_{K_{\gamma,T}} \lesssim \sup_{0 \leq \tau \leq T} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,T}}.
\]
(3.14)
3.1.2. Estimation of $\mathcal{B}_3(v, \theta)$. As for $\mathcal{B}_i(v, \theta)$, $i = 1, 2$, we take the $L^p$ norm to $\triangle_j \mathcal{B}_3(v, \theta)$.

\[
\|\triangle_j \mathcal{B}_3(v, \theta)(t)\|_{L^p} \lesssim \int_0^t \left[ 2^j e^{-(t-s)2\gamma_j} \sum_{k \geq j-2} \|\triangle_k \theta(s)\|_{L^p} \|\triangle_k \theta(s)\|_{L^p} \right] ds \\
\lesssim \int_0^t \left[ 2^j e^{-(t-s)2\gamma_j} \sum_{k \geq j-2} \|\triangle_k \theta(s)\|_{L^p} 2^{k^2} \|\triangle_k \theta(s)\|_{L^p} \right] ds \\
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \int_0^t \left[ 2^j e^{-(t-s)2\gamma_j} s^{-\beta} \sum_{k \geq j-2} 2^{k^2} \|\triangle_k \theta(s)\|_{L^p} \right] ds,
\]

(3.15)

where we use (2.7) to replace $\|\triangle_k \theta(s)\|_{L^\infty}$ by $2^{k^2} \|\triangle_k \theta(s)\|_{L^p}$ at the second inequality. We multiply (3.15) by $2^{j(\frac{2}{p}+1-\gamma)}$. Then,

\[
2^{j(\frac{2}{p}+1-\gamma)} \|\triangle_j \mathcal{B}_3(v, \theta)(t)\|_{L^p} \\
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \int_0^t \left[ 2^j (\frac{2}{p}+1-\gamma) s^{-\beta} \sum_{k \geq j-2} 2^{j-k}(\frac{2}{p}+1-\gamma) 2^{k^2} \|\triangle_k \theta(s)\|_{L^p} \right] ds \\
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \int_0^t \left[ (t-s)^{-\frac{1}{2}}(\frac{2}{p}+1-\gamma) s^{-\beta} \sum_{k \geq j-2} 2^{j-k}(\frac{2}{p}+1-\gamma) 2^{k^2} \|\triangle_k \theta(s)\|_{L^p} \right] ds.
\]

(3.16)

We take the $l^q$ norm to (3.16). Since $\frac{2}{p}+1-\gamma-\frac{2}{q} > 0$, by applying Young’s inequality to $\sum k a_{k-j} b_k$, where $a_j = 2^{-j(\frac{2}{p}+1-\gamma-\frac{2}{q})}$ and $b_j = 2^{j(\frac{2}{p}+1-\gamma)} \|\triangle_k \theta(s)\|_{L^p}$, we have

\[
\|\mathcal{B}_3(v, \theta)(t)\|_{B^\frac{2}{p}+1-\gamma}_{p,q} \lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \sup_{0 < \tau \leq t} \left[ \|\theta(\tau)\|_{B^\frac{2}{p}+1-\gamma}_{p,q} \right] \int_0^t \left[ (t-s)^{-\frac{1}{2}}(\frac{2}{p}+1-\gamma) s^{-\beta} \right] ds \\
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,t}} \int_0^t \left[ (t-s)^{-\frac{1}{2}}(\frac{2}{p}+1-\gamma) s^{-\beta} \right] ds \\
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \|\theta\|_{K_{\gamma,t}}.
\]

(3.17)

Next, we estimate $\mathcal{B}_3(v, \theta)$ in $B^\frac{2}{p}+1-\gamma+\alpha$. By multiplying (3.15) by $2^{j(\frac{2}{p}+1-\gamma+\alpha)}$ and following the calculation in (3.16), we have

\[
2^{j(\frac{2}{p}+1-\gamma+\alpha)} \|\triangle_j \mathcal{B}_3(v, \theta)(t)\|_{L^p} \\
\lesssim \sup_{0 < \tau \leq t} \left[ \tau^\beta \|\theta(\tau)\|_{L^p} \right] \times \int_0^t \left[ (t-s)^{-\frac{1}{2}}(\frac{2}{p}+1-\gamma+\alpha) s^{-\beta} \sum_{k \geq j-2} 2^{j-k}(\frac{2}{p}+1-\gamma+\alpha) 2^{k^2} \|\triangle_k \theta(s)\|_{L^p} \right] ds.
\]

(3.18)
We recall the equation of $\Theta$ Proof of Theorem 1.2.

3.2.1. □

The claim follows from Hormander’s multiplier theorem, e.g., [35].

We take the

\[ \text{Proof.} \]

Lemma 3.3.

this kernel is bounded by a constant independent of $k$.

Clearly, $\text{consider the operator } E := e^{-[(t-s)\frac{1}{\gamma} + s \frac{1}{\gamma} - t \frac{\gamma}{p}]} A_1$ for $0 \leq s \leq t$. Then $E$ is either the identity operator or is an $L^1$ kernel whose $L^1$ norm is bounded independent of $s, t$.

Proof. Clearly, $a := (t-s)\frac{1}{\gamma} + s \frac{1}{\gamma} - t \frac{\gamma}{p}$ is non-negative for $s \leq t$. In case $a = 0$, $E = e^{-aA_1}$ is the identity operator, while if $a > 0$, $E = e^{-aA_1}$ is a Fourier multiplier with symbol $\hat{E}(\xi) = \prod_{i=1}^{d} e^{-a|\xi_i|}$. Thus, the kernel of $E$ is given by the product of one dimensional Poisson kernels $\prod_{i=1}^{d} \frac{a}{\pi(a^2 + x_i^2)}$. The $L^1$ norm of this kernel is bounded by a constant independent of $a$.

\[ \text{Lemma 3.3. The operator } E = e^{\frac{\gamma}{p}A_1 - \frac{s}{\gamma}aA_1}, \gamma > 1, \text{ is a Fourier multiplier which maps boundedly } L^p \to L^p, 1 < p < \infty, \text{ and its operator norm is uniformly bounded with respect to } a \geq 0. \]

Proof. When $a = 0$, $E$ is the identity operator. When $a > 0$, then $E$ is Fourier multiplier with symbol $\hat{E}(\xi) = e^{a\frac{\gamma}{2} |\xi|^\gamma - \frac{1}{2} a |\xi|^\gamma}$. Since $\hat{E}(\xi)$ is uniformly bounded for all $\xi$ and decays exponentially for $|\xi| \gg 1$, the claim follows from Hormander’s multiplier theorem, e.g., [35].

Prove of Theorem 1.2. We are now ready to prove a large time existence of $\Theta$ in the $L^p$ space. We recall the equation of $(\Theta, V)$:

\[ \Theta(t) = e^{t(\frac{\gamma}{p}A_1 - tA_1^\gamma)} \theta_0 - \int_0^t \left[ e^{(\frac{\gamma}{p}A_1 - (s-t)A_1^\gamma) \nabla \cdot (e^{-s\frac{1}{p}A_1 V e^{-s\frac{\gamma}{p}A_1} \Theta})} \right] ds \]

\[ := e^{t(\frac{\gamma}{p}A_1 - tA_1^\gamma)} \theta_0 - B(\Theta, V). \]
By Lemma 3.3, it is easy to show that the linear part is equivalent to
\[ e^{-\frac{1}{2}t\Lambda^\gamma} \theta_0 \]
which only depends of \( \theta \), not on \( \Theta \). Therefore, if we show that the nonlinear part \( \mathcal{B}(\Theta, V) \) is also equivalent to
\[
\int_0^t \left[ e^{-\frac{1}{2}(t-s)\Lambda^\gamma} \nabla \cdot (V\Theta)(s) \right] ds
\]
in terms of estimation, then we can follow [9] to complete the proof.

We rewrite \( \mathcal{B}(V, \Theta) \) as follows.
\[
\mathcal{B}(V, \Theta)(t) = \int_0^t \left[ e^{(t-s)^\frac{1}{2} \Lambda_1} \right. \left. - e^{(t-s)^\frac{1}{2} \Lambda_1} \nabla e^{-\frac{1}{2}(t-s)\Lambda^\gamma} \cdot e^{s^\frac{1}{2} \Lambda_1} (e^{-s^\frac{1}{2} \Lambda_1} V e^{-s^\frac{1}{2} \Lambda_1} \Theta)(s) \right] ds.
\]
We now express \( (t^\frac{1}{2} - s^\frac{1}{2}) \) as
\[ -(t-s)^\frac{1}{2} - t^\frac{1}{2} + s^\frac{1}{2} \]
By Lemma 3.2, (3.24) can be estimated by
\[
\mathcal{B}(V, \Theta)(t) \lesssim \int_0^t \left[ e^{(t-s)^\frac{1}{2} \Lambda_1} \right. \left. - e^{(t-s)^\frac{1}{2} \Lambda_1} \nabla e^{-\frac{1}{2}(t-s)\Lambda^\gamma} \cdot e^{s^\frac{1}{2} \Lambda_1} (e^{-s^\frac{1}{2} \Lambda_1} V e^{-s^\frac{1}{2} \Lambda_1} \Theta)(s) \right] ds.
\]
By Lemma 3.3, the right-hand side of (3.25) can be replaced by
\[
\mathcal{B}(V, \Theta)(t) \lesssim \int_0^t \left[ \nabla e^{-\frac{1}{2}(t-s)\Lambda^\gamma} \cdot e^{s^\frac{1}{2} \Lambda_1} (e^{-s^\frac{1}{2} \Lambda_1} V e^{-s^\frac{1}{2} \Lambda_1} \Theta)(s) \right] ds.
\]
Using
\[ \| \nabla e^{-t\Lambda^\gamma} f \|_{L^r} \lesssim t^{-\frac{1}{2} - \frac{1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \| f \|_{L^q}, \]
we estimate (3.26) as
\[
\| \mathcal{B}(V, \Theta)(t) \|_{L^r} \lesssim \int_0^t (t-s)^{-\frac{1}{2} - \frac{1}{2} \left( \frac{1}{q} - \frac{1}{2} \right)} \| e^{s^\frac{1}{2} \Lambda_1} (e^{-s^\frac{1}{2} \Lambda_1} V e^{-s^\frac{1}{2} \Lambda_1} \Theta)(s) \|_{L^q} ds.
\]
To estimate the right-hand side of (3.27), we introduce the bilinear operators \( B_t \) of the form
\[
B_t(f, g) = e^{t^\frac{1}{2} \Lambda_1} (e^{-s^\frac{1}{2} \Lambda_1} f e^{-s^\frac{1}{2} \Lambda_1} g)
\]
(3.28)
\[
= \int_{R^2} \int_{R^2} e^{ix \cdot (\xi + \eta)} e^{t^\frac{1}{2} \left( |\xi| - |\xi|_1 - |\eta|_1 - |\eta| \right)} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.
\]
Recall that for a vector \( \xi = (\xi_1, \xi_2) \), we denoted \( |\xi|_1 = |\xi_1| + |\xi_2| \). For \( \xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \), we now split the domain of integration of the above integral into sub-domains depending on the sign of \( \xi_j, \eta_j \) and \( \xi_j + \eta_j \). In order to do so, we introduce the operators acting on one variable (see page 253 in [30]) by
\[
K_1 f = \frac{1}{2\pi} \int_0^\infty e^{ix\xi} \hat{f}(\xi) d\xi, \quad K_{-1} f = \frac{1}{2\pi} \int_0^\infty e^{ix\xi} \hat{f}(\xi) d\xi.
K_{-1} f = \frac{1}{2\pi} \int_0^\infty e^{ix\xi} \hat{f}(\xi) d\xi.
\]
Let the operators \( L_{t,-1} \) and \( L_{t,1} \) be defined by
\[
L_{t,1} f = f, \quad L_{t,-1} f = \frac{1}{2\pi} \int_{R^2} e^{ix \xi} e^{-2t|\xi|} \hat{f}(\xi) d\xi.
\]
For \( \vec{\alpha} = (\alpha_1, \alpha_2), \vec{\beta} = (\beta_1, \beta_2) \in \{-1, 1\}^2 \), denote the operator
\[
Z_{t,\vec{\alpha},\vec{\beta}} = K_{\beta_1} L_{t,\alpha_1,\beta_1} \otimes \cdots \otimes K_{\beta_2} L_{t,\alpha_2,\beta_2}, \quad K_{\vec{\alpha}} = k_{\alpha_1} \otimes K_{\alpha_2}.
\]
The above tensor product means that the \( j \)-th operator in the tensor product acts on the \( j \)-th variable of the function \( f(x_1, x_2) \). A tedious (but elementary) calculation now yields the following identity:
\[
(3.39) \quad B_t(f, g) = \sum_{(\vec{\alpha},\vec{\beta},\vec{\gamma}) \in \{-1, 1\}^{2 \times 2}} K_{\alpha_1} K_{\alpha_2} (Z_{t,\vec{\alpha},\vec{\beta}} f Z_{t,\vec{\alpha},\vec{\gamma}} g).
\]
We note that the operators \( K_{\vec{\alpha}}, Z_{t,\vec{\alpha},\vec{\beta}} \) defined above, being linear combinations of Fourier multipliers (including Hilbert transform) and the identity operator, commute with \( \Lambda_1 \) and \( \Lambda \). Moreover, they are bounded linear operators on \( L^p, 1 < p < \infty \) and the corresponding operator norm of \( Z_{t,\vec{\alpha},\vec{\beta}} \) is bounded independent of \( t \geq 0 \). Therefore,
\[
(3.30) \quad \|B_t(f, g)\|_{L^q} \lesssim \|fg\|_{L^p}
\]
We apply the above argument to the right-hand side of (3.27) to conclude that
\[
(3.31) \quad \|\mathcal{B}(V, \Theta)(t)\|_{L^r} \lesssim \int_0^t (t-s)^{-\frac{1}{2} - \frac{\gamma}{p} (\frac{1}{2} - \frac{1}{q})} \|V(s)\|_{L^1} \|\Theta(s)\|_{L^r} \, ds.
\]
We now follow the proof in [9] line by line to complete the proof.

### 3.2.2. Proof of Theorem 1.3.
As for the proof of theorem 1.1, we only need to obtain the following a priori estimate:
\[
(3.32) \quad \|\Theta\|_{K_{s,T}} \lesssim \|\theta_0\|_{B^{\frac{3}{2}+1-\gamma}_{p,q}} + \sup_{0 < \tau \leq T} \left[ \tau^\beta \|\Theta(\tau)\|_{L^r} \right] \|\Theta\|_{K_{s,T}}.
\]
By Lemma 3.3, the linear estimation is obvious. By following the proof of Theorem 1.2, we can estimate the nonlinear term as those of \( \theta \). For the reader’s convenience, we provide a few lines.

We take \( \triangle_j \) to \( \mathcal{B}(V, \Theta) \) in (3.22) and take the \( L^p \) norm. By Lemma 3.2 and Lemma 3.3,
\[
(3.33) \quad \|\triangle_j e^{t\Lambda_1} \mathcal{B}(V, \Theta)\|_{L^p} \lesssim \int_0^t \left[ e^{-\frac{1}{2}(t-s)\gamma} \right] ds \left[ e^{s\frac{1}{2}\Lambda_1} \|\Theta(e^{-s\frac{1}{2}\Lambda_1} V)\|_{L^p} \right] ds.
\]
We decompose the product \( e^{-s\frac{1}{2}\Lambda_1} V e^{-s\frac{1}{2}\Lambda_1} \) as paraproduct:
\[
T(e^{-s\frac{1}{2}\Lambda_1} V) e^{-s\frac{1}{2}\Lambda_1} + T(e^{-s\frac{1}{2}\Lambda_1} \Theta) e^{-s\frac{1}{2}\Lambda_1} V + R(e^{-s\frac{1}{2}\Lambda_1} V, e^{-s\frac{1}{2}\Lambda_1} \Theta).
\]
Then,
\[
(3.34) \quad \|\triangle_j e^{t\Lambda_1} \mathcal{B}(V, \Theta)\|_{L^p} \lesssim \int_0^t \left[ e^{-\frac{1}{2}(t-s)\gamma} \right] ds \left[ e^{s\frac{1}{2}\Lambda_1} \|\Theta(e^{-s\frac{1}{2}\Lambda_1} S_j V)\|_{L^p} \right] ds
\]
\[
+ \int_0^t \left[ e^{-\frac{1}{2}(t-s)\gamma} \right] ds \left[ e^{s\frac{1}{2}\Lambda_1} \|\Theta(e^{-s\frac{1}{2}\Lambda_1} S_j \Theta e^{-s\frac{1}{2}\Lambda_1} \triangle_j V)\|_{L^p} \right] ds
\]
\[
+ \int_0^t \sum_{k \geq j} \left[ e^{-\frac{1}{2}(t-s)\gamma} \right] ds \left[ e^{s\frac{1}{2}\Lambda_1} \|\Theta(e^{-s\frac{1}{2}\Lambda_1} \triangle_k V e^{-s\frac{1}{2}\Lambda_1} \triangle_k \Theta)\|_{L^p} \right] ds
\]
By using (3.30), we can estimate (3.34) as
\[\left\| \Delta_j e^{t\frac{1}{2} \Lambda_1 \mathcal{R}(\mathbf{V}, \Theta)} \right\|_{L^p} \lesssim \int_0^t \left[ e^{-\frac{1}{2} (t-s)^2 \frac{2}{3}^j} \left( \left\| S_j \mathbf{V}(s) \Delta_j \Theta(s) \right\|_{L^p} + \left\| S_j \Theta(s) \Delta_j \mathbf{V}(s) \right\|_{L^p} \right) \right] ds \\
+ \int_0^t e^{-\frac{1}{2} (t-s)^2 \frac{2}{3}^j} \sum_{k \geq j-2} \left[ \left\| \Delta_k \mathbf{V}(s) \Delta_k \Theta(s) \right\|_{L^p} \right] ds \]

Therefore, we can follow the calculations line by line from (3.6) to (3.21) in the proof of Theorem 1.1 to complete the proof.

4. Critical Case: Proof of Theorem 1.4 and Theorem 1.5

To prove Theorem 1.4 and Theorem 1.5, we need several lemmas. First, to estimate \( \mathbf{v} \) in \( L^\infty \), we need the following representation.

**Lemma 4.1. Oseen Kernel [30]:** The operator \( O_t = \mathcal{R} \mathcal{K}_t \) is a convolution operator whose kernel \( \tilde{K}_t \) satisfies
\[ \tilde{K}_t(x) = \frac{1}{t^d} \tilde{K} \left( \frac{x}{t} \right) \]
for a smooth function \( \tilde{K} \) such that for all \( \alpha \in \mathbb{N}^d \), \( (1 + |x|)^{d+\alpha} \partial^\alpha \tilde{K} \in L^\infty \). In particular, \( \tilde{K} \) is in \( L^p \) for \( p > 1 \).

Next, we provide the \( L^p \) bounds of the Poisson kernel and its Oseen kernel in two dimensions. The proof is easily obtained by their representation.

**Lemma 4.2. For any \( 1 < p < \infty \),**
\[ \| P_t \|_{L^p} \lesssim t^{-2(1-\frac{1}{p})}, \quad \| \mathcal{R} P_t \|_{L^p} \lesssim t^{-2(1-\frac{1}{p})}. \]

Finally, to deal with the time singularities appearing in Lemma 4.2, we need the following lemma.

**Lemma 4.3. Hardy-Littlewood-Sobolev Inequality [31]:** Let \( 0 < \lambda < d, \ \frac{1}{p} + \frac{\lambda}{d} + \frac{1}{q} = 2 \). Then,
\[ \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \frac{1}{|x-y|^\lambda} g(y) dy dx \right| \lesssim \| f \|_{L^p} \| g \|_{L^q}. \]

In particular, for the one dimensional case,
\[ \sup_{t > 0} \left| \int_0^t \frac{1}{t-s} a(s) ds \right| \lesssim \| a \|_{L^2}. \]

4.1. Existence: Proof of Theorem 1.4. For the reader’s convenience, we will repeat the calculations in [3]. We begin with the estimation in \( H^1 \cap L^\infty \).

4.1.1. \( H^1 \) bound. By taking one derivative \( \nabla \) to (1.1),
\[ \nabla \theta_t + \mathbf{v} \cdot \nabla \nabla \theta + \nabla \mathbf{v} \cdot \nabla \Theta - \Lambda \nabla \Theta = 0. \]

We multiply (4.1) by \( \nabla \theta \) and integrate over \( \mathbb{R}^2 \) to do the energy estimate. Then,
\[ \frac{1}{2} \frac{d}{dt} \| \nabla \theta \|_{L^2}^2 + \| \nabla^2 \theta \|_{L^2}^2 \leq \int_{\mathbb{R}^2} |\nabla \mathbf{v}| \| \nabla \theta \| \| \nabla \mathbf{v} \| \ dx \leq \| \nabla \theta \|_{L^2} \| \nabla \mathbf{v} \|_{L^1}^2. \]
By the Sobolev embedding $\dot{H}^{\frac{1}{2}} \subset L^4$ in two dimensions, we can replace $\|\nabla v\|_{L^4}$ in (4.2) by $\|\nabla^\frac{3}{2} v\|_{L^2}$.

$$\frac{d}{dt} \|\nabla \theta\|_{L^2}^2 + \|\nabla^\frac{3}{2} \theta\|_{L^2}^2 \lesssim \|\nabla \theta\|_{L^2} \|\nabla^\frac{3}{2} v\|_{L^2}^2.$$  

Since $v = (-\mathbb{P}_2 \theta, \mathbb{P}_1 \theta)$, and the Riesz transforms map $H^s$ to $H^s$,

$$\frac{d}{dt} \|\nabla \theta\|_{L^2}^2 + \|\nabla^\frac{3}{2} \theta\|_{L^2}^2 \lesssim \|\nabla \theta\|_{L^2} \|\nabla^\frac{3}{2} \theta\|_{L^2}^2.$$  

Integrating (4.3) in time, we have

$$\|\nabla \theta\|_{L^\infty L^2}^2 + \|\nabla^\frac{3}{2} \theta\|_{L^2_t L^2}^2 \lesssim \|\nabla \theta_0\|_{L^2}^2 + \|\nabla \theta\|_{L^\infty L^2} \|\nabla^\frac{3}{2} \theta\|_{L^2_t L^2}^2,$$

which implies that there exists a global-in-time small solution $\theta$ in $L^\infty_t \dot{H}^1 \cap L^2_t \dot{H}^{\frac{1}{2}}$ if initial data $\theta_0$ is sufficiently small in $\dot{H}^1$.

4.1.2. $L^\infty$ bound. To obtain the $L^\infty$ bound of $\theta$, we express $\theta$ as the integral form:

$$\theta(t) = e^{-t\Lambda} \theta_0 - \int_0^t \left[ \nabla e^{-(t-s)\Lambda} \cdot (v \theta)(s) \right] ds.$$  

By taking the $L^\infty$ norm, we have

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \int_0^t \|e^{-(t-s)\Lambda} (v \cdot \nabla \theta)(s)\|_{L^\infty} ds$$

$$\lesssim \|\theta_0\|_{L^\infty} + \int_0^t \|e^{-(t-s)\Lambda} \|_{L^\frac{1}{2}} \|v(s)\|_{L^\infty} \|\nabla \theta(s)\|_{L^4} ds.$$  

By Lemma 4.2,

$$\|\theta(t)\|_{L^\infty} \lesssim \|\theta_0\|_{L^\infty} + \int_0^t \frac{1}{\sqrt{t-s}} \|v(s)\|_{L^\infty} \|\nabla \theta(s)\|_{L^4} ds.$$  

By Lemma 4.3 and the Sobolev embedding $\dot{H}^1 \subset L^4$, we finally have

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \|v\|_{L^\infty L^\infty} \|\nabla \theta\|_{L^2_t L^4}$$

$$\lesssim \|\theta_0\|_{L^\infty} + \|v\|_{L^\infty L^\infty} \|\nabla^\frac{3}{2} \theta\|_{L^2_t L^2}.$$  

Next, we estimate the velocity field $v$ in $L^\infty$. For notational simplicity, we let $v = \mathbb{R} \theta$. We take $\mathbb{R}$ to (4.5).

$$v(t) = e^{-t\Lambda} v_0 - \int_0^t \mathbb{R} e^{-(t-s)\Lambda} (v \cdot \nabla \theta)(s) ds.$$  

By following the estimates of $\theta$ obtained from (4.6) to (4.8), we have

$$\|v(t)\|_{L^\infty} \lesssim \|v_0\|_{L^\infty} + \int_0^t \|\mathbb{R} e^{-(t-s)\Lambda} \|_{L^\frac{1}{2}} \|v(s)\|_{L^\infty} \|\nabla \theta(s)\|_{L^4} ds$$

$$\lesssim \|v_0\|_{L^\infty} + \int_0^t \frac{1}{\sqrt{t-s}} \|v(s)\|_{L^\infty} \|\nabla \theta(s)\|_{L^4} ds$$

$$\lesssim \|v_0\|_{L^\infty} + \|v\|_{L^\infty L^\infty} \|\nabla^\frac{3}{2} \theta\|_{L^2_t L^2}.$$
In sum, by (4.4), (4.8), and (4.9), we obtain that
\[
\|\theta\|_{L^\infty_t L^\infty_x} + \|v\|_{L^\infty_t L^\infty_x} + \|\theta\|_{L^\infty_t \dot{H}^1} + \|\nabla^3 \theta\|_{L^2_t L^2}
\]
\[
\lesssim \|\theta_0\|_{L^\infty_x} + \|v_0\|_{L^\infty_x} + \|\theta_0\|_{\dot{H}^1} + \left(\|\theta\|_{L^\infty_t L^\infty_x} + \|v\|_{L^\infty_t L^\infty_x} + \|\theta\|_{L^\infty_t \dot{H}^1} + \|\nabla^3 \theta\|_{L^2_t L^2}\right)^2,
\]
which implies that the existence of a global-in-time solution \(\theta\) and \(v\) in \(L^\infty_t (L^\infty_x \cap \dot{H}^1) \cap L^2_t \dot{H}^{3/2}\) provided that \(\|\theta_0\|_{L^\infty_x} + \|v_0\|_{L^\infty_x} + \|\theta_0\|_{\dot{H}^1}\) is sufficiently small.

4.1.3. \(B^2_{p,q}\) bound. We take \(\triangle_j\) to (4.5) and take the \(L^p\) norm. By (2.8), we have
\[
\|\triangle_j \theta(t)\|_{L^p_x} \lesssim e^{-t\|	riangle_j \theta\|_{L^p}} + \int_0^t e^{-(t-s)\|	riangle_j (v \cdot \nabla \theta)\|_{L^p}} ds.
\]
By taking the \(L^\infty\) norm in time, we obtain
\[
\|\triangle_j \theta\|_{L^\infty_t L^p_x} \lesssim \|\triangle_j \theta_0\|_{L^p_x} + \|\triangle_j (v \cdot \nabla \theta)\|_{L^1_t L^p_x}.
\]
We multiply (4.12) by \(2^{j/2}\) and take the \(l^q\) norm. Then,
\[
\|\theta\|_{L^\infty_t B^{\frac{2}{p},q}_{p,q}} \lesssim \|\theta_0\|_{B^{\frac{2}{p},q}_{p,q}} + \|v \cdot \nabla \theta\|_{L^1_t B^{\frac{2}{p},q}_{p,q}}.
\]
Similarly, by taking the \(L^1\) norm in time to (4.12), multiplying by \(2^{j(1 + \frac{2}{q})}\), and taking the \(l^q\) norm, we obtain
\[
\|\theta\|_{L^1_t B^{\frac{2}{p},1+1}_{p,q}} \lesssim \|\theta_0\|_{B^{\frac{2}{p},1+1}_{p,q}} + \|v \cdot \nabla \theta\|_{L^1_t B^{\frac{2}{p},1+1}_{p,q}}.
\]
By adding (4.13) and (4.14), we finally have
\[
\|\theta\|_{L^\infty_t B^{\frac{2}{p},1+1}_{p,q}} + \|\theta\|_{L^1_t B^{\frac{2}{p},1+1}_{p,q}} \lesssim \|\theta_0\|_{B^{\frac{2}{p},1+1}_{p,q}} + \|v \cdot \nabla \theta\|_{L^1_t B^{\frac{2}{p},1+1}_{p,q}}.
\]
We now estimate the nonlinear term \(\|v \cdot \nabla \theta\|_{L^1_t B^{\frac{2}{p},1+1}_{p,q}}\) by using the paraproduct of \(v\) and \(\theta\). We represent the product \(v \theta\) by
\[
v \theta = T_v \theta + T_\theta v + R(\theta, v),
\]
and we apply the operator \(\triangle_j\) to (4.16). By (2.6),
\[
\triangle_j (v \theta) = S_j v \triangle_j \theta + S_j \theta \triangle_j v + \sum_{k \geq j-2} \triangle_k v \triangle_k \theta.
\]
By taking the \(L^p\) norm to (4.17), we have
\[
\|\triangle_j (v \theta)\|_{L^p_x} \lesssim \|S_j v\|_{L^\infty_x} \|\triangle_j \theta\|_{L^p_x} + \|S_j \theta\|_{L^\infty_x} \|\triangle_j v\|_{L^p_x} + \sum_{k \geq j-2} \|\triangle_k v\|_{L^\infty_x} \|\triangle_k \theta\|_{L^p_x}
\]
\[
= \|S_j v\|_{L^\infty_x} \|\triangle_j \theta\|_{L^p_x} + \|S_j \theta\|_{L^\infty_x} \|\triangle_j v\|_{L^p_x}
\]
\[
+ \sum_{k \geq j-2} 2^{-k(\frac{2}{p}+1)} \|\triangle_k v\|_{L^\infty_x} 2^{k(\frac{2}{p}+1)} \|\triangle_k \theta\|_{L^p_x}.
\]
By taking the $L^1$ norm in time to (4.18),
\begin{equation}
\|\triangle_j (v\theta)\|_{L^1_tL^p} \lesssim \|S_j v\|_{L^{\infty}_tL^\infty} \|\triangle_j \theta\|_{L^1_tL^p} + \|S_j \theta\|_{L^{\infty}_tL^\infty} \|\triangle_j v\|_{L^1_tL^p} + \sum_{k \geq j-2} 2^{-k(\frac{2}{p}+1)} \|\triangle_k v\|_{L^{\infty}_tL^\infty} 2^{k(\frac{2}{p}+1)} \|\triangle_k \theta\|_{L^1_tL^\infty}.
\end{equation}

We multiply (4.19) by $2^{j(1+\frac{2}{p})}$. Then,
\begin{equation}
2^{j(1+\frac{2}{p})} \|\triangle_j (v\theta)\|_{L^1_tL^p} \lesssim 2^{j(1+\frac{2}{p})} \left( \|S_j v\|_{L^{\infty}_tL^\infty} \|\triangle_j \theta\|_{L^1_tL^p} + \|S_j \theta\|_{L^{\infty}_tL^\infty} \|\triangle_j v\|_{L^1_tL^p} \right) + \sum_{k \geq j-2} 2^{j-k(\frac{2}{p}+1)} \|\triangle_k v\|_{L^{\infty}_tL^\infty} 2^{k(\frac{2}{p}+1)} \|\triangle_k \theta\|_{L^1_tL^\infty}.
\end{equation}

We take the $l^q$ norm to (4.20). Since $1 + \frac{2}{p} > 0$, by applying Young’s inequality to $\sum_{k \geq j-2} a_k b_k$, where $a_j = 2^{-j(1+\frac{2}{p})}$ and $b_j = 2^{j(1+\frac{2}{p})} \|\triangle_k \theta\|_{L^1_tL^p}$, we have
\begin{equation}
\|v \cdot \nabla \theta\|_{L^1_tB^{\frac{2}{p}+1}_{p,q}} \lesssim \|v\|_{L^{\infty}_tL^\infty} \|\theta\|_{L^1_tB^{\frac{2}{p}+1}_{p,q}} + \|\theta\|_{L^{\infty}_tL^\infty} \|v\|_{L^1_tB^{\frac{2}{p}+1}_{p,q}}.
\end{equation}

Since $v = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta)$ and Besov spaces are bounded under the Riesz transformations,
\begin{equation}
\|v \cdot \nabla \theta\|_{L^1_tB^{\frac{2}{p}+1}_{p,q}} \lesssim \left( \|\theta\|_{L^{\infty}_tL^\infty} + \|v\|_{L^{\infty}_tL^\infty} \right) \|\theta\|_{L^1_tB^{\frac{2}{p}+1}_{p,q}}.
\end{equation}

Combining (4.15) and (4.22), we finally have
\begin{equation}
\|\theta\|_{L^\infty_tB^\frac{2}{p}_{p,q}} + \|\theta\|_{L^1_tB^{\frac{2}{p}+1}_{p,q}} \lesssim \|\theta_0\|_{B^\frac{2}{p}_{p,q}} + \left( \|\theta\|_{L^{\infty}_tL^\infty} + \|v\|_{L^{\infty}_tL^\infty} \right) \|\theta\|_{L^1_tB^{\frac{2}{p}+1}_{p,q}},
\end{equation}

which completes the a priori estimate of the solution in $\tilde{L}^\infty_tB^\frac{2}{p}_{p,q} \cap \tilde{L}^1_tB^{\frac{2}{p}+1}_{p,q}$ by using the smallness of $\|\theta\|_{L^{\infty}_tL^\infty} + \|v\|_{L^{\infty}_tL^\infty}$ derived from (4.10).

4.2. **Analyticity: Proof of Theorem 1.5.** To show Gevrey regularity of Theorem 1.4, we recall
\begin{equation}
\Theta(t) = e^{\frac{1}{4} t \Lambda_1} \theta(t), \quad V(t) = e^{\frac{1}{4} t \Lambda_1} v(t)
\end{equation}
and the equation they satisfy:
\begin{equation}
\Theta(t) = e^{\frac{1}{4} t \Lambda_1 - \frac{1}{4} t \Lambda} \theta_0 - \int_0^t \left[ e^{\frac{1}{4} t \Lambda_1 - (t-s) \Lambda} \left( e^{-\frac{1}{4} s \Lambda_1} V \cdot \nabla e^{-\frac{1}{4} s \Lambda_1} \Theta \right)(s) \right] ds.
\end{equation}

In order to reduce (4.25) to
\begin{equation}
\Theta(t) = e^{-\frac{1}{4} t \Lambda} \theta_0 - \int_0^t \left[ e^{-\frac{1}{4} (t-s) \Lambda} \left( V \cdot \nabla \Theta \right)(s) \right] ds
\end{equation}
in terms of estimation, we need to show that the variation of Lemma 3.2 and Lemma 3.3 for $\gamma = 1$ works. Lemma 3.2 is identically applicable to the case $\gamma = 1$. Moreover, $e^{\frac{1}{4} t \Lambda_1 - \frac{1}{4} t \Lambda}$ is a Fourier multiplier which
maps boundedly $L^p \mapsto L^p$, $1 < p < \infty$, and its operator norm is uniformly bounded with respect to $t \geq 0$ because the symbol $e^{\frac{t}{2} |\xi| - \frac{1}{2} t |\xi|}$ is uniformly bounded and decays exponentially for $|\xi| \gg 1$.

Therefore, along the lines of the estimation of $\theta$ in $L^\infty$ and $\dot{B}_{p,q}^\frac{2}{p}$, we can find the following two bounds immediately from (4.26):

\[
\|\Theta\|_{L^\infty_t L^\infty_x} + \|V\|_{L^\infty_t L^\infty_x} 
\lesssim \|\theta_0\|_{L^\infty_x} + \|v_0\|_{L^\infty_x} + \left(\|\Theta\|_{L^\infty_t L^\infty_x} + \|V\|_{L^\infty_t L^\infty_x}\right)\|\nabla^\frac{3}{2} \Theta\|_{L^2_t L^2_x},
\]

and

\[
\|\Theta\|_{\dot{B}_{p,q}^\frac{2}{p}} + \|\Theta\|_{\dot{B}_{p,q}^{\frac{2}{p}+1}} \lesssim \|\theta_0\|_{\dot{B}_{p,q}^{\frac{2}{p}}} + \left(\|\Theta\|_{L^\infty_t L^\infty_x} + \|V\|_{L^\infty_t L^\infty_x}\right)\|\Theta\|_{\dot{B}_{p,q}^{\frac{2}{p}+1}}.
\]

It remains to derive the $H^1$ estimation of $\Theta$, which will be obtained by the energy method.

\[
\frac{1}{2} \frac{d}{dt} \int_{R^2} |\nabla \Theta|^2 dx = \int_{R^2} \nabla \Theta \cdot \nabla \Theta_t dx = \int_{R^2} \nabla \Theta \cdot \nabla \left( e^{\frac{1}{4} t \Lambda_1} \theta \right)_t dx
\]

\[
= \int_{R^2} \nabla \Theta \cdot \nabla \left( \frac{1}{4} \Lambda_1 \Theta - e^{\frac{1}{4} t \Theta} (v \cdot \nabla \Theta) - \Lambda \Theta \right) dx
\]

\[
= \int_{R^2} \nabla \Theta \cdot \nabla \left( \frac{1}{4} \Lambda_1 - \Lambda \right) \Theta dx - \int_{R^2} \nabla \Theta \cdot \nabla e^{\frac{1}{4} t \Lambda_1} (e^{-\frac{1}{4} t \Lambda_1} V \cdot \nabla e^{-\frac{1}{4} t \Lambda_1} \Theta) dx.
\]

By using $\frac{1}{4} |\xi| < \frac{1}{2} |\xi|$ and the boundedness of $B_t (V, \Theta)$,

\[
\frac{1}{2} \frac{d}{dt} \int_{R^2} |\nabla \Theta|^2 dx \lesssim -\frac{1}{2} \|\nabla^\frac{3}{2} \Theta\|_{L^2_x}^2 + \|\nabla^\frac{3}{2} \Theta\|_{L^2_x} \|\nabla^\frac{3}{2} (\Theta \mathcal{R} \Theta)\|_{L^2_x}.
\]

By applying the product rule ([36]) to $\Lambda^\frac{3}{2} (\Theta \mathcal{R} \Theta)$, we finally have

\[
\frac{1}{2} \frac{d}{dt} \int_{R^2} \left| \nabla \Theta \right|^2 dx \lesssim -\frac{1}{2} \|\Lambda^\frac{3}{2} \Theta\|_{L^2_x}^2 + \|\Lambda^\frac{3}{2} \Theta\|_{L^2_x}^2 \left(\|\Theta\|_{L^\infty_x} + \|V\|_{L^\infty_x}\right).
\]

Integrating (4.31) in time,

\[
\|\nabla \Theta\|_{L^\infty_t L^2_x}^2 + \|\nabla^\frac{3}{2} \Theta\|_{L^2_t L^2_x}^2 \lesssim \|\nabla \theta_0\|_{L^2_x}^2 + \left(\|\Theta\|_{L^\infty_t L^\infty_x} + \|V\|_{L^\infty_t L^\infty_x}\right) \|\nabla^\frac{3}{2} \Theta\|_{L^2_t L^2_x}^2.
\]

By (4.27), (4.28), and (4.32), we complete the proof.

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