Exercise 1.
1. Does the following sequence have a limit

\[ u_n = 1 + (-1)^n \cos n \sin \frac{1}{n^2} \]

Solution. The sequence converges to 1. Indeed \((-1)^n \cos n\) is bounded; \(\sin \frac{1}{n^2}\) converges to 0 as \(1/n^2 \to 0\), and \(\sin\) is continuous at 0 with \(\sin(0) = 0\).

2. Assume that \(u_n\) defined

\[ u_{n+1} = \frac{e^{u_n}}{4} \]

is converging. Calculate the limit \(l\) and justify.
Solution. If \(u_n \to l\) then \(u_{n+1} \to l\) as well. By the continuity of \(\exp\), one needs to have

\[ l = e^{l}/4. \]

The study of \(f(x) = x - e^{x}/4\) shows that it is increasing till \(\log 4\) and then decreasing. On the other hand, \(f(1) > 0\) and the limit of \(f\) at \(+\infty\) is \(-\infty\) while its limit at \(-\infty\) is also \(-\infty\). Hence there are exactly two possible limits \(l\).

Exercise 2.
1. Show that \(x e^x\) is increasing on \(R_+\).
Solution. Just differentiate!

2. Use integrals to give a lower and upper bound for

\[ \sum_{k=1}^{n} k e^k. \]

Solution. As \(x e^x\) is increasing, one has that

\[ \int_{0}^{n} x e^x dx \leq \sum_{k=1}^{n} k e^k \leq \int_{1}^{n+1} x e^x dx. \]

3. Calculate the anti-derivative of \(x e^x\) and conclude the bound. Hint: Try a combination of \(x e^x\) and \(e^x\).
Solution. By linear combination \((xe^x - e^x)' = xe^x\) and thus
\[(n - 1)e^n \leq \sum_{k=1}^{n} ke^k \leq ne^{n+1}.

Exercise 3.
1. Assume that \(f\) is continuous on \([0, 1]\) with \(f([0, 1]) \subset [0, 1]\). Show that there exists \(x_0 \in [0, 1]\) s.t. \(f(x_0) = x_0\).
Solution. We study the function \(h(x) = f(x) - x\). It is continuous and moreover \(h(0) = -f(0) \leq 0\) as \(f(0) \in [0, 1]\). Similarly \(h(1) = 1 - f(1) \geq 0\) as \(f(1) \in [0, 1]\). By the intermediate value theorem, there is some \(x_0\) s.t. \(h(x_0) = 0\), which concludes.
2. Assume that \(f\) and \(g\) are continuous on \([0, 1]\) with \(f([0, 1]) \subset [0, 1]\) and \(g([0, 1]) = [0, 1]\). Show that there exists \(x_0 \in [0, 1]\) s.t. \(f(x_0) = g(x_0)\).
Solution. We follow the same idea and define \(h(x) = g(x) - f(x)\). Since \(g([0, 1]) = [0, 1]\), there exist \(x_1\) and \(x_2\) s.t. \(g(x_1) = 0\) and \(g(x_2) = 1\). Then \(h(x_1) \leq 0\) and \(h(x_2) \geq 0\). Using again the intermediate value theorem, we conclude that there is \(x_0\) s.t. \(h(x_0) = 0\).
3. Assume that \(g\) is continuous and differentiable on \([0, 1]\) with \(g([0, 1]) = [0, 1]\) but that \(g(0) \neq 0\) and \(g(1) \neq 0\). Prove that there exists \(x_0 \in (0, 1)\) s.t. \(g'(x_0) = 0\).
Solution. Since \(g([0, 1]) = [0, 1]\), there exists \(x_0\) s.t. \(g(x_0) = 0\) which is a minimum of \(g\). As \(g(0) \neq 0\) and \(g(1) \neq 0\), then \(x_0 \in (0, 1)\). By Rolle’s theorem, one then has that \(g'(x_0) = 0\).

Exercise 4.
1. Prove that the following function is continuous on \(\mathbb{R}\)
\[f(x) = e^{-1/x} \quad \text{if } x > 0, \quad f(x) = 0 \quad \text{if } x \leq 0.
Solution. For any \(\varepsilon > 0\), as \(e^y\) has limit 0 at \(-\infty\), there exists \(R < 0\) s.t. for all \(y < R\) then \(|e^y| < \varepsilon\). Now choose \(\delta = -1/R\). For any \(|x| \leq \delta\), if \(x < 0\) then \(f(x) = 0\). If \(x > 0\) then \(-1/x < -1/\eta = R\) and thus \(|f(x)| < \varepsilon\).
2. Prove that \(f\) is differentiable on \(\mathbb{R}\) and calculate its derivative at every point.
Solution. We have to calculate
\[
\lim_{x \to 0} \frac{f(x)}{x}.
\]
We use the same argument as in 1, just by noticing that $y e^y$ still converges to 0 at $-\infty$.

3. (Difficult) Prove that $f$ is infinitely differentiable on $\mathbb{R}$.
Solution. Prove by induction that the $k$-th derivative of $f$ has the form
\[ f^{(k)}(x) = P(1/x) e^{-1/x} \quad \text{if } x > 0, \quad f^{(k)}(x) = 0 \quad \text{if } x \leq 0, \]
where $P$ is a polynomial.

Exercise 5.
1. Assume that $g$ is integrable on $[0, 1]$, that $f$ is positive continuous on $[0, 1]$. Prove that $fg$ is integrable on the same interval.
Solution. Take any sequence of partitions $P_n$ with gap $(P_n) \to 0$. Observe that on any consecutive points $x^n_i, x^n_{i+1}$ of $P_n$

\[ \sup_{[x^n_i, x^n_{i+1}]} f g \leq \inf_{[x^n_i, x^n_{i+1}]} f \sup_{[x^n_i, x^n_{i+1}]} g + \eta_n, \]

with

\[ |\eta_n| \leq \sup_{[0, 1]} |g| \sup_i \left( \sup_{[x^n_i, x^n_{i+1}]} f - \inf_{[x^n_i, x^n_{i+1}]} f \right). \]

Observe that $g$ is bounded on $[0, 1]$ as it is integrable. Since $f$ is continuous, it is uniformly continuous on $[0, 1]$ and

\[ \sup_{[x^n_i, x^n_{i+1}]} f - \inf_{[x^n_i, x^n_{i+1}]} f \to 0, \quad \text{as } n \to \infty. \]

Therefore $\eta_n \to 0$. But

\[ \inf_{[x^n_i, x^n_{i+1}]} f g \geq \inf_{[x^n_i, x^n_{i+1}]} f \inf_{[x^n_i, x^n_{i+1}]} g. \]

Thus

\[ U(f g, P_n) - L(f g, P_n) \leq \eta_n + \sum_{i=0}^{n-1} \inf_{[x^n_i, x^n_{i+1}]} f \left( \sup_{[x^n_i, x^n_{i+1}]} g - \inf_{[x^n_i, x^n_{i+1}]} g \right), \]

or

\[ U(f g, P_n) - L(f g, P_n) \leq \eta_n + \sup_{[0, 1]} f (U(g, P_n) - L(g, P_n)). \]

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and the difference converges to 0.

2. We now only assume that $f$ is integrable and positive. For any $n$, we define the partition $P_n = (0, 1/n, 2/n, \ldots, n/n = 1)$. What can be said about

$$\varepsilon_n = U(f, P_n) - L(f, P_n)?$$

**Solution.** As per the definition of integrability, $\varepsilon_n$ converges to 0 since the gap of the partition $P_n$ converges to 0.

3. For every $n$ we denote by $I_n$ the set of indices $i$ s.t.

$$\sup_{[i/n, (i+1)/n]} f \geq \sqrt{\varepsilon_n} + \inf_{[i/n, (i+1)/n]} f.$$ 

Denote by $N_n$ the cardinal of $I_n$ and prove that

$$N_n \leq n \sqrt{\varepsilon_n}.$$ 

**Solution.** Observe that

$$U(f, P_n) - L(f, P_n) \geq \sum_{i \in I_n} \frac{1}{n} \left( \sup_{[i/n, (i+1)/n]} f - \inf_{[i/n, (i+1)/n]} f \right).$$

Therefore

$$\varepsilon_n \geq N_n \frac{\sqrt{\varepsilon_n}}{n},$$

which gives the desired inequality.

4. Show that $fg$ is integrable.

**Solution.** We follow the steps of question 1. If $i \notin I_n$ then we have similarly

$$\sup_{[i/n, (i+1)/n]} f g \leq \inf_{[i/n, (i+1)/n]} f \sup_{[i/n, (i+1)/n]} g + \sqrt{\varepsilon_n} \sup_{[0, 1]} |g|.$$ 

Therefore as in 1,

$$U(f g, P_n) - L(f g, P_n) \leq \sqrt{\varepsilon_n} \sup_{[0, 1]} |g| + \frac{1}{n} \sum_{i \notin I_n} \inf_{[i/n, (i+1)/n]} f \left( \sup_{[i/n, (i+1)/n]} g - \inf_{[i/n, (i+1)/n]} g \right)$$

$$+ \frac{1}{n} \sum_{i \in I_n} \left( \sup_{[i/n, (i+1)/n]} f g - \inf_{[i/n, (i+1)/n]} f g \right).$$
As before

\[
\frac{1}{n} \sum_{i \notin I_n} \inf_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} f \left( \sup_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} g - \inf_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} g \right) \leq \frac{1}{n} \sup_{[0,1]} |f| \sum_{i \notin I_n} \left( \sup_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} g - \inf_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} g \right) \\
\leq \frac{1}{n} \sup_{[0,1]} |f| \sum_{i=0}^{n-1} \left( \sup_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} g - \inf_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} g \right) \\
\leq \sup_{[0,1]} |f| \left( U(g, P_n) - L(g, P_n) \right).
\]

But now by question 3

\[
\frac{1}{n} \sum_{i \in I_n} \left( \sup_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} f g - \inf_{\left[\frac{i}{n}, \frac{i+1}{n}\right]} f g \right) \leq 2 \sup_{[0,1]} |f g| \frac{N_n}{n} \leq 2 \sqrt{\varepsilon_n} \sup_{[0,1]} |f g|.
\]

Summing up all terms, one concludes that \( U(f g, P_n) - L(f g, P_n) \) converges to 0.