Global weak solutions of PDEs for compressible media: A compactness criterion to cover new physical situations

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Abstract This short paper is an introduction of the memoir recently written by the two authors (see [D.Bresch., P.–E. Jabin, arXiv:1507.04629, (2015)]) which concerns the resolution of two longstanding problems: Global existence of weak solutions for compressible Navier–Stokes equations with *thermodynamically unstable pressure* and with *anisotropic stress tensor*. We focus here on a Stokes-like system which can for instance model flows in a compressible tissue in biology or in a compressible porous media in petroleum engineering. This allows to explain, on a simpler but still relevant and important system, the tools recently introduced by the authors and to discuss the important results that have been obtained on the compressible Navier–Stokes equations. It is finally a real pleasure to dedicate this paper to G. MÉTIVIER for his 65's Birthday.

1 Introduction

We consider in this paper a model which has been developed for flows in a compressible tissue in biology (see [3], [6]) or in compressible porous media in petroleum engineering (see [10]). The most simple system involves a density ρ that is transported,

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

by a velocity field u described by a Stokes-like equation

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$$-\mu \Delta u + \alpha u + \nabla P(\rho) = S,$$

with $\mu, \alpha > 0$.

For simplicity we consider periodic boundary conditions, namely both equations are posed for $x \in \Omega = \Pi^d$. This is also the reason for the damping term αu to control u without imposing any additional condition on S. The corresponding PDE is usually named Brinkman equation. It accounts for flow through medium where the grains are porous themselves.

In this short paper, we explain how to consider non-monotone pressure laws P for this system (complex pressure laws (attractive and repulsive)) to obtain the existence of global weak-solutions. Note that in particular biological systems frequently exhibit preferred ranges of densities for instance attractive interactions for low densities and repulsive at higher ones.

To get such global existence of weak solutions result, the two authors have recently revisited (see [4]) the classical compactness theory on the density by obtaining precise quantitative regularity estimates: This requires a more precise analysis of the structure of the equations combined to a novel approach to the compactness of the continuity equation (by introducing appropriate weights). We quote at the end of the article some of the precise results obtained in [4] on the compressible Navier-Stokes systems but we of course refer the reader to [4] for all the details and possible extensions for instance including temperature conductivity dependency.

2 Equations and main result

As mentioned above, we work on the torus Π^d. This is only for simplicity in order to avoid discussing boundary conditions or the behavior at infinity.

2.1 Statements of the result

We present in this section our main existence result concerning System (1). As usual for global existence of weak solutions to nonlinear PDEs, one has to prove stability estimates for sequences of approximate solutions and construct such approximate sequences. The main contribution in this paper and the major part of the proofs concern the stability procedure and more precisely the compactness of the density. We refer to [4] for details and the way to construct the approximate solutions sequence. As per the introduction, we consider the following system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\mu \Delta u + \alpha u + \nabla P(\rho) = S, \end{cases} \tag{1}$$

with $\mu, \alpha > 0$, a pressure law P which is continuous on $[0,+\infty)$, P locally Lipschitz on $(0,+\infty)$ with $P(0) = 0$ such that there exists $C > 0$ with

$$C^{-1}\rho^\gamma - C \le P(\rho) \le C\rho^\gamma + C, \tag{2}$$

and for all $s \ge 0$

$$|P'(s)| \le \bar{P} s^{\gamma-1}. \tag{3}$$

One then has global existence

Theorem 1. *Assume that $S \in L^2(0,T;\, H^{-1}(\Pi^d))$ and the initial data ρ_0 satisfies the bound*

$$\rho_0 \ge 0, \qquad 0 < M_0 = \int_{\Pi^d} \rho_0 < +\infty, \qquad E_0 = \int_{\Pi^d} \rho_0 e(\rho_0)\, dx < +\infty,$$

where $e(\rho) = \int_{\rho^\star}^{\rho} P(s)/s^2 ds$ with $\rho^\star$ a constant reference density. Let the pressure law P satisfies (25) *and* (26) *with $\gamma > 1$. Then there exists a global weak solution (ρ, u) of the compressible system* (1) *with*

$$\rho \in L^\infty(0,T; L^\gamma(\Pi^d)) \cap L^{2\gamma}((0,T)\times \Pi^d), \qquad u \in L^2(0,T; H^1(\Pi^d)).$$

Remark 1. Let us note that we do not try to optimize the regularity of S which could be far less smooth. The objective of this short note being to be an introduction to [4] focusing on the new compactness criterion.

3 Sketch of the new compactness method

We present in the section the tool which has been used in [4] and which is the cornerstone to prove compactness on the density. The interested reader is also referred to [1], [2], [13] for more on the corresponding critical spaces. This tool is really appropriate to cover more general equation of state or stress tensor form compared to the more standard defect measure criterion used in [11], [8], [9], [12] for instance.

3.1 The compactness criterion

We start by a well known result providing compactness of a sequence

Proposition 1. *Let ρ_k be a sequence uniformly bounded in some $L^p((0,T)\times \Pi^d)$ with $1 \le p < \infty$. Assume that $\mathscr{K}_h$ is a sequence of smooth, positive, bounded functions s.t.*

$$i. \quad \forall \eta > 0, \quad \sup_h \int_{|x|\geq \eta} \mathscr{K}_h(x)\,dx < \infty, \qquad supp\, K_h \in B(0,R), \tag{4}$$

$$ii. \quad \|\mathscr{K}_h\|_{L^1(\Pi^d)} \longrightarrow +\infty. \tag{5}$$

Assume that $\partial_t \rho_k \in L^q(0,T,W^{-1,q}(\Pi^d))$ (with $q>1$) for any smooth compact set Ω, uniformly in k and

$$\limsup_k \sup_{t\in[0,T]} \Big[\frac{1}{\|\mathscr{K}_h\|_{L^1}} \int_{\Pi^{2d}} \mathscr{K}_h(x-y)\,|\rho_k(t,x)-\rho_k(t,y)|^p\,dx\,dy\Big] \longrightarrow 0, \text{ as } h\to 0, \tag{6}$$

then ρ_k is compact in $L^p_{loc}((0,T)\times\Pi^d)$. Conversely if ρ_k is compact in $L^p_{loc}((0,T)\times\Pi^d)$ then the above quantity converges to 0 with h.

For reader's convenience, we just quickly recall why (6) implies the compactness in space (by simply forgetting the time dependency). Denote $\bar{\mathscr{K}_h}$ the normalized kernel

$$\bar{\mathscr{K}_h} = \frac{\mathscr{K}_h}{\|\mathscr{K}_h\|_{L^1}}.$$

Write

$$\begin{aligned} \|\rho_k - \bar{\mathscr{K}_h}\star_x \rho_k\|^p_{L^p} &\leq \frac{1}{\|\mathscr{K}_h\|^p_{L^1}} \int_{\Pi^d}\Big(\int_{\Pi^d} \mathscr{K}_h(x-y)|\rho_k(t,x)-\rho_k(t,y)|dx\Big)^p dy \quad (7)\\ &\leq \frac{1}{\|\mathscr{K}_h\|_{L^1}} \int_{\Pi^{2d}} \mathscr{K}_h(x-y)|\rho_k(t,x)-\rho_k(t,y)|^p dx\,dy, \end{aligned}$$

which converges to zero uniformly in k as the limsup is 0 for the sup in time. On the other-hand for a fixed h, $\overline{\mathscr{K}}_h \star_x u_k$ is compact in k so for example for any $z>0$

$$\begin{aligned} \|\rho_k - \rho_k(.+z)\|_{L^p} &\leq 2\,\|\rho_k - \bar{\mathscr{K}_h}\star_x \rho_k\|_{L^p} + \|\bar{\mathscr{K}_h}\star_x \rho_k - \bar{\mathscr{K}_h}\star_x \rho_k(.+z)\|_{L^p} \quad (8)\\ &\leq 2\,\|\rho_k - \bar{\mathscr{K}_h}\star_x \rho_k\|_{L^p} + |z|\,\|\rho_k\|_{L^p}\,\|\bar{\mathscr{K}_h}\|_{W^{1,\infty}}. \quad (9) \end{aligned}$$

This shows by optimizing in h that

$$\sup_k \|\rho_k - \rho_k(.+z)\|_{L^p} \longrightarrow 0, \quad \text{as } |z|\to 0.$$

proving the compactness in space by the Rellich criterion. Concerning the compactness in time, one just has to use the uniform bound on $\partial_t \rho_k$.

The $\mathscr{K}_{h_0}$ functions. Define K_h a sequence of non negative functions,

$$K_h(x) = \frac{1}{(h+|x|)^a}, \quad \text{for } |x|\leq 1/2,$$

with some $a>d$ and K_h non negative, independent of h for $|x|\geq 2/3$, with support in $B(0,3/4)$ and periodized such as to belong in $C^\infty(\Pi^d\setminus B(0,3/4))$.

For convenience, let us denote

$$\overline{K}_h(x) = \frac{K_h(x)}{\|K_h\|_{L^1}}.$$

For $0 < h_0 < 1$, the important quantity to be used in Proposition 1 will be

$$\mathscr{K}_{h_0}(x) = \int_{h_0}^{1} \overline{K}_h(x)\, \frac{dh}{h}$$

where

$$K_h(x) = \frac{1}{(h+|x|)^a}, \quad \text{for } |x| \le 1/2.$$

Remark the important property: $\|\mathscr{K}_{h_0}\|_{L^1} \sim |\log h_0|$.

4 Proof of Theorem 1

As usually the proof of global weak solutions of PDEs is divided in three steps:

- *A priori* energy estimates and control of unknowns,
- Stability of weak sequences: Compactness,
- Construction of approximate solutions.

4.1 Energy estimates and control of unknowns.

Energy estimate. Le us multiply the Stokes equation by u and integrate by parts, we get

$$\mu \int_{\Pi^d} |\nabla u_k|^2 + \alpha \int_{\Pi^d} |u_k|^2 + \int_{\Pi^d} \nabla P(\rho_k) \cdot u = \int_{\Pi^d} S_k \cdot u_k.$$

Now we write the equation satisfied by $\rho_k e(\rho_k)$ where $e(\rho_k) = \int_{\rho_{\rm ref}}^{\rho_k} P(s)/s^2 ds$, with $\rho_{\rm ref}$ a constant reference density, we get

$$\partial_t(\rho e(\rho)) + \operatorname{div}(\rho e(\rho) u) + P(\rho)\operatorname{div} u = 0.$$

Integrating in space and adding to th first equation we get

$$\frac{d}{dt}\int_{\Pi^d} \rho_k e(\rho_k) + \mu \int_{\Pi^d} |\nabla u_k|^2 = \int_{\Pi^d} S_k \cdot u_k.$$

One only needs $S_k \in L^2([0,\ T],\ H^{-1}(\Pi^d))$ uniformly and using the behavior of P, then we get the uniform bound

$$\rho_k^\gamma \in L^\infty(0,T;L^1(\Pi^d)), \qquad u_k \in L^2(0,T;H^1(\Pi^d)).$$

Extra integrability on ρ_k. When now considering the compressible system (1), the divergence $\mathrm{div} u_k$ is given

$$\mathrm{div} u_k = \frac{1}{\mu} P(\rho_k) + \frac{1}{\mu} \Delta^{-1} \mathrm{div} R_k$$

with $R_k = S_k - \alpha u_k$. Therefore, since $\rho_k \in L^\infty(0,T;L^\gamma(\Pi^d))$, if we multiply by ρ_k^θ, we get

$$I = \int_0^T \int_{\Pi^d} P(\rho_k)\rho_k^\theta = \mu \int_0^T \int_{\Pi^d} \mathrm{div} u_k \rho_k^\theta - \int_0^T \int_{\Pi^d} \Delta^{-1} \mathrm{div} R_k \, \rho_k^\theta$$

which is easily bounded as follows

$$I \le \left[\mu \|\mathrm{div} u_k\|_{L^2((0,T)\times\Pi^d)} + \|\Delta^{-1} \mathrm{div} R_k\|_{L^2((0,T)\times\Pi^d)}\right] \|\rho_k^\theta\|_{L^2((0,T)\times\Pi^d)}$$

Thus using the behavior of P and information on u_k and R_k, we get for large density

$$\int_0^T \int_{\Pi^d} (\rho^{\gamma+\theta}) \le C + \varepsilon \int_0^T \int_{\Pi^d} (\rho^{2\theta}).$$

Thus we get a control on $\rho_k^{\gamma+\theta}$ if $\theta \le \gamma$. Therefore, we get $\rho_k \in L^p((0,T)\times\Pi^d)$ with $p > 2$ is $\gamma > 1$.

Remark 2. Note that for the barotropic compressible Navier-Stokes equations, we get

$$\frac{1}{2}\frac{d}{dt} \int_{\Pi^d} \rho |u_k|^2 + \frac{d}{dt} \int_{\Pi^d} \rho_k e(\rho_k) + \mu \int_{\Pi^d} |\nabla u_k|^2 = 0.$$

and

$$\int_0^T \int_{\Pi^d} \rho_k^{\gamma+\theta} < +\infty$$

for $\theta \le 2\gamma/d - 1$ where d is the space dimension. The constraint on γ in [4] is different because of more restrictive integrability information (due to the presence of the total time derivative).

4.2 Stability of weak sequences: Compactness

We will prove the following result which is the main part of the proof

Proposition 2. *Assume* (ρ_k, u_k) *satisfy system* (1) *in a weak sense with a pressure law satisfying* (2)–(3) *and with the following weak regularity*

$$\sup_k \|\rho_k^\gamma\|_{L_t^\infty L_x^1} < \infty, \qquad \sup_k \|\rho_k\|_{L_{t,x}^p} < \infty \quad \textit{with } p \le 2\gamma,$$

and

$$\sup_k \|u_k\|_{L^2_t H^1_x} < \infty.$$

If the source term S_k *is compact in* $L^2([0,\ T],\ H^{-1}(\Pi^d))$ *and the initial density sequence* $(\rho_k)_0$ *is assumed to be compact and hence satisfies*

$$\limsup_k \Big[\frac{1}{\|K_h\|_{L^1}}\int_{\Pi^{2d}} K_h(x-y)|(\rho_k^x)_0-(\rho_k^y)_0|\Big] = \varepsilon(h) \to 0 \text{ as } h \to 0,$$

then ρ_k *is locally compact.*

Remark 3. Here and in the following, we use the convenient notation $\rho_k^x = \rho_k(t,x)$, $\rho_k^y = \rho_k(t,y)$ and $(\rho_k^x)_0 = \rho_k(t=0,x)$, $(\rho_k^y)_0 = \rho_k(t=0,y)$.

Proof. Of course, we know that due to the weak regularity of $\mathrm{div} u_k$, we cannot expect to simply propagate the regularity assumed on the density. The idea is to accept to lose some of it by introducing appropriate weights. More precisely, we consider weights w_k such that $w_k|_{t=0} = 1$ and thus in particular, since ρ_k^0 is compact

$$\limsup_k \Big[\frac{1}{|\log h_0|}\int_{\Pi^{2d}} \mathscr{K}_{h_0}(x-y)|(\rho_k^x)_0-(\rho_k^y)_0|\Big]((w_k^x)_0+(w_k^y)_0) \to 0 \text{ as } h \to 0.$$

Remark that

$$\frac{1}{|\log h_0|}\int_{h_0}^1 \frac{\varepsilon(h)}{h}dh \to 0 \text{ when } h_0 \to 0.$$

Let us now choose weights satisfying PDEs which are dual to the continuity equation

$$\begin{cases} \partial_t w_k^x + u_k^x \cdot \nabla_x w_k^x + \lambda D_k^x w_k^x = 0, \\ w_k^x|_{t=0} = (w_k^x)_0 = 1, \end{cases} \tag{10}$$

and

$$\begin{cases} \partial_t w_k^y + u_k^y \cdot \nabla_y w_k^y + \lambda D_y^x w_k^y = 0, \\ w_k^y|_{t=0} = (w_k^y)_0 = 1, \end{cases} \tag{11}$$

with λ a constant parameter to be chosen later on and appropriate positive damping terms D_k^x and D_k^y. We first study the propagation of the quantity

$$R_{h_0}(t) = \int_{\Pi^{2d}} \mathscr{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,(w^x+w^y)\,dx\,dy = \frac{1}{\|K_h\|_{L^1}}\int_{h_0}^1 R(t)\frac{dh}{h}$$

where

$$R(t) = \int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x-\rho_k^y|\,(w^x+w^y)\,dx\,dy.$$

We show that it is possible to choose D_k^x, D_k^y and λ such that

$$\limsup_k \Big[\frac{1}{|\log h_0|}\int_{\Pi^{2d}} \mathscr{K}_{h_0}(x-y)|\rho_k^x-\rho_k^y|\Big](w_k^x+w_k^y) \to 0 \text{ as } h_0 \to 0$$

as initially. Then, we will need properties on w_k^x and w_k^y to conclude that we also have

$$\limsup_k \Big[\frac{1}{|\log h_0|}\int_{\Pi^{2d}} \mathscr{K}_{h_0}(x-y)|\rho_k^x-\rho_k^y|\Big] \to 0 \text{ as } h_0\to 0$$

which is the criterion giving compactness. Thus the proof is divided in two parts.

I) *First step: Propagation of a weighted regularity.* Using the transport equation, we obtain that

$$\begin{aligned}&\partial_t|\rho_k^x-\rho_k^y| + \mathrm{div}_x\,(u_k^x\,|\rho_k^x-\rho_k^y|) + \mathrm{div}_y\,(u_k^y\,|\rho_k^x-\rho_k^y|)\\ &\le \tfrac12(\mathrm{div}_x u_k^x + \mathrm{div}_y u_k^y)\,|\rho_k^x-\rho_k^y| - \tfrac12(\mathrm{div}_x u_k^x - \mathrm{div}_y u_k^y)\,(\rho_k(x)+\rho_k(y))\,s_k,\end{aligned} \tag{12}$$

where $s_k = sign\,(\rho_k(x)-\rho_k(y))$. Remark that these calculations can be justified for a fixed k through the DiPerna-Lions theory on renormalized solutions because the densities and the gradient of the velocity are in L^2 in space and time. From this equation on $|\rho_k^x-\rho_k^y|$, we deduce by symmetry that

$$\begin{aligned}\frac{d}{dt}R(t) = &\int_{\Pi^{2d}} \nabla K_h(x-y)\,(u_k^x-u_k^y)\,|\rho_k^x-\rho_k^y|\,(w^x+w^y)) \qquad (13)\\ &-\int_{\Pi^{2d}} K_h(x-y)\,(\mathrm{div} u_k^x - \mathrm{div} u_k^y)\,(\rho_k^x+\rho_k^y)\,s_k\,w^x\\ &+2\int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x-\rho_k^y|\,\left(\partial_t w_k^x + u_k^x\cdot\nabla_x w^x + \frac12\,\mathrm{div}_x u_k^x\,w_k^x\right)\\ =\ & A_1+A_2+A_3.\end{aligned}$$

First term. The first term will lead to non symmetric contributions. By definition of K_h, we have

$$|z||\nabla K_h(z)| \le C K_h(z).$$

We hence write

$$\begin{aligned}A_1 = &\int_{\Pi^{2d}} \nabla K_h(x-y)\cdot(u_k^x-u_k^y)\,|\rho_k^y-\rho_k^y|\,(w_k^x+w_k^y)\\ &\le C\int_{\Pi^{2d}} K_h(x-y)\,(D_{|x-y|}u_k^x + D_{|x-y|}u_k^y)\,|\rho_k^x-\rho_k^y| w_k^x,\end{aligned} \tag{14}$$

where we have used here

$$|u(x)-u(y)| \le C\,|x-y|\,(D_{|x-y|}u_k^x + D_{|x-y|}u_k^y),$$

for an operator $D_{|x-y|}$; this inequality is fully described in Lemma 1 in the appendix.

The key problem is the $D_h u_k^y w_k^x$ term which one will have to control by the term $M|\nabla u_k^x| w_k^x$ in the penalization. This is where integration over h and the use of translation properties of operator will be used. For that we will add and subtract an appropriate quantity to obtain a symmetric expression.

Denoting $z=x-y$, we have

$$\int_{h_0}^1 \frac{A_1}{\|K_h\|_L^1}\frac{dh}{h} \ \le C\int_{h_0}^1\int_0^t\int_{\Pi^{2d}} \overline{K_h}(z)\|D_{|z|}u_k(\cdot)-D_{|z|}u_k(\cdot+z)\|_{L^2}\frac{dh}{h} \tag{15}$$

$$+C\int_0^t\int_{\Pi^{2d}}\mathcal{K}_{h_0}(x-y)D_{|x-y|}u_k(x)\,|\rho_k^x-\rho_k^y|\,w_k^x.$$

Using Lemma 2 which bounds $D_{|x-y|}u_k^x$ by the Maximal operator $M\,|\nabla u_k|(x)$, we deduce that

$$\begin{aligned}\int_{h_0}^1\frac{A_1}{\|K_h\|_L^1}\frac{dh}{h} &\le C\int_{h_0}^1\int_0^t\int_{\Pi^{2d}}\overline{K_h}(z)\|D_{|z|}u_k(\cdot)-D_{|z|}u_k(\cdot+z)\|_{L^2}\frac{dh}{h} \qquad (16)\\ &+C\int_0^t\int_{\Pi^{2d}}\mathcal{K}_{h_0}(x-y)M|\nabla u_k^x|\,|\rho_k^x-\rho_k^y|w_k^x.\end{aligned}$$

The second term will be absorbed using the weight definition. But the first quantity has to be controlled using the property of the translation of operator D_h and for this reason, this calculation is critical as it is the one which imposes the scales in $\mathcal{K}_{h_0}$.

Second term. Use the relation between $\operatorname{div} u_k^x$ (respectively $\operatorname{div} u_k^y$) with ρ_k^x (respectively ρ_k^y), to obtain

$$A_2=-\int_{\Pi^{2d}}K_h(x-y)\,(p(\rho_k^x)-p(\rho_k^y))\,(\rho_k^x+\rho_k^y)\,s_k\,w^x+Q_h(t)$$

where $Q_h(t)$ encodes the compactness in space of $\Delta^{-1}\operatorname{div}R_k$ and therefore may be forgotten for simplicity as

$$\frac{1}{|\log h_0|}\int_0^t\int_{\Pi^{2d}}\mathcal{K}_{h_0}(x-y)Q_h(t)\to 0 \text{ as } h_0\to 0,$$

as R_k is compact in $L_t^2H_x^{-1}$ and hence $\Delta^{-1}\operatorname{div}R_k$ is compact in $L_{t,x}^2$ by the gain of one derivative.

The bad term $p(\rho_k^y)w_k^x$ cannot a priori be bounded directly with weights. Hence we have to work a little on the expression A_2.

First note that we have $\rho_k^x+\rho_k^y\ge|\rho_k^x-\rho_k^y|$.

– Case 1: The case where $p(\rho_k^x)-p(\rho_k^y))(\rho_k^x-\rho_k^y)\ge 0$. Then we have the right sign for the contribution namely a negative sign.

– Case 2: The case $p(\rho_k^x)-p(\rho_k^y))(\rho_k^x-\rho_k^y)<0$ and $\rho_k^y\le\rho_k^x/2$ or $\rho_k^y\ge 2\rho_k^x$.

Assume we are in the case $\rho_k^y\ge 2\rho_k^x$, then

$$(p(\rho_k^x)-p(\rho_k^y))(\rho_k^x+\rho_k^y)s_k\ge -C\,(\rho_k^x)^\gamma\,|\rho_k^x-\rho_k^y|,$$

since $p(\xi)\le p(0)+C\xi^{\gamma-1}\xi\le C\xi^\gamma$. If we now look at the case $p(\rho_k^x)\le p(\rho_k^y)$ and $\rho_k^y\le\rho_k^x/2$, then we again bound

$$(p(\rho_k^x)-p(\rho_k^y))(\rho_k^x+\rho_k^y)s_k\ge -C\,(\rho_k^x)^\gamma\,|\rho_k^x-\rho_k^y|.$$

— Case 3: The case where $p(\rho_k^x)-p(\rho_k^y)$ and $\rho_k^x-\rho_k^y$ have different signs but $\rho_k^x/2\le\rho_k^y\le 2\rho_k^x$. Then it is easy to get again

$$(p(\rho_k^x) - p(\rho_k^y))(\rho_k^x + \rho_k^y)s_k \geq -C\,(1 + (\rho_k^x)^\gamma)\,|\rho_k^x - \rho_k^y|.$$

Therefore we get the following interesting bound:

$$A_2 \leq C \int K_h(x-y)\,(1 + (\rho_k^x)^\gamma))\,|\rho_k^x - \rho_k^y|\,w_k^x.$$

Third term. Using the equations satisfied by w_k^x and w_k^y, we have

$$\begin{aligned} A_3 = & \int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,\left(\partial_t w_k^x + u_k^x \cdot \nabla_x w^x + \frac{1}{2}\,\mathrm{div}_x u_k^x\, w_k^x\right) \\ & \leq \int_{\Pi^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,\left(-\lambda D_k^x + \frac{1}{2}\,\mathrm{div}_x u_k^x\right) w_k^x. \end{aligned} \tag{17}$$

Conclusion of the first step. Collecting the three steps, we get

$$\begin{aligned} R_{h_0}(t) - R_{h_0}(0) \ & \leq C \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \overline{K_h}(z) \|D_{|z|} u_k(\cdot) - D_{|z|} u_k(\cdot + z)\|_{L^2} \frac{dh}{h} \\ & + C \int_0^t \int_{\Pi^{2d}} \mathscr{K}_{h_0}(x-y) M |\nabla u_k^x|\,|\rho_k^x - \rho_k^y|\,w_k^x \\ & + C \int_0^t \int_{\Pi^{2d}} \mathscr{K}_{h_0}(x-y)(1 + (\rho_k^x)^\gamma))|\rho_k^x - \rho_k^y| w_k^x \\ & + \int_0^t \int_{\Pi^{2d}} \mathscr{K}_{h_0}(x-y)\,|\rho_k^x - \rho_k^y|\,\left(-\lambda D_k^x + \frac{1}{2}\,\mathrm{div}_x u_k^x\right) w_k^x. \end{aligned} \tag{18}$$

Therefore we choose

$$D_k^x = M |\nabla u_k^x| + |div_x u_k^x| + (\rho_k^x)^\gamma,$$

with a similar formula for D_k^y. Then for λ large enough, we get

$$\begin{aligned} R_{h_0}(t) - R_{h_0}(0) \ & \leq C \int_{h_0}^1 \int_0^t \int_{\Pi^{2d}} \overline{K_h}(z) \|D_{|z|} u_k(\cdot) - D_{|z|} u_k(\cdot + z)\|_{L^2} \frac{dh}{h} \\ & + C \int_0^t R_{h_0}(\tau) d\tau. \end{aligned} \tag{19}$$

We now use translation property implied by the square functions given in Appendix, and more precisely using Lemma 3, we may write

$$R_{h_0}(t) - R_{h_0}(0) \ \leq C\,|\log h_0|^{1/2} \int_0^t \|u(\tau,.)\|_{H_x^1}\, d\tau + C \int_0^t R_{h_0}(\tau) d\tau. \tag{20}$$

Therefore using that u_k is uniformly bounded in $L^2(0,T;H^1(\Pi^d))$ and using the assumption on $R_{h_0}(0)$, then by Gronwall Lemma, we get that

$$\limsup_k \sup_{t\in[0,\,T]} \frac{R_{h_0}}{|\log h_0|} \longrightarrow 0, \quad \text{as } h_0 \to 0,$$

which is the desired propagation property.

II) *Second step.* We now have to control the weights so as to remove them. Namely we want to prove that

$$\limsup_k [\frac{1}{|\log h_0|}\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy] \to 0 \text{ as } h_0 \to 0$$

and not only

$$\limsup_k [\frac{1}{|\log h_0|}\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,(w_k^x+w_k^y)dx\,dy] \to 0 \text{ as } h_0 \to 0.$$

Remark that from its equation, the weight also satisfies

$$\partial_t|\log w_k| + u_k\cdot\nabla|\log w_k| = \lambda D_k,$$

with

$$D_k = M|\nabla u_k| + |\mathrm{div} u_k| + (\rho_k)^\gamma.$$

Thus multiplying by ρ_k and using the mass or continuity equation, we get

$$\frac{d}{dt}\int_{\Pi^d} \rho|\log w_k| = \lambda\int_{\Pi^d} \rho D_k.$$

Note that $u_k \in L^2(0,T;H^1(\Pi^d))$ and $\rho_k \in L^{2\gamma}$ with $\gamma > 1$, thus the right-hand side is uniformly bounded.

Denoting $\omega = \{x : w_k \le \eta\}$, note that

$$\begin{aligned}\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy &= \int_{h_0}^1\int_{\Pi^{2d}} \overline{K}_h(x-y)|\rho_k^x-\rho_k^y|\frac{dh}{h}\\ &= \int_{h_0}^1\int_{x\in\omega_\eta^c \text{ or } y\in\omega_\eta^c} \overline{K}_h(x-y)|\rho_k^x-\rho_k^y|\frac{dh}{h}\\ &+\int_{h_0}^1\int_{x\in\omega_\eta \text{ and } y\in\omega_\eta} \overline{K}_h(x-y)|\rho_k^x-\rho_k^y|\frac{dh}{h}\\ &= B_1+B_2.\end{aligned}$$

It suffices to observe that

$$B_1 \le \frac{1}{\eta}R_{h_0}$$

while by the property of the weights w_k

$$B_2 \le 2\int_{h_0}^1\int_{\Pi^{2d}} \overline{K}_h(x-y)\,\rho_k\,1_{w_k\le\eta}\,\frac{dh}{h} \le C\,\frac{|\log h_0|}{|\log\eta|}\int_{\Pi^d}\rho_k\,|\log w_k|\,dx \le C\,\frac{|\log h_0|}{|\log\eta|}.$$

Combining the estimates, one obtains

$$\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy \le C\Big(\frac{\int_{h_0}^1 \varepsilon(h)\frac{dh}{h}+|\log h_0|^{1/2}}{\eta}+\frac{\|\mathcal{K}_{h_0}\|_{L^1}}{|\log\eta|}\Big)$$

and therefore

$$\frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}}\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy$$
$$\le C\left(\frac{\frac{1}{|\log h_0|}\displaystyle\int_{h_0}^1 \varepsilon(h)\frac{dh}{h}+|\log h_0|^{-1/2}}{\eta}+\frac{1}{|\log\eta|}\right).$$

Denoting $\overline{\varepsilon}(h_0)=\int_{h_0}^1 \varepsilon(h)/h\,dh$ and optimizing η, we get

$$\frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}}\int_{\Pi^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy \le \frac{C}{|\log\big(|\log h_0|^{-1/2}+\overline{\varepsilon}(h_0)\big)|^{1/2}}$$

and therefore the result holds.

Remark. The choice of appropriate weights is important in the proof. It really depends on the system under consideration. In [4], we can find various choices depending on pressure laws or anisotropy in the viscous tensor. These weights penalize in some sense bad trajectories.

4.3 Construction of approximate solutions.

Our starting point for global existence is the following regularized system

$$\begin{cases} \partial_t\rho_k+\mathrm{div}(\rho_k u_k)=\alpha_k\Delta\rho_k, \\ -\mu\Delta u_k-(\lambda+\mu)\nabla\mathrm{div}u_k+\nabla P_\varepsilon(\rho_k)+\alpha_k\nabla\rho_k\cdot\nabla u_k=S, \end{cases} \tag{21}$$

with the fixed source term S and the fixed initial data

$$\rho_k|_{t=0}=\rho^0. \tag{22}$$

The pressure P_ε is define as follows:

$$P_\varepsilon(\rho)=p(\rho) \text{ if } \rho\le c_{0,\varepsilon}, \qquad P_\varepsilon(\rho)=p(C_{0,\varepsilon})+C(\rho-c_{0,\varepsilon})^\beta \text{ if } \rho\ge c_{0,\varepsilon},$$

with large enough β. As usual the equation of continuity is regularized by means of an artificial viscosity term and the momentum balance is replaced by a Faedo-Galerkin approximation to eventually reduce the problem on X_n, a finite-dimensional vector space of functions.

This approximate system can then be solved by a standard procedure: The velocity u_k of the approximate momentum equation is looked as a fixed point of a suitable integral operator. Then given u_k, the approximate continuity equation is solved directly by means of the standard theory of linear parabolic equations. This methodology concerning the compressible Navier–Stokes equations is well explained and described in the reference books [9], [12]. We omit the rest of this classical (but tedious) procedure and we assume that we have well posed and smooth solutions to (21)–(22).

We now use the classical energy and extra bounds estimates detailed in the previous section. Note that they remain the same in spite of the added viscosity in the continuity equation. This is the reason in particular for the added term $\alpha_k \nabla\rho_k \cdot \nabla u_k$ in the momentum equation to keep the same energy balance. Let us summarize the *a priori* estimates that are obtained

$$\sup_{k,\varepsilon}\sup_t \int_{\Pi^d} \rho_k^\gamma\, dx < \infty, \qquad \sup_{k,\varepsilon}\int_0^T \int_{\Pi^d} |\nabla u_k|^2\, dx\, dt < \infty,$$

and

$$\sup_{k,\varepsilon}\int_0^T \int_{\Pi^d} \rho_k^p(t,x)\, dx\, dt < \infty$$

for all $p \le 2\gamma$. From those bounds it is straightforward to deduce that $\rho_k u_k$ belong to $L^q_{t,x}$ for some $q > 1$, uniformly in k and ε. Therefore using the continuity equation bounds on $\partial_t \rho_k$. We have now to show the compactness of ρ_k in L^1 and we can use the procedure mentioned in [7] letting α_k goes to zero. Then extracting converging subsequences, we can pass to the limit in every term (by classical approach) and obtain the existence of weak solutions to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\mu \Delta u + \alpha u + \nabla P_\varepsilon(\rho) = S. \end{cases} \tag{23}$$

It remains then to pass to the limit with respect to ε. This is done using the stability procedure developed in the previous subsection concerning compactness for general pressure laws.

5 The compressible Navier-Stokes equations

We state in this section the main existence results that have been obtained in [4]. There exist several differences and complications compared to the global existence result we proved in this short paper due in particular to the presence of the total time derivative. This leads to more restrictions on the coefficient γ in the pressure law. It could be interesting to try to extend our results with better gamma exponent using the renormalization procedure in [8] or with anisotropy in the stress tensor.

I) The isotropic compressible Navier–Stokes equations with general pressure laws. Let us consider the isotropic compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div} u + \nabla P(\rho) = \rho f, \end{cases} \tag{24}$$

with $2\mu/d + \lambda$, a pressure law P which is continuous on $[0, +\infty)$, P locally Lipschitz on $(0, +\infty)$ with $P(0) = 0$ such that there exists $C > 0$ with

$$C^{-1}\rho^\gamma - C \leq P(\rho) \leq C\rho^\gamma + C \tag{25}$$

and for all $s \geq 0$

$$|P'(s)| \leq \bar{P} s^{\tilde{\gamma}-1}. \tag{26}$$

One then has global existence

Theorem 2. *Assume that the initial data u_0 and $\rho_0 \geq 0$ satisfy the bound*

$$E_0 = \int_{\Pi^d} \Big(\rho_0 \frac{|u_0|^2}{2} + \rho_0 e(\rho_0)\Big)\, dx < +\infty.$$

Let the pressure law P satisfies (25) *and* (26) *with*

$$\gamma > \big(\max(2, \tilde{\gamma}) + 1\big)\, \frac{d}{d+2}. \tag{27}$$

Then there exists a global weak solution of the compressible Navier–Stokes system (24) *satisfying the initial data conditions*

$$\rho|_{t=0} = \rho_0, \qquad \rho u|_{t=0} = \rho_0 u_0.$$

Moreover the solution satisfies the explicit regularity estimate

$$\int_{\Pi^{2d}} 1_{\rho_k(x) \geq \eta}\, 1_{\rho_k(y) \geq \eta}\, K_h(x-y)\, \chi(\delta \rho_k) \leq \frac{C\, \|K_h\|_{L^1}}{\eta^{1/2}\, |\log h|^{\theta/2}},$$

for some $\theta > 0$ where χ is a C^2 function such that $\chi(\xi) = |\xi|^2$ if $|\xi| \leq 1/2$ and $\chi(\xi) = |\xi|$ if $|\xi| > 1$.

II) A non-isotropic compressible Navier–Stokes equations. We consider an example of non-isotropic compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}\,(A(t)\,\nabla u) - (\mu + \lambda)\nabla \mathrm{div} u + \nabla P(\rho) = 0, \end{cases} \tag{28}$$

with $A(t)$ a given smooth and symmetric matrix, satisfying

$$A(t) = \mu\, Id + \delta A(t), \quad \mu > 0, \quad \frac{2}{d}\mu + \lambda - \|\delta A(t)\|_{L^\infty} > 0. \tag{29}$$

We again take P continuous on $[0,+\infty)$ with $P(0) = 0$ but require it to be monotone after a certain point

$$C^{-1}\rho^{\gamma-1} - C \le P'(\rho) \le C\rho^{\gamma-1} + C. \tag{30}$$

with $\gamma > d/2$. The second main result that we obtain is

Theorem 3. *Assume that the initial data u_0 and $\rho_0 \ge 0$ satisfies the bound*

$$E_0 = \int_{\Pi^d} \big(\rho^0 \frac{|u_0|^2}{2} + \rho_0 e(\rho_0)\big)\, dx < +\infty.$$

Let the pressure P satisfies (30) with

$$\gamma > \frac{d}{2}\left[\left(1+\frac{1}{d}\right) + \sqrt{1+\frac{1}{d^2}}\right].$$

There exists a universal constant $C_\star$ such that if

$$\|\delta A\|_\infty \le C_\star (2\mu + \lambda),$$

then there exists a global weak solution of the compressible Navier–Stokes equation (28) *satisfying the initial data conditions*

$$\rho|_{t=0} = \rho_0, \qquad \rho u|_{t=0} = \rho_0 u_0.$$

The isotropic energy inequality is replaced by the following anisotropic energy

$$E(\rho,u)(\tau) + \int_0^\tau \int_\Omega (\nabla_x u^T A(t)\, \nabla u + (\mu+\lambda)\, |\mathrm{div} u|^2) \le E_0.$$

6 Appendix

In this appendix, let us give different results which are used in the paper. The interested reader is referred to [4] for details and proofs. These concern Maximal functions, Square functions and translation of operators. First we remind the well known inequality

$$|\Phi(x) - \Phi(y)| \le C\,|x-y|\,(M|\nabla\Phi|(x) + M|\nabla\Phi|(y)), \tag{31}$$

where M is the localized maximal operator

$$M f(x) = \sup_{r\le 1} \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x+z)\, dz. \tag{32}$$

Let us mention several mathematical properties that may be proved, see [4]. First one has

Lemma 1. *There exists $C > 0$ s.t. for any $u \in W^{1,1}(\Pi^d)$, one has*

$$|u(x) - u(y)| \leq C\,|x-y|\,(D_{|x-y|}u(x) + D_{|x-y|}u(y)),$$

where we denote

$$D_h u(x) = \frac{1}{h}\int_{|z|\leq h} \frac{|\nabla u(x+z)|}{|z|^{d-1}}\,dz.$$

Note that this result implies the estimate (31) as

Lemma 2. *There exists $C > 0$, for any $u \in W^{1,p}(\Pi^d)$ with $p \geq 1$*

$$D_h\, u(x) \leq C\,M|\nabla u|(x).$$

The key improvement in using D_h is that small translations of the operator D_h are actually easy to control

Lemma 3. *Let $u \in H^1(\Pi^d)$ then have the following estimates*

$$\int_{h_0}^1 \int_{\Pi^d} \overline{K}_h(z)\,\|D_{|z|}\,u(.) - D_{|z|}\,u(.+z)\|_{L^2}\,dz\,\frac{dh}{h} \leq C\,|\log h_0|^{1/2}\,\|u\|_{H^1}. \qquad (33)$$

This lemma is critical and explain why we propagate a quantity integrated with respect to h with a weight dh/h namely with the Kernel $\mathcal{K}_{h_0}$. The full proof is rather classical and can be found in [4].

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