Exercise 1.
1. Assume that \( u_n \) is periodic, that is for some \( L \in \mathbb{N} \) \( u_{n+L} = u_n \). Prove that
\[
A_n = \frac{u_1 + \ldots + u_n}{n}
\]
converges.
2. Find an example of \( u_n \) s.t. \( \frac{u_1 + \ldots + u_n}{n} \) converges but not \( u_n \).

Exercise 2.
1. We define for a given \( u_0 \) and \( a \in \mathbb{R} \), \( a \neq 1 \)
\[
u_{n+1} = a u_n + b,
\]
Denote \( v_n = u_n + b/(a - 1) \). Write the induction relation for \( v_n \) and find an explicit expression for \( u_n \).
2. What if \( a = 1 \)?
3. Prove that the following sequence is increasing
\[
u_0 = 1, \quad v_1 = 2, \quad v_{n+2} = 2 v_{n+1} - v_n.
\]

Exercise 3.
Assume that \( u_n \) is bounded.
1. Recall why it is possible to find \( l_\sigma \) and a subsequence \( u_{\sigma(n)} \) converging to \( l \).
2. Prove that if \( u_n \) does not converge to some limit \( l \) then one can find \( a \neq l \)
and a subsequence s.t. \( u_{\sigma(n)} \) converges to \( a \).
2. Assume now that every converging subsequence of \( u_n \) converges to the same limit \( l \). Deduce that \( u_n \) converges to \( l \).

Exercise 4.
1. Assume that \( \sum a_n \) is absolutely converging and that \( \sigma : \mathbb{N} \to \mathbb{N} \) is strictly increasing. Prove that
\[
\sum_{n=0}^{N} |a_{\sigma(n)}| \leq \sum_{n=0}^{\sigma(N)} |a_n|.
\]
Conclude that \( \sum a_{\sigma(n)} \) is also absolutely converging.
2. Find an example of \( a_n \) and strictly increasing \( \sigma \) s.t. \( \sum a_n \) is converging
but \(\sum a_{\sigma(n)}\) is not.

3. Assume that \(\sum a_n\) is absolutely converging and that \(\sigma : \mathbb{N} \to \mathbb{N}\) is one-to-one. Prove again that \(\sum a_{\sigma(n)}\) is absolutely converging.

**Exercise 5.**

For any sequence \((a_n)\), we denote

\[
\|a_n\|_1 = \sum_{n=0}^{\infty} |a_n|,
\]

if \(\sum a_n\) is absolutely converging and \(\|a_n\|_1 = +\infty\) otherwise.

1. Assume that \((a^k_n)\) (the sequence of sequences \((a^k_1, \ldots, a^k_n, \ldots)\)) is such that

\[
\|a^k_n\|_1 = \sum_{n=0}^{\infty} |a^k_n| \to 0 \quad \text{as } k \to \infty.
\]

Prove that for any fixed \(n\), \(a^k_n \to 0\) as \(k \to \infty\).

2. Assume now conversely that for any fixed \(n\), \(a^k_n \to 0\) as \(k \to \infty\) and that

\[
\sup_k \sum_{n=0}^{\infty} n |a^k_n| < \infty.
\]

Prove that

\[
\|a^k_n\|_1 = \sum_{n=0}^{\infty} |a^k_n| \to 0 \quad \text{as } k \to \infty.
\]

3. Is the result of the previous question still true if one only assumes that

\[
\sup_k \sum_{n=0}^{\infty} |a^k_n| < \infty?
\]