MATH 141H Exam 1 Preparation Solution

1. Let \( f(x) = 3x - x^2 \) and let \( R \) denote the region bounded by the graph of \( f \) and the \( x \) axis.

   (1) Draw the graph of \( f \) and find the \( x \) intercepts.
   
   (2) Find the volume of the solid generated by revolving the region \( R \) about the \( x \) axis.
   
   (3) Find the volume of the solid generated by revolving the region \( R \) about the \( y \) axis.
   
   (4) Find the moment \( M_y \) of the region \( R \) about the \( y \) axis.
   
   (5) Find the moment \( M_x \) of the region \( R \) about the \( x \) axis.

   **Solution:** (1)

   \[
   y = 3x - x^2
   \]

   \[
   \begin{align*}
   (2)\ V &= \int_0^3 \pi (3x - x^2)^2 dx = \pi \int_0^3 (9x^2 - 6x^3 + x^4) dx = \pi \left( 3x^3 - \frac{3x^4}{2} + \frac{x^5}{5} \right) \bigg|_0^3 = \frac{81}{10} \pi. \\
   (3)\ V &= 2\pi \int_0^3 (3x - x^2) dx = 2\pi \int_0^3 (3x^2 - x^3) dx = 2\pi \left( x^3 - \frac{x^4}{4} \right) \bigg|_0^3 = \frac{91}{2} \pi. \\
   (4) \text{ and } (5): \text{ By definition of moments and their relation to the disc/shell method,} \\
   M_y &= \frac{1}{2\pi} \times (3) = \frac{91}{4}, \quad M_x = \frac{1}{2\pi} \times (2) = \frac{81}{20}.
   \end{align*}
\]

2. A tank has the shape of a solid generated by revolving about the \( y \) axis the curve \( y = x^3 \) for \( x \) in \([0, 2]\), and is full of water weighting 62.5 pounds per cubic foot.

   (1) Draw a picture of the situation.
   
   (2) Calculate the cross-sectional area in terms of \( y \).
   
   (3) Write down the integral for work \( W \) required to bring water to the top of the tank until there is a depth of 1 foot of water in the tank with respect to the \( y \) integral.
3. Let $R$ be the region enclosed by the graphs of the functions $f(x) = 4|x|$ and $g(x) = x^2$.

(1) Draw the graphs and find the $x$ coordinates of intersecting points.

(2) Calculate the area of the region $R$.

(3) Find the moment $M_y$ of the region $R$ about the $y$ axis and the moment $M_x$ of the region $R$ about the $x$ axis. (You may use symmetry where appropriate; otherwise you must show integral calculus work.)

(4) Calculate the center of mass.

**Solution:** (1)

$$A = \int_{-4}^{4} (4|x| - x^2)\,dx = 2\int_{0}^{4} (4x - x^2)\,dx = 2\left(2x^2 - \frac{x^3}{3}\right)\Big|_{0}^{4} = \frac{64}{3}.$$
(3) Since the graph is symmetric with respect to the y axis, $M_y = 0$.

\[
M_x = \frac{1}{2} \int_{-4}^{4} [(4|x|)^2 - (x^2)^2] \, dx = \int_{0}^{4} (16x^2 - x^4) \, dx = \left( \frac{16x^3}{3} - \frac{x^5}{5} \right) \bigg|_0^4 = \frac{2048}{15}.
\]

(4) The center of mass $(\bar{x}, \bar{y})$ is given by $\bar{x} = \frac{M_y}{A} = 0$, $\bar{y} = \frac{M_x}{A} = \frac{32}{5}$.

---

4. Consider the curve $C$ with parametrization

\[
x = \frac{2}{3} \sin^\frac{3}{2} t, \quad y = \sin t, \quad 0 \leq t \leq \frac{\pi}{2}.
\]

Find the length of the curve.

**Solution:** Since

\[
\frac{dx}{dt} = \sin^{\frac{3}{2}} t \cos t, \quad \frac{dy}{dt} = \cos t \Rightarrow \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = \sin t \cos^2 t + \cos^2 t = \cos^2 t(\sin t + 1),
\]

The length is

\[
L = \int_{0}^{\frac{\pi}{2}} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0}^{\frac{\pi}{2}} |\cos t| \sqrt{\sin t + 1} dt.
\]

Since $\cos t \geq 0$ on the interval $[0, \frac{\pi}{2}]$,

\[
L = \int_{0}^{\frac{\pi}{2}} \cos t \sqrt{\sin t + 1} dt = \int_{0}^{\frac{\pi}{2}} \cos t \sqrt{\sin t + 1} dt.
\]

Let $u = \sin t + 1 \Rightarrow du = \cos t \, dt$. Therefore,

\[
L = \int_{0}^{\frac{\pi}{2}} \cos t \sqrt{\sin t + 1} dt = \int_{u(0)}^{u(\frac{\pi}{2})} \sqrt{u} \, du = \int_{1}^{2} \sqrt{u} \, du = \frac{2}{3} u^{\frac{3}{2}} \bigg|_1^{2} = \frac{2}{3} (2^{\frac{3}{2}} - 1).
\]

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5. Given $f(x) = 2x^\frac{3}{2}$ find the length of the graph of $f$ on the interval $[0, 2]$.

**Solution:** Since

\[
f'(x) = 2 \cdot \frac{3}{2} x^{\frac{1}{2}} = 3\sqrt{x} \Rightarrow 1 + (f'(x))^2 = 1 + 9x,
\]

The length is

\[
L = \int_{0}^{2} \sqrt{1 + (f'(x))^2} \, dx = \int_{0}^{2} \sqrt{1 + 9x} \, dx.
\]

Let $u = 1 + 9x \Rightarrow du = 9 \, dx$. Then,

\[
L = \int_{0}^{2} \sqrt{1 + 9x} \, dx = \frac{1}{9} \int_{u(0)}^{u(2)} \sqrt{u} \, du = \frac{1}{9} \int_{1}^{19} \sqrt{u} \, du = \frac{2}{27} u^{\frac{3}{2}} \bigg|_1^{19} = \frac{2}{27} (19^{\frac{3}{2}} - 1).
\]
6. Sketch the graph of the equation \( r = 1 + 2 \sin \theta \).

*Solution:* Since \( \sin(\pi - \theta) = \sin \theta \), the graph is symmetric with respect to the \( y \) axis.

<table>
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<th>( \theta )</th>
<th>(-\frac{\pi}{2})</th>
<th>(-\frac{\pi}{6})</th>
<th>0</th>
<th>(\frac{\pi}{6})</th>
<th>(\frac{\pi}{2})</th>
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<tbody>
<tr>
<td>( r )</td>
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<td>1</td>
<td>2</td>
<td>3</td>
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</table>

7. Sketch the graph of the equation \( r^2 = \cos \theta \).

*Solution:* Since \( \cos(-\theta) = \cos \theta \), the graph is symmetric with respect to the \( x \) axis. Since \( \cos \theta \leq 0 \) on \( [\frac{\pi}{2}, \pi] \), we skip it. By using the symmetry, we complete the sketch of the graph as follows.
8. Sketch the graph of the equation $r^2 = \cos 2\theta$ and calculate the area enclosed by the curve.

Solution: Since $\cos(-2\theta) = \cos 2\theta$, the graph is symmetric with respect to the $x$ axis. Since $\cos 2\theta \leq 0$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$, we skip it.

<table>
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<th>$\theta$</th>
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<th>$\frac{\pi}{4}$</th>
<th>$\sim$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>1</td>
<td>0</td>
<td>$\times$</td>
<td>0</td>
<td>1</td>
</tr>
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</table>

The area enclosed by the graph is

$$A = 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta = 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta = \sin 2\theta \bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2.$$