Exercise 1. Uniqueness of the QR decomposition
1. Recall the definition of an orthogonal matrix. Check that the transpose of an orthogonal matrix and the product of two orthogonal matrices are orthogonal.
2. Show that the inverse of an upper triangular matrix is also upper triangular. Show that the product of two upper triangular matrices is upper triangular.
3. Assume that $A$ is orthogonal and upper triangular. Show that $A = I$.
4. Conclude that the QR decomposition is unique.

Exercise 2. Jacobi and Gauss-Seidel
1. What are the Jacobi and Gauss-Seidel methods?
2. Let $A$ be a symmetric positive definite matrix and decompose $A = U + R$ according to Gauss-Seidel. Assume $x$ is an eigenvector of $U^{-1}R$ with eigenvalue $\lambda$. Show that

\[(1 + \lambda) x^T M x = x^T A x,\]

and deduce that $\lambda \neq -1$.
3. Prove independently that $2x^T M x \geq x^T A x$.
4. Conclude that $|\lambda| < 1$ and that the method converges.

Exercise 3. Various
1. What is the general formula for the polynomial $L$ s.t. $L(x_i) = f(x_i)$ for $i = 0 \ldots k$?
2. Recall the trapezoidal and Simpson rules. For which functions are they exact?

Exercise 4. Taylor expansions, Polynomial interpolation and numerical integration
1. Assume that $f \in C^{k+1}$, what is the Taylor expansion of $f$ around point $x$ at order $k$?
2. Given points $x_0, \ldots, x_k$ with $x_{i+1} = x_i + h$, denote by $L$ the polynomial of degree $k$ with $L(x_i) = f(x_i)$. Can you give an estimate on

\[\sup_{[x_0, x_k]} |L(x) - f(x)|\]
in terms of $h$?
3. Describe the Newton-Cotes method using Lagrange interpolation with $k + 1$ points.
4. Give a bound on the error for the previous method.