Math 141H Homework 5 Solution (Section 7.1-7.3)

Show all your work. Jumping to the right answer without minimum reasoning deserves no credit.

1. Find the inverse of \( f(x) = \frac{e^x - 1}{e^x + 1} \).

Solution:
\[ y = \frac{e^x - 1}{e^x + 1} \implies x = \frac{e^y - 1}{e^y + 1} \implies xe^y + x = e^y - 1 \implies e^y = 1 + \frac{x}{1 - x}. \]
Therefore, the inverse is \( f^{-1}(x) = \ln\left(\frac{1 + x}{1 - x}\right) \).

2. Let \( f(x) = x^4 - 4x - 1 \).

(1) What is the biggest interval \( I \) containing 0 for which \( f \) has an inverse? (Note: \( I \) is allowed to extend out to \( \infty \), to \( -\infty \), or both, as appropriate.)

(2) What is the domain of the inverse function \( f^{-1}(x) \)?

Solution (1) To find intervals on which \( f \) is increasing or decreasing, we take the derivative to \( f \). Since
\[ f'(x) = 4x^3 - 4 = 4(x - 1)(x^2 + x + 1), \]
\( f \) is increasing on \([1, \infty)\) and decreasing on \((-\infty, 1]\). Therefore, the biggest interval containing 0 for which \( f \) has an inverse is \((-\infty, 1]\). (2) The domain of \( f^{-1}(x) \) is the range of \( f(x) \). Since
\[ \lim_{x \to -\infty} f(x) = \infty, \quad f(1) = -4 \]
and \( f \) is decreasing, the range of \( f \) is \([-4, \infty)\).

3. Let \( f(x) = \int_0^x \cos^2 t \, dt \) for all \( x \in [0, 100] \).

(1) Show that \( f \) has an inverse.

(2) Let \( c = f(\frac{\pi}{4}) \). Find \( (f^{-1})'(c) \).

Solution (1) Since \( f'(x) = \cos^2 x > 0 \) except for a finitely many \( x \)'s in \([0, 100] \), \( f \) has an inverse. (2) Since \( f'(\frac{\pi}{4}) = \cos^2(\frac{\pi}{4}) = \frac{1}{2} \),
\[ (f^{-1})'(c) = \frac{1}{f'(\frac{\pi}{4})} = 2. \]
4. Let \( f(x) = \frac{e^x + 2}{e^x + 4} \).

(1) Show that \( f \) has an inverse.

(2) Find \( (f^{-1})'(\frac{3}{5}) \).

**Solution** (1) Since
\[
f'(x) = \frac{e^x(e^x + 4) - e^x(e^x + 2)}{(e^x + 4)^2} = \frac{2e^x}{(e^x + 4)^2} > 0,
\]
\( f \) has an inverse.

(2) We first find \( a \) such that \( f(a) = \frac{3}{5} \).

\[
\frac{e^a + 2}{e^a + 4} = \frac{3}{5} \implies e^a = 1 \implies a = 0.
\]
Therefore,
\[
(f^{-1})'(\frac{3}{5}) = \frac{1}{f'(0)} = \frac{1}{\frac{2}{25}} = \frac{25}{2}.
\]

5. Find the derivative of the following functions.

(1) \( f(x) = (1 + x^2)^{1+x^2} \)

(2) \( f(x) = (\sin x)^{5x} \)

(3) \( f(x) = (1 + \frac{1}{x})^x \)

**Solution** (1) Since \( f(x) = e^{(1+x^2)\ln(1+x^2)} \),
\[
f'(x) = e^{(1+x^2)\ln(1+x^2)} \cdot \left[ 2x \ln(1+x^2) + (1+x^2) \frac{2x}{1+x^2} \right] = (1+x^2)^{1+x^2} \cdot \left[ 2x \ln(1+x^2) + 2x \right].
\]

(2) Since \( f(x) = e^{5x\ln(\sin x)} \),
\[
f'(x) = e^{5x\ln(\sin x)} \cdot \left[ 5 \ln(\sin x) + 5x \frac{\cos x}{\sin x} \right] = (\sin x)^{5x} \cdot \left[ 5 \ln(\sin x) + \frac{5x \cos x}{\sin x} \right].
\]

(3) Since \( f(x) = e^{x\ln(1+\frac{1}{x})} \),
\[
f'(x) = e^{x\ln(1+\frac{1}{x})} \cdot \ln \left( 1 + \frac{1}{x} \right) + x \cdot \frac{-1}{1 + \frac{1}{x}} = \left( 1 + \frac{1}{x} \right)^x \cdot \left[ \ln \left( 1 + \frac{1}{x} \right) - \frac{1}{1 + x} \right].
\]
6. Find the integral.

(1) \[ \int_{0}^{1} \frac{e^t - e^{-t}}{e^t + e^{-t}} \, dt \]

(2) \[ \int \frac{e^{-x} \ln(1 + e^{-x})}{1 + e^{-x}} \, dx \]

Solution (1) Let \( u = e^t + e^{-t} \implies du = (e^t - e^{-t}) \, dt \). Therefore,

\[ \int_{0}^{1} \frac{e^t - e^{-t}}{e^t + e^{-t}} \, dt = \int_{u(0)}^{u(1)} \frac{du}{u} = \int_{2}^{e + \frac{1}{e}} \frac{du}{u} = \ln u \bigg|_{e + \frac{1}{e}}^{2} = \ln \left( e + \frac{1}{e} \right) - \ln 2. \]

(2) Let \( u = 1 + e^{-x} \implies du = (-e^{-x}) \, dx \). Therefore,

\[ \int \frac{e^{-x} \ln(1 + e^{-x})}{1 + e^{-x}} \, dx = -\int \frac{\ln u}{u} \, du. \]

Let \( v = \ln u \implies dv = \frac{1}{u} \, du \). Therefore,

\[ \int \frac{e^{-x} \ln(1 + e^{-x})}{1 + e^{-x}} \, dx = -\int \frac{\ln u}{u} \, du = -\int v \, dv = -\frac{1}{2} v^2 + C = -\frac{1}{2} \left( \ln \left( 1 + e^{-x} \right) \right)^2 + C \]

7. Find the area of the region between the graph of \( y = x \cdot 2x^2 \) and the \( x \) axis on \([1, 2]\).

Solution: The area is

\[ \int_{1}^{2} x \cdot 2x^2 \, dx \]

Let \( u = x^2 \implies du = 2x \, dx \). Therefore,

\[ \int_{1}^{2} x \cdot 2x^2 \, dx = \frac{1}{2} \int_{u(1)}^{u(2)} 2u \, du = \frac{1}{2} \int_{1}^{4} 2^u \, du = \frac{1}{2} \ln 2 \cdot 2^u \bigg|_{1}^{4} = \frac{7}{\ln 2}. \]