Exercise 1. Give the \( LU \) decomposition of
\[
A = \begin{bmatrix}
2 & 1 & 1 \\
4 & 8 & 2 \\
6 & 12 & 6
\end{bmatrix}.
\]

Solution. We first do the elimination on the first column: \( R_2 \rightarrow R_2 - 2R_1 \) and \( R_3 \rightarrow R_3 - 3R_1 \). This gives the coefficients \( L_{21} = 2 \) and \( L_{31} = 3 \) and leads to the matrix
\[
\begin{bmatrix}
2 & 1 & 1 \\
0 & 6 & 0 \\
0 & 9 & 3
\end{bmatrix}.
\]

Now we eliminate on the second column with: \( R_3 \rightarrow R_3 - 3R_2/2 \) leading to \( L_{32} = 3/2 \) and
\[
U = \begin{bmatrix}
2 & 1 & 1 \\
0 & 6 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

Collecting all the information on \( L \)
\[
L = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & 3/2 & 1
\end{bmatrix}.
\]
Exercise 2. Let $A$ be a diagonal matrix of size $N$. How many multiplications/divisions are needed to solve $Ax = y$?

Solution. The solution is given by $x_i = y_i/A_{ii}$. Therefore there are $N$ divisions.
Exercise 3. We wish to use Newton’s method to find a root to
\[ f(x) = x - (e^x - 1). \]

1. Prove that \( f \) has exactly one root.

   Solution. We calculate \( f'(x) = 1 - e^x \) which shows that \( f'(x) < 0 \) if \( x > 0 \)
   and \( f'(x) > 0 \) if \( x < 0 \). Therefore \( f \) has a unique global maximum at 0 with value \( f(0) = 0 \) and 0 is the only root.

2. Recall what Newton’s method consists in.

   Solution. Newton’s method consists in calculating the sequence of values \( x_k \) given by the recursive relation \( x_{k+1} = N_f(x_k) \) with
   \[ N_f(x) = x - \frac{f(x)}{f'(x)}. \]
3. We define \( N_f = x - \frac{f(x)}{f'(x)} \). Calculate \( N'_f(x) \) and prove that it is continuous on \( \mathbb{R} \) with \( N'_f(0) = \frac{1}{2} \).

Solution. Just calculate

\[
N_f(x) = x - \frac{x - e^x + 1}{1 - e^x}.
\]

The only possible problem is where \( 1 - e^x = 0 \) that is at \( x = 0 \) and \( N'_f \) is continuous at any other point. Now observe that

\[
x - e^x + 1 = -\frac{x^2}{2} + O(x^3), \quad 1 - e^x = -x + O(x^2).
\]

Therefore around 0

\[
N_f(x) = x - \frac{x}{2} + O(x^2), \quad N'_f(x) = 1 - \frac{1}{2} + O(x),
\]

which shows that \( N_f \) and \( N'_f \) are continuous and \( N'_f(0) = 1/2 \).

4. Deduce that there exists \( \varepsilon > 0 \) s.t. if \( |x_0| < \varepsilon \) then the sequence defined by \( x_{k+1} = N_f(x_k) \) converges to 0.

Solution. We have seen in class that this was the case for a recursive method \( x_{k+1} = N_f(x_k) \) if \( |N_f(\bar{x})| < 1 \) which is true here by the previous question as \( \bar{x} = 0 \). It is also possible to prove again that result. By the continuity of \( N'_f \), choose \( \varepsilon \) s.t. for instance

\[
\max_{[-\varepsilon, \varepsilon]} |N'_f(x)| \leq 3/4.
\]

Then if \( |x_k| < \varepsilon \),

\[
|x_{k+1}| = |N_f(x_k)| \leq |x_k| \max_{[-\varepsilon, \varepsilon]} |N'_f(x)| \leq \frac{3}{4} |x_k|.
\]

This shows by induction that if \( |x_0| < \varepsilon \) then \( |x_k| < \varepsilon \) and in a second step that then \( |x_k| \leq |x_0| (3/4)^k \); hence the convergence.
5. What is the order of convergence?

*Solution. The order is* $1$ as

\[
\frac{|x_{k+1}|}{|x_k|} = \frac{|N_f(x_k)|}{|x_k|} \rightarrow |N'_f(0)| = \frac{1}{2}.
\]
Exercise 4 The 1 and $\infty$ norms for vectors are defined as

$$\|u\|_1 = \sum_{i=1}^{N} |u_i|, \quad \|u\|_\infty = \max_{i=1...N} |u_i|. $$

1. Recall the definitions of the 1 and $\infty$ norms for matrices, coherent or induced by the above vector norms.

Solution. The corresponding norms are

$$\|A\|_1 = \max_j \sum_i |A_{ij}|, \quad \|A\|_\infty = \max_i \sum_j |A_{ij}|. $$

2. Show that if $u$ is a vector of $\mathbb{R}^N$ and $A$ a matrix of size $N$

$$\|u\|_\infty \leq \|u\|_1 \leq N \|u\|_\infty, \quad \|A\|_1 \leq N \|A\|_\infty. $$

It is obvious that $\|u\|_\infty \leq \|u\|_1$. On the other hand as each element $|u_i|$ is less than $\|u\|_\infty$

$$\|u\|_1 = \sum_i |u_i| \leq \sum_i \|u\|_\infty \leq N \|u\|_\infty. $$

For the inequality on norms, note that by the using the previous inequalities

$$\sup_{u \neq 0} \frac{\|Au\|_1}{\|u\|_1} \leq N \sup_{u \neq 0} \frac{\|Au\|_\infty}{\|u\|_1} \leq N \sup_{u \neq 0} \frac{\|A\|_\infty \|u\|_\infty}{\|u\|_1} \leq N \|A\|_\infty, $$

as $\|u\|_\infty \leq \|u\|_1$. 

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