# Clustering and asymptotic behavior in opinion formation

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**Abstract**

We investigate the long time behavior of models of opinion formation. We consider the case of compactly supported interactions between agents which are also non-symmetric, including for instance so-called Krause model. Because of the finite range of interaction, convergence to a unique consensus is not expected in general. We are nevertheless able to prove the convergence to a final equilibrium state composed of possibly several local consensus; This result had so far only been conjectured through numerical evidence. Because of the non symmetry in the model, the analysis is delicate and is performed in two steps: First using entropy estimates to prove the formation of stable clusters and then studying the evolution in each cluster. We study both discrete and continuous in time models and give rates of convergence when those are available.

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# 1 Introduction

Thanks to the development of social media, the dynamics of opinion formation has recently generated much interest [1,4,10,11,17,20,21]. Through a complex network of interactions between individuals emerge groups with different opinions raising several questions; for instance how groups are formed and how many of them will survive throughout time.

Different models have been introduced to study opinion dynamics [2,3,10,12,13,15–17,21]. In this paper, we focus on a widely-used model referred to as the *consensus model* [7–9,13,14]. The consensus model describes the evolution of $N$ agents which tend to have a similar *opinion* as the one of their close neighbors. Each opinion is represented by a quantity $x_i$ (a scalar or a vector) and evolves according to the following dynamics:

$$\dot{x}_i = \frac{\sum_j \phi_{ij}(x_j - x_i)}{\sum_j \phi_{ij}} , \quad \phi_{ij} = \phi(|x_j - x_i|^2). \tag{1.1}$$

Here, $\phi$ is the so-called influence function, it is a non-negative function strictly positive at the origin (i.e. $\phi(0) > 0$). The greater $\phi_{ij}$ is, the more the agent $i$ is influenced by $j$ (and vice-versa). Without loss of generality, one can assume that $\phi(0) = 1$.

Several studies have been conducted to study numerically the long time behavior of the consensus model [5,7,19]. It has been observed that the dynamics generate concentration of opinions (also called *clusters*) and that, after a transient period, the configuration stabilizes. The goal of this manuscript is to prove analytically those observations. One of the main difficulty to study the consensus model is the lack of conserved quantities. For instance, the *total momentum*, $\frac{1}{N}\sum_{i=1}^{N} x_i$, is not preserved in time. This lack of conservation is due to the non-symmetry in the interaction. Writing the model as

$$\dot{x}_i = a_{ij}(x_j - x_i), \quad a_{ij} = \frac{\phi_{ij}(x_j - x_i)}{\sum_j \phi_{ij}}, \tag{1.2}$$

we find out that $a_{ij} \neq a_{ji}$ in general. Thus, the influence between agents is not symmetric.

Depending on the interaction function $\phi$, one can have very different dynamics. When the interaction function $\phi$ is globally supported (i.e. $\phi(r) > 0$ for all $r \geq 0$) meaning that every agent interacts with each other, the dynamics is easily understood: the system converges to a so-called *consensus* [19]. In other words, there exists one opinion $x_\infty$ such that

$$x_i(t) \stackrel{t\to\infty}{\longrightarrow} x_\infty \quad , \quad \text{for all } i. \tag{1.3}$$

However the case with a compactly supported interaction function $\phi$ is more relevant and realistic from a modeling perspective. Agents then only interact with those having relatively similar opinions and this makes the corresponding analysis more delicate. One immediately sees that the previous simple scenario cannot always take place: Just take two agents with initial positions $x_1(0)$ and $x_2(0)$ s.t. $\phi(|x_1 - x_2|) = 0$. However in that elementary case, we still have formation of what one would call *local consensus*. This means that the opinions $x_i(t)$ still converge in time to some limit $\bar{x}_i$. All the limits $\bar{x}_i$ are not necessarily equal though.

The question we wish to investigate in this paper is whether we always have local consensus or if some other, more complicated, asymptotic behaviors can manifest. It turns out that for the interaction kernels $\phi$ used in practice, we can prove that local consensus always takes place.

**Theorem 1** *Assume that $\phi \in L^\infty(\mathbb{R}^d)$ with compact support in $[0,1]$. Assume moreover that $\phi \in W^{1,\infty}([0,\ 1-\varepsilon])$ and is strictly positive on $[0,\ 1-\varepsilon]$ for any $\varepsilon > 0$. Finally assume that*

$$|\phi'(r)|^2 \leq C\phi(r), \qquad \text{if} \quad 0 \leq r < 1. \tag{1.4}$$

*Then there exists $\{\bar{x}_i\}$ s.t. for all $i$, $x_i(t) \longrightarrow \bar{x}_i$ as $t \to \infty$. And moreover for any $i, j$, either $\bar{x}_i = \bar{x}_j$ or $|\bar{x}_i - \bar{x}_j| \geq 1$.*

The proof of the theorem is split in two parts. In a first step, using an energy method, we prove that the system converges to *cluster formations*. The energy method is however not sufficient to guarantee the convergence since the set of constant energy is not discrete. Thus, in a second step, we study the evolution of each *cluster* and prove their convergence asymptotically in time.

Let us comment briefly on the assumptions on $\phi$. It is very reasonable to assume that it is bounded and locally Lipschitz; it is always possible to choose the support of length 1. A crucial point is that the support of $\phi$ be simply connected, which means the interval $[0,1]$ here. For essentially every application, this assumption is satisfied since there is no reason for the interaction to suddenly vanish at one point in $[0,1]$. Mathematically this of course helps a lot. For instance it makes the set of stationary states much simpler. We point out that we *do not assume that $\phi$ in continuous*. Obviously given the other assumptions the only possible point of discontinuity is at $r = 1$. Condition (1.4) guarantees that $\phi$ is Lipschitz and has a limit on the left and on the right at $r = 1$. But this leaves open the possibility that

$$\phi(1-) = \lim_{r<1,\, r\to 1} \phi(x) > \lim_{r>1,\, r\to 1} \phi(x) = \phi(1+) = 0.$$

This is important as such discontinuous profiles are used for some applications.

The less obvious assumption is Condition (1.4). It is automatically satisfied if $\phi(1-) > 0$, or if $\phi$ is $C^2$ with $\phi''(1) \neq 0$ as $|\phi'(x)|^2/\phi(x) \overset{x\to 1^-}{\longrightarrow} 2\phi''(1) < \infty$. Nevertheless there are examples of interaction functions $\phi$ that do not satisfy (1.4), for instance $\phi(r) = (1-r)^\alpha$ on $[0,1]$ with $0 < \alpha < 2$ (see figure 1). While those interactions do not seem to be used in applications, they raise the question of whether Th. 1 could be proved without Condition (1.4).

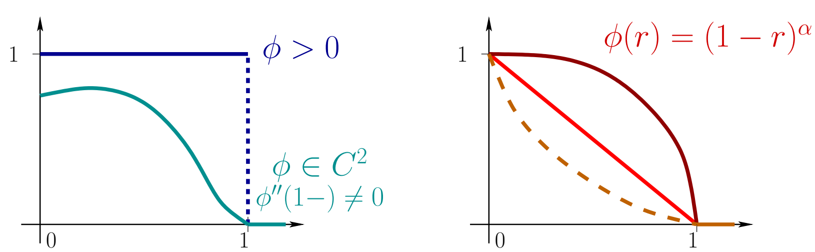


Figure 1: **Left:** interaction functions $\phi$ satisfying the condition (1.4) of Th. 1: $\phi > 0$ (blue) and $\phi \in C^2$ with $\phi''(1-) \neq 0$ (cyan). **Right:** the interaction functions $\phi(r) = (1-r)^\alpha$ with $0 < \alpha < 2$ do not satisfy condition (1.4).

Given the convergence provided by Th. 1, a natural question is whether one could obtain rates of convergence; in the sense of an explicit time decay depending only on some explicit norms of the initial data. In higher dimension or if $\phi(1-) = 0$, this is not possible in general, the speed of convergence depends intrinsically on the initial distribution (see Remark 3.4). However in the other case, one may prove

**Theorem 2** *Assume $d = 1$ and that $\phi \in W^{1,\infty}([0,1))$ with* $\inf_{[0,1)} \phi > 0$ i.e. $\phi(1-) > 0$. *Then there exist $\bar{x}_i$ depending on the full initial distribution such that*

$$|x_i(t) - \bar{x}_i| \leq C\, e^{-\lambda\,(t-t_0)}, \quad \forall t \geq t_0,$$

*where the constants $C$ and $\lambda$ depend only on $N$ and $\phi$, the time $t_0$ depends only on $N$, $\phi$ and the diameter of the initial support.*

**Remark 1.1** *This theorem gives exponential convergence toward equilibrium with rates depending on the interaction function $\phi$, $N$ and the size of the initial support. We kept the formulation with a time $t_0$ (instead of including $e^{\lambda\, t_0}$ in the constant $C$) in order to better represent the two parts in the dynamics. First of all the agents regroup themselves in clusters. A characteristic of the 1-d dynamics (and only in 1-d) is that an agent in a cluster never again interact with an agent in another cluster after this first step. After this first step agents within a cluster interact just as if $\phi$ was bounded was below on $\mathbb{R}$ and there is exponential convergence: This is the second step. However it is possible to estimate the time $t_0$ that the first step takes only if $\phi(1-) > 0$.*

**Remark 1.2** *It is easy to give quantitative estimates on $t_0$, for instance*

$$t_0 \leq \frac{4\, R^2\, N^2\, \|\phi\|_{L^\infty}}{\inf_{[1/2,\ 1)} \phi},$$

*where $R$ is the diameter of the initial support.*

We also tackle the convergence of a discrete version of the consensus model (1.1). Discretizing the equation with an explicit Euler method and a time step $\Delta t = 1$ yields the so-called Krause model [14]:

$$x_i^{n+1} = \frac{\sum_j \phi_{ij}^n x_j^n}{\sum_j \phi_{ij}^n}\ , \quad \phi_{ij}^n = \phi(|x_j^n - x_i^n|^2). \tag{1.5}$$

This dynamics of this discrete dynamics has been studied in [7,19]. In [6], it has been conjectured that the dynamics converges toward a stationary state. We propose in the following to prove rigorously that this conjecture is in fact true.

**Theorem 3** *Assume that $\phi \in L^\infty(\mathbb{R}^d)$ with compact support in* $[0,\ 1]$ *and that $\phi \in W^{1,\infty}([0, 1-\varepsilon])$ and is strictly positive on* $[0,\ 1-\varepsilon]$ *for any $\varepsilon > 0$. Moreover assume again* (1.4) *namely*

$$|\phi'(r)|^2 \leq C\phi(r), \qquad \text{if} \quad 0 \leq r < 1.$$

*Suppose additionally that $\phi$ is non increasing and concave down.*
*Then there exists $\{\bar{x}_i\}$ s.t. for all $i$, $x_i^n \longrightarrow \bar{x}_i$ as $n \to \infty$. And moreover for any $i, j$, either $\bar{x}_i = \bar{x}_j$* or *$|\bar{x}_i - \bar{x}_j| \geq 1$.*

The paper is organized as follows: in the section 2 we introduce several tools (e.g. convex hull, Lyapunov function) to study the dynamics and deduce the convergence of the consensus model in 1D. The convergence in multi-dimension is tackled in the section 3 where we improve some inequalities introduced previously. Finally, in section 4, we establish the convergence of the discrete version of the consensus model.

**Notations.** We denote by $\langle a, b \rangle$ the usual inner product in $\mathbb{R}^d$. In the sequel $C$ will denote a numerical constant whose exact value may change from line to line and which might depend on the initial conditions $x_i(0)$ but which will always be *independent of the time*. We try as much as possible to keep track of the explicit dependence on the number of agents $N$ in the computations; when this is too complicated, we will use $C_N$ for a constant depending also on $N$. Of course this is slightly ambiguous as the initial conditions obviously depend on $N$. The logic here is that if all $x_i(0)$ are taken in a ball of diameter $R$ of order 1 then $C$ will remain of order 1 no matter how large $N$ can be.

# 2 Cluster formation

To analyze the long time behavior of the consensus model, we introduce a decreasing Lyapunov functional $E$. This functional allows us to identify the stationary states of our system. Moreover, since $E$ is a Lyapunov function, the solution has to come closer and closer to these stationary states. As a consequence, the dynamics will create cluster formation.

This property will not be sufficient to deduce the convergence of the dynamics. However, in dimension 1, the cluster formation combined with a simple convexity argument is enough to ensure the convergence of the dynamics. In higher dimension, the proofs are more intricate and are presented in the next section.

## 2.1 Convexity

Denote by $\Omega(t)$ the convex hull of $\{x_i(t)\}_i$. A striking feature of the dynamics is that $\Omega(t)$ is contracting in time

**Proposition 2.1** *The convex hull $\Omega(t)$ of the solution $\{x_i\}_i$ satisfies $\Omega(s) \subset \Omega(t)$ for any $s \geq t$. Thus, there exists a convex compact $\Omega_\infty$ such that*

$$\Omega(t) \stackrel{t\to\infty}{\longrightarrow} \Omega_\infty. \qquad (2.1)$$

**Proof.** As $\Omega(t)$ is the convex hull of a finite number of points, it is polygonal. Take any extremal point or vertex $x_i(t)$ of $\Omega(t)$. There is a finite number of linearly independent vectors $u_k$ (equal to the number of facets to which $x_i$ belongs) s.t. in the neighborhood of $x_i$, a point $y$ belongs to $\Omega(t)$ iff

$$\langle y - x_i(t),\ u_k \rangle \geq 0 \quad \text{for all } k.$$

Use Formulation (1.2) and for any $k$ compute

$$\langle \dot{x}_i(t),\ u_k \rangle = \sum_j a_{ij} \, \langle x_j - x_i,\ u_k \rangle.$$

Now note that each $a_{ij}$ is positive. In addition since $x_j \in \Omega(t)$ then $\langle x_j - x_i,\ u_k \rangle \geq 0$. Therefore

$$\langle \dot{x}_i(t),\ u_k \rangle \geq 0.$$

This exactly characterizes the fact that $\Omega(t)$ is contracting (see figure 2). □

From the Prop. 2.1, we deduce that the agents $\{x_i(t)\}_i$ stay on a compact set $\Omega_0$ with $\Omega_0$ the convex hull of the initial configuration $\{x_i(0)\}_i$.

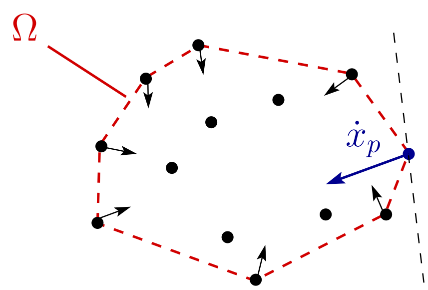


Figure 2: The convex hull of $\{x_i\}_i$ is decreasing in time.

## 2.2 The Lyapunov functional

We now introduce the main ingredient to study the long time behavior of the consensus model. Define

$$E(\{x_i\}_i) = \sum_{ij} \Phi(|x_j - x_i|^2) \qquad \text{with} \qquad \Phi(r) = \int_0^r \phi(s)\, ds. \tag{2.2}$$

The functional $E$ is decreasing in time.

**Proposition 2.2** *Let $\{x_i\}_i$ be the solution of the consensus model (1.1). Then $E(t) = E(\{x_i(t)\}_i)$ satisfies:*

$$\frac{d}{dt}E = -4 \sum_{ij} \phi_{ij} |\dot{x}_i|^2, \tag{2.3}$$

$$\frac{d}{dt}E \leq -\frac{\left(\sum_{ij} \phi_{ij} |x_j - x_i|^2\right)^2}{\sum_{ij} \phi_{ij} |x_i|^2}. \tag{2.4}$$

*Hence, we have*

$$\sum_i \int_0^\infty |\dot{x}_i(t)|^2 \, dt \leq C\, N^2, \tag{2.5}$$

$$\int_0^\infty \phi_{ij}^2 |x_j(t) - x_i(t)|^4 \, dt \leq C_N \qquad \textit{for any } i, j. \tag{2.6}$$

**Proof.** Using the symmetry $\phi_{ij} = \phi_{ji}$, we find

$$\frac{d}{dt}E = 2\sum_{ij} \Phi'(|x_j - x_i|^2)\langle \dot{x}_j - \dot{x}_i, x_j - x_i\rangle = -4\sum_{ij} \phi_{ij}\langle \dot{x}_i, x_j - x_i\rangle.$$

The equality $\sum_j \phi_{ij}\dot{x}_i = \sum_j \phi_{ij}(x_j - x_i)$ yields (2.3). Noticing that $\phi_{ii} = 1$, we deduce (2.5).

Denote $\sigma_i = \sum_j \phi_{ij}$. From the Cauchy-Schwarz inequality, we deduce:

$$\left(\sum_i \sigma_i \dot{x}_i \cdot x_i\right)^2 \leq \left(\sum_i \sigma_i |\dot{x}_i|^2\right)\left(\sum_i \sigma_i |x_i|^2\right).$$

Using once again that $\sigma_i \dot{x}_i = \sum_j \phi_{ij}(x_j - x_i)$ and eq. (2.3), we find:

$$\left(\sum_{ij} \phi_{ij}(x_j - x_i)\cdot x_i\right)^2 \leq \left(-\frac{1}{4}\frac{d}{dt}E\right)\left(\sum_i \sigma_i |x_i|^2\right).$$

Using the symmetry $\phi_{ij} = \phi_{ji}$, we deduce (2.4). To obtain (2.6), we use that $\sigma_i$ and $|x_i|^2$ are uniformly bounded in time. □

**Remarks 2.3** *The inequality* $\int_0^\infty |\dot{x}_i|^2 \, dt \leq C$ *does not allow to conclude that* $\{x_i(t)\}_t$ *converges in time (e.g.* $x_i(t) = \sin t^\alpha$ *with* $0 < \alpha < \frac{1}{2}$*). We need a stronger estimate such as* $\int_0^\infty |\dot{x}_i| \, dt \leq C$*, which is of course not available from Prop. 2.2.*

From Prop. 2.2, we deduce that the stationary states of the system have to satisfy $dE/dt = 0$ which implies that $\phi_{ij}|x_j - x_i|^2 = 0$ for all $i$, $j$. In other words, the set of stationary states is given by (see figure 3):

$$\omega = \Big\{ \{x_i\}_i \;/\quad |x_j - x_i| = 0 \;\text{ or }\; |x_j - x_i| \geq 1 \Big\}. \tag{2.7}$$

Our next step is to prove that the solution $\{x_i\}_i$ is getting closer to this set $\omega$. Note that since the set $\omega$ is not discrete this will *not* imply any convergence.

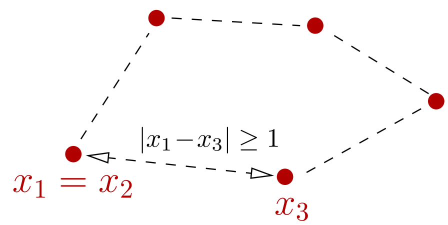


Figure 3: A stationary state of the consensus model: two agents $x_p$ and $x_q$ has to be either at the same location (e.g. $x_1 = x_2$) or at a distance greater than 1 (e.g. $|x_1 - x_3| \geq 1$).

## 2.3 Clustering

Using the functional $E$, one can deduce that the system is going to create *clusters*: two agents will stay either *very close* or *far away*. More precisely

**Proposition 2.4** *Let $\{x_i\}_i$ be the solution of the consensus model (1.1). For any $\varepsilon > 0$, there exists a time $T_\varepsilon > 0$ such that for any $i,\ j$*

$$|x_j(t) - x_i(t)| \in [0,\ \varepsilon] \cup [1-\varepsilon,\ \infty) \quad \text{for any } t \geq T_\varepsilon. \tag{2.8}$$

*Moreover, if $\phi(1-) > \phi(1+)$ , the set:*

$$I_\varepsilon = \{t \geq 0 \ /\ \text{ there exist } i,\ j \text{ satisfying } |x_j(t) - x_i(t)| \in [\varepsilon, 1)\} \tag{2.9}$$

*has finite Lebesgue measure (i.e. $|I_\varepsilon| < \infty$).*

**Corollary 2.5** *For any $i$ and $j$, the distance $|x_i(t) - x_j(t)|$ satisfies:*

$$\lim_{t\to\infty} |x_j(t) - x_i(t)| = 0 \qquad \text{or} \qquad \underline{\lim}_{t\to\infty} |x_j(t) - x_i(t)| \geq 1. \tag{2.10}$$

**Proof.** We prove the proposition by contradiction. As there are only a finite number of couples $i,\ j$, we may fix the indices $i$ and $j$ and assume that there exists $\{t_n\}_n$ such that $|x_j(t_n) - x_i(t_n)| \in [\varepsilon, 1-\varepsilon]$. Since $\phi$ is strictly positive on $[0, 1)$, there exists $\delta > 0$ satisfying $\phi > \delta$ on $[\varepsilon, 1-\varepsilon]$. Note that

$$|\dot{x}_i| \leq N\, R\, \|\phi\|_{L^\infty},$$

with $R$ the maximal diameter of the $x_k$, according to Prop. 2.1. Hence we have uniform continuity in time of $\{x_i(t)\}_i$ and there exists $\Delta t$ such that $\phi(|x_j - x_i|^2) \geq \delta/2$ for any $[t_n, t_n + \Delta t]$. We deduce that $\phi_{ij}^2 |x_j - x_i|^4 \geq \delta' > 0$ on this interval, with $\delta' = \varepsilon^4\, \delta^2/4$. Thus,

$$\int_0^\infty \phi_{ij}^2 |x_j - x_i|^4\, dt \ \geq\ \sum_n \int_{t_n}^{t_n+\Delta t} \phi_{ij}^2 |x_j - x_i|^4\, dt \ \geq\ \sum_n \delta' \Delta t = +\infty,$$

which gives a contradiction with proposition 2.2.

Now if the interaction function $\phi$ is discontinuous with $\phi(1-) > \phi(1+)$, for any $t \in I_\varepsilon$, there exists $p, q$ such that $1-\varepsilon \leq |x_p(t) - x_q(t)| < 1$ which implies $\phi_{pq} \geq m > 0$. Therefore for $\tilde{m} = m^2 (1-\varepsilon)^4$ and any $t \in I_\varepsilon$

$$\sum_{i,j} \phi_{ij}^2 |x_j(t) - x_i(t)|^4 \geq \phi_{pq}^2 |x_p(t) - x_q(t)|^4 \geq \widetilde{m} \; > \; 0.$$

As a consequence, we obtain

$$\int_0^\infty \sum_{i,j} \phi_{ij}^2 |x_j - x_i|^4 \, dt \; \geq \; \int_{I_\varepsilon} \sum_{i,j} \phi_{ij}^2 |x_j - x_i|^4 \, dt \; \geq \; \widetilde{m} |I_\varepsilon|,$$

From (2.6), we deduce that $|I_\varepsilon| < \infty$. □

As a consequence Corollary 2.5, we can regroup the agents $\{x_i\}_i$ into *clusters* denoted by $\{\mathcal{C}_k\}_k$. Two agents $p$ and $q$ belong to the same clusters $\mathcal{C}_k$ if they satisfy $\lim_{t\to\infty} |x_p(t) - x_q(t)| = 0$. In other words, we have defined an equivalence relation:

$$p \sim q \quad \text{if} \quad \lim_{t\to\infty} |x_p(t) - x_q(t)| = 0. \tag{2.11}$$

The clusters $\mathcal{C}_k$ are simply the connected components of this equivalence relation.

From the proposition 2.4, we deduce that for any $\varepsilon > 0$, there exists $T_\varepsilon$ such that (see figure 4):

$$\text{for all } t \geq T_\varepsilon\,, \quad |x_j - x_i| \leq \varepsilon \text{ if } i \sim j \quad \text{ and } \quad |x_j - x_i| \geq 1-\varepsilon \text{ if } i \not\sim j.$$

We have to notice that the number and the size of the clusters $\{\mathcal{C}_k\}_k$ are unknown initially (i.e. at time $t = 0$). The clusters are formed after $T_\varepsilon$ with $\varepsilon < 1/2$.

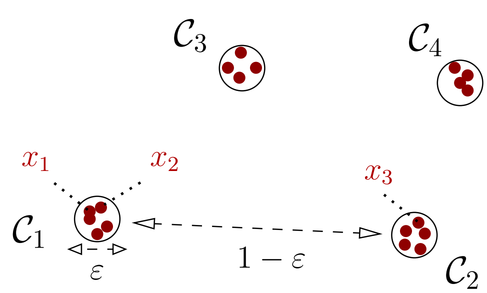


Figure 4: The distance between two agents in a cluster $\mathcal{C}_k$ converges to zero (e.g. $x_1$ and $x_2$). The distance between two clusters becomes greater or equal to 1.

## 2.4 Convergence in dimension 1

The consensus model in dimension 1 has some special properties, for instance we can *order* the agents (i.e. $x_1 \leq x_2 \leq \cdots \leq x_n$) and this order is preserved by the dynamics. As a consequence, in dimension 1, Propositions 2.1 and 2.4 are sufficient to prove that the consensus model convergences toward a stationary state. Hence, we can prove Th. 1 in dimension 1 under very general assumptions.

**Theorem 4** *Assume that* $\phi \in L^\infty(\mathbb{R})$ *with* $\phi$ *bounded from below on any interval* $[0,\ 1-\varepsilon]$ *and compactly supported in* $[0,1]$*. Then the solution to the consensus model (1.1) in dimension* 1 *converges as* $t \longrightarrow +\infty$*.*

*In others words, for any* $i$*, there exists* $\bar{x}_i$ *such that* $x_i(t) \stackrel{t\to\infty}{\longrightarrow} \bar{x}_i$*. Moreover, the stationary state* $\{\bar{x}_i\}_i$ *satisfies:*

$$\text{for any } i,j, \qquad \bar{x}_i = \bar{x}_j \ \text{ or } \ |\bar{x}_i - \bar{x}_j| \geq 1.$$

**Proof.** We order the agents such that $x_1 \leq x_2 \leq \cdots \leq x_N$. Since the convex hull converges in time (Proposition 2.1), we immediately deduce that $x_1$ and $x_N$ converge. To prove that $x_2, \ldots, x_{N-1}$ converge, we proceed by induction.

Assume that the agents $x_1, ..., x_i$ and $x_{N-i+1}, ..., x_N$ converge and let us prove that $x_{i+1}$ and $x_{N-i}$ converge as well. We prove it for $x_{i+1}$, we can proceed similarly for $x_{N-i}$. Due to the cluster formation, we have either:

$$\lim_{t\to\infty} |x_{i+1}(t) - x_i(t)| = 0 \qquad \text{or} \qquad \underline{\lim}_{t\to\infty} |x_{i+1}(t) - x_i(t)| \geq 1,$$

in other words $i+1 \sim i$ or $i+1 \not\sim i$. If $i+1 \sim i$, then since $x_i(t)$ converges, $x_{i+1}$ will converge as well.

For the case that $i+1 \not\sim i$, there are two possible scenarios. First, there exists $T > 0$ such that $x_{i+1}(T) - x_i(T) > 1$. Then for $k \leq i$, $x_k(T) \leq x_i(T) < x_{i+1}(T) - 1$ and hence $\phi_{ik} = 0$. Therefore $x_{i+1}$ can only interact with agents $i+1 \leq j$. By the ordering property, this means $x_{i+1}(T) \leq x_j(T)$ and hence $\dot{x}_{i+1} \geq 0$. Similarly $x_i$ only interacts with agents $k \leq i$ and $\dot{x}_i \leq 0$.

Thus, we have

$$\frac{d}{dt}(x_{i+1} - x_i)|_T \geq 0.$$

This actually implies that $x_{i+1}(t) - x_i(t) > 1$ for all $t \geq T$. So after time $T$ the dynamics is completely disconnected: Agents $i+1, \ldots, N$ only interact between themselves. Therefore we can apply Prop. 2.1 once again to $\{x_{i+1}, \ldots, x_N\}$ and deduce that $x_{i+1}$ converges.

The other scenario is that $x_{i+1}(t) - x_i(t) \leq 1$ for all $t$. Since $i+1 \not\sim i$, we have $\underline{\lim}_{t\to\infty} |x_{i+1}(t) - x_i(t)| \geq 1$. Hence, $\lim_{t\to\infty} |x_{i+1}(t) - x_i(t)| = 1$. In dimension 1, since $x_i \leq x_{i+1}$, this leaves the only possibility $\lim_{t\to\infty} x_{i+1}(t) - x_i(t) = 1$ and thus $x_{i+1}$ converges. $\square$

**Remark 2.6** *We have to notice that several properties of dimension* 1 *have been used in the proof of theorem 4. For instance, in* $\mathbb{R}^2$*, if two agents* $i$ *and* $j$ *from different clusters (*$i \not\sim j$*) are disconnected at time* $T$ *(i.e.* $|x_i(T) - x_j(T)| > 1$*), they may be connected at a latter time (i.e.* $|x_i(t) - x_j(t)| < 1$ *with* $t \geq T$*) (see figure 5). This is a* key feature *of the multidimensional case vs the 1-dimensional: In dimension* 1*, connectivity once lost is never recovered. This property immensely simplifies the possible dynamics.*

*Moreover, having* $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 1$ *with* $x_i$ *converging does not imply that* $x_j$ *converges in* $\mathbb{R}^d$ *with* $d \geq 2$*. For these reasons, one has to introduce new tools to study the asymptotic behavior of the consensus model in multi-dimension.*

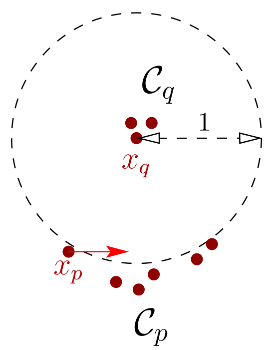


Figure 5: In dimension $d \geq 2$, two agents $p$ and $q$ from different clusters can be disconnected once and reconnected latter on.

## 2.5 Rate of convergence in dimension 1

In the particular case where $\phi(1-) > 0$, we may refine even further the previous one-dimensional analysis. First of all we formalize and quantify the previous remark about the loss of connectivity

**Proposition 2.7** *Assume that* $\phi$ *is bounded from below on* $[0,\ 1)$ *and let* $\{x_i\}_i$ *be the solution to the consensus model* (1.1)*. Then there exists* $t_0 > 0$ *depending only on* $\phi$*,* $N$ *and the size of the initial support s.t. for any* $t_0 > 0$ *the clusters are formed and* non longer interact*, that is*

$$|x_i - x_j| \leq \frac{1}{2} \text{ if } i \sim j, \quad |x_i - x_j| \geq 1 \text{ if } i \not\sim j.$$

**Proof.** We start by proving that connectivity once lost is never recovered. For simplicity order the agents s.t. $x_i \leq x_j$ if $i \leq j$. Assume that at some time $T$ and for some $i < j$ one has $x_i \leq x_j - 1$. Denote $I_1 = (-\infty,\ x_i(T)]$ and $I_2 = [x_j(T),\ +\infty)$. Then $I_1$ and $I_2$ are convex, at $T$ for all $p \leq i$, $x_p \in I_1$ and for all $q \geq j$, $x_q \in I_2$.

Therefore by Proposition 2.1, this is preserved for any $t \geq T$, *i.e.* for all $p \leq i$, $x_p \in I_1$ and for all $q \geq j$, $x_q \in I_2$. In particular for any $t \geq T$ then $x_i \leq x_j - 1$.

Now let us denote by $t_0$ the first time s.t.

$$|x_i - x_j| \leq \frac{1}{2} \text{ if } i \sim j, \quad |x_i - x_j| \geq 1 \text{ if } i \not\sim j.$$

By the previous remark, this remains true for all $t \geq t_0$ and moreover for any $t < t_0$, there exists $i$ and $j$ s.t.

$$\frac{1}{2} < |x_i - x_j| < 1,$$

that is for some $C_\phi = \inf_{[1/2,\ 1)} \phi$ depending only on $\phi$, $\phi_{ij} \geq C$. That implies that for any $t < t_0$ then

$$-\frac{\left(\sum_{ij} \phi_{ij}\, |x_j - x_i|^2\right)^2}{\sum_{ij} \phi_{ij}\, |x_j|^2} \leq -\frac{C_\phi^2}{4\, R^2},$$

with $R$ the diameter of the initial support. Therefore by Proposition 2.2

$$E(t_0) \leq E(0) - \frac{C_\phi^2}{4\, R^2}\, t_0.$$

Observe that $E(0) \leq N^2\, \|\phi\|_{L^\infty}$ and this controls $t_0$ as claimed

$$t_0 \leq \frac{4\, R^2\, N^2\, \|\phi\|_{L^\infty}}{\inf_{[1/2,\ 1)} \phi}.$$

□

Therefore after $t_0$ the dynamics in each cluster are completely independent. The only remaining question is how fast can one prove convergence in that case. But this case was extensively studied in [19] and we recall one of the main result in this paper

**Theorem 5** *(Motsch-Tadmor) There exists $C$ and $\lambda > 0$ depending only on $N$ and $\phi$ s.t. if $|x_i(0)| \leq 1/2$ for all $i$ then there exists $x_\infty$ s.t.*

$$\sup_i |x_i(t) - x_\infty| \leq C\, e^{-\lambda\, t}.$$

After $t_0$, we may study the dynamics independently in every cluster $\mathcal{C}_k$. Denote by $\bar{x}_k$ the center of mass of cluster $\mathcal{C}_k$. Since for every $i,\ j \in \mathcal{C}_k$, at $t_0$: $|x_i(t_0) - x_j(t_0)| \leq 1/2$ then

$$|x_i(t_0) - \bar{x}_k| \leq 1/2, \quad \text{for any } i \in \mathcal{C}_k.$$

Now since the dynamics is invariant by translation, it is possible to apply Theorem 5 starting from $t_0$, on every $i \in \mathcal{C}_k$. Thus there exists $x_\infty^k$ s.t.

$$\sup_{i \in calC_k} |x_i(t) - x_\infty^k| \leq C\, e^{-\lambda\,(t-t_0)}, \quad \forall\; t \geq t_0,$$

which concludes the proof.

# 3 Convergence in multi-dimension

In multi-dimension, the consensus model can have a more complicated dynamics than in 1D since *connectivity* can be lost and recovered later on. Hence, we have to improve the upper-bound found in Prop. 2.2 to establish the convergence of the dynamics.

## 3.1 A refined estimate

The additional assumption on the interaction function (1.4) allows to improve the proposition 2.2.

**Proposition 3.1** *Assume that $\phi$ satisfies (1.4). Then, the solution $\{x_i\}_i$ satisfies:*

$$\int_0^\infty \sum_{i,j} \phi_{ij} |x_j - x_i|^2 \, dt \leq C\, N^3. \tag{3.1}$$

With respect to Prop. 2.2, Prop. 3.1 improves the order: For instance it controls $|x_j - x_i|^2$ instead of $|x_j - x_i|^4$.
**Proof.** From the equality $\sum_j \phi_{ij} \dot{x}_i = \sum_j \phi_{ij}(x_j - x_i)$, we deduce

$$\int_0^T \sum_{ij} \phi_{ij} \dot{x}_i x_i \, dt = \int_0^T \sum_{ij} \phi_{ij}(x_j - x_i) x_i \, dt.$$

Thus,

$$\int_0^T \sum_{ij} \phi_{ij} \frac{d}{dt} |x_i|^2 \, dt = -\int_0^T \sum_{ij} \phi_{ij} |x_j - x_i|^2 \, dt. \tag{3.2}$$

Integrating by parts the left-hand side yields

$$\sum_{ij} \int_0^T \phi_{ij} \frac{d}{dt} |x_i|^2 \, dt = \sum_{ij} \phi_{ij} |x_i|^2 \Big|_0^T - \sum_{ij} \int_0^T \phi'_{ij} 2 \langle \dot{x}_j - \dot{x}_i \,,\, x_j - x_i \rangle |x_i|^2 \, dt.$$

Since $\{x_i(t)\}_i$ is uniformly bounded, we obtain:

$$\left|\sum_{ij}\int_0^T \phi_{ij}\frac{d}{dt}|x_i|^2\,dt\right| \leq C\,N^2 + C\sum_{ij}\int_0^T |\phi'_{ij}||x_j - x_i||\dot{x}_j - \dot{x}_i|\,dt$$
$$\leq C\,N^2 + C\sum_{ij}\left(\int_0^T |\phi'_{ij}|^2|x_j - x_i|^2\,dt\right)^{1/2}\left(\int_0^T |\dot{x}_j - \dot{x}_i|^2\,dt\right)^{1/2}$$

using Cauchy-Schwarz inequality. Recall that $\sum_i \int_0^T |\dot{x}_i|^2\,dt \leq C\,N^2$ by estimate (2.5). Thus, since $\phi$ satisfies (1.4), we deduce

$$\left|\sum_{ij}\int_0^T \phi_{ij}\frac{d}{dt}|x_i|^2\,dt\right| \leq C\,N^2 + C\,N^{3/2}\left(\sum_{ij}\int_0^T \phi_{ij}|x_j - x_i|^2\,dt\right)^{1/2}.$$

Combining the last inequality with (3.2) yields:

$$\sum_{ij}\int_0^T \phi_{ij}|x_j - x_i|^2\,dt \leq C\,N^2 + C\,N^{3/2}\left(\sum_{ij}\int_0^T \phi_{ij}|x_j - x_i|^2\,dt\right)^{1/2}.$$

To conclude, we denote $\beta(t) = \sum_{ij}\int_0^T \phi_{ij}|x_j - x_i|^2\,dt$. We have:

$$\beta(t) \leq C\,N^2 + CN^{3/2}\sqrt{\beta(t)}.$$

We deduce that the function $\beta(t)$ has to be bounded independently of time $t$ which yields (3.1). $\square$

## 3.2 Dynamics of the cluster centers

Thanks to Prop. 3.1, we have a control of $|x_j(t) - x_i(t)|^2$. But to prove the convergence of the dynamics, we have to go one step further and control $|x_j(t) - x_i(t)|$. With this aim, we introduce $\{y_k\}_k$ the centers of the clusters $\mathcal{C}_k$ (see figure 6):

$$y_k = \frac{1}{N_k}\sum_{i\in\mathcal{C}_k} x_i. \tag{3.3}$$

There are cancellations in $\dot{y}_k$, enough to gain one order and to conclude about the convergence of $y_k$.

**Proposition 3.2** *Assume $\phi$ satisfies (1.4). Then the center $y_k$ of each cluster $\mathcal{C}_k$ verifies*

$$\int_0^T |\dot{y}(t)|\,dt \leq C\,N^3 + C\,N\int_0^T D(t)\,dt, \tag{3.4}$$

*with $D(t) = \frac{1}{2}\sum_{i,j}\phi_{ij}|x_j(t) - x_i(t)|^2$.*

*Therefore, by Prop. 3.1, $y_k(t)$ converges in time for any $k$.*

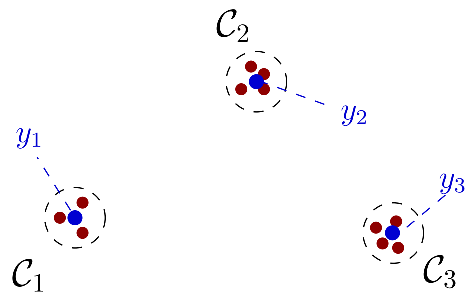


Figure 6: The centers $y_k$ (3.3) of 3 clusters $\mathcal{C}_k$.

**Proof.** To simplify the notation, we focus our attention on the first cluster $\mathcal{C}_1$.

$$\frac{d}{dt}y_1 = \frac{1}{N_1}\sum_{i\in\mathcal{C}_1}\dot{x}_i = \frac{1}{N_1}\sum_{i\in\mathcal{C}_1,\, j\in[\![1,N]\!]} a_{i,j}(x_j - x_i).$$

We regroup the agents $j$ by clusters as well

$$\frac{d}{dt}y_1 = \frac{1}{N_1}\left(\sum_{i\in\mathcal{C}_1,\, j\in\mathcal{C}_1} a_{i,j}(x_j - x_i) \;+\; \sum_{i\in\mathcal{C}_1,\, j\notin\mathcal{C}_1} a_{i,j}(x_j - x_i)\right).$$

We start by looking at the case $i \in \mathcal{C}_1$ and $j \notin \mathcal{C}_1$. For any $t \geq T_{1/2}$, we have $|x_j - x_i| > 1/2$, thus:

$$|a_{i,j}(x_j - x_i)| \leq C\phi_{ij}|x_j - x_i| \leq 2C\phi_{ij}|x_j - x_i|^2 \qquad \text{for } t \geq T_{1/2}.$$

Therefore

$$\begin{aligned}\sum_{i\in\mathcal{C}_1,\, j\notin\mathcal{C}_1}\int_0^T a_{i,j}|x_j - x_i|\,dt \;&\leq\; C\,N^2 + \sum_{i\in\mathcal{C}_1,\, j\notin\mathcal{C}_1}\int_{T_{1/2}}^T a_{i,j}|x_j - x_i|\,dt \\ &\leq\; C\,N^2 + C\int_0^T D(t)\,dt. \end{aligned} \tag{3.5}$$

The case where $i \in \mathcal{C}_1$, $j \in \mathcal{C}_1$ is more delicate. Our key point is the following identity:

$$\sum_{i\in\mathcal{C}_1,\, j\in\mathcal{C}_1} a_{i,j}(x_j - x_i) = \sum_{i,j\in\mathcal{C}_1,\, j>i}(a_{i,j} - a_{j,i})(x_j - x_i). \tag{3.6}$$

Using the symmetry $\phi_{ij} = \phi_{ji}$, we find:

$$a_{i,j} - a_{j,i} = \frac{\phi_{i,j}}{\sum_k \phi_{i,k}} - \frac{\phi_{j,i}}{\sum_k \phi_{j,k}} = \phi_{i,j}\left(\frac{\sum_k(\phi_{j,k} - \phi_{i,k})}{\sum_k \phi_{i,k}\cdot\sum_k \phi_{j,k}}\right). \tag{3.7}$$

We are forced to treat differently the cases where $\phi$ is continuous or not at $r = 1$.

• $\phi$ **continuous at** $r = 1$.

By the assumptions of Th. 1, $\phi$ is Lipschitz on $[0,1)$. Moreover by assumption (1.4), $\phi'$ is bounded on the neighborhood of 1. Finally $\phi$ is continuous at 1 and thus $\phi$ is Lipschitz on $\mathbb{R}_+$. This implies

$$\begin{aligned}|\phi_{j,k} - \phi_{i,k}| &= \Big|\phi(|x_k - x_j|) - \phi(|x_k - x_i|)\Big| \le C\Big(|x_k - x_j| - |x_k - x_i|\Big). \\ &\le C|x_j - x_i|,\end{aligned}$$

using the triangular inequality. Thus, one deduces that

$$|a_{i,j} - a_{j,i}| \le C\, N\, \phi_{i,j}|x_j - x_i|.$$

Coming back to eq. (3.6), we obtain:

$$\sum_{i\in\mathcal{C}_1,\, j\in\mathcal{C}_1} a_{i,j}|x_j - x_i| \ \le \ C\, N \sum_{i\in\mathcal{C}_1,\, j\in\mathcal{C}_1} \phi_{i,j}|x_j - x_i|^2$$

and deduce that

$$\sum_{i\in\mathcal{C}_1,\, j\in\mathcal{C}_1} \int_0^T a_{i,j}|x_j - x_i|\, dt \le C\, N \int_0^T D(t)\, dt. \tag{3.8}$$

Combining (3.5) and (3.8) concludes the proof for $\phi$ continuous at $r = 1$.

• $\phi$ **has a jump at** $r = 1$.

When $\phi$ is discontinuous, the inequality $|a_{i,j} - a_{j,i}| \le C\phi_{i,j}|x_j - x_i|$ is no longer true. Indeed, if there is a $k$ such that $|x_i - x_k| > 1$ and $|x_j - x_k| < 1$, one may very well have that $|a_{i,j} - a_{j,i}|$ is of order 1. For this reason, we need to use another approach to obtain (3.8). Fortunately, in that case what was lost in smoothness is compensated by the fact that $\phi$ is now bounded from below on $[0,\ 1)$.

From (3.7), we have

$$\begin{aligned}\int_0^T \sum_{i\in\mathcal{C}_1,\, j\in\mathcal{C}_1} a_{i,j}|x_j - x_i|\, dt \ &\le \ C\int_0^T \sum_{i,j,k\in\mathcal{C}_1} |\phi_{i,k} - \phi_{j,k}|\cdot|x_j - x_i|\, dt \\ &\qquad + \ C\int_0^T \sum_{i,j\in\mathcal{C}_1,\, k\notin\mathcal{C}_1} |\phi_{i,k} - \phi_{j,k}|\cdot|x_j - x_i|\, dt. \\ &\le \ I_1 + I_2.\end{aligned}$$

For $k \in \mathcal{C}_1$, we use that $\phi$ is Lipschitz in $[0, \frac{1}{2}]$ and we conclude once again that $|\phi_{i,k} - \phi_{j,k}| \le C\, N\, |x_j - x_i|$. Thus,

$$\begin{aligned}I_1 \ &\le \ C\, N^3 + C\int_{T_{1/2}}^T \sum_{i,j,k\in\mathcal{C}_1} |\phi_{i,k} - \phi_{j,k}|\cdot|x_j - x_i|\, dt \\ &\le \ C\, N^3 + C\, N\int_{T_{1/2}}^T \sum_{i,j,k\in\mathcal{C}_1} \phi_{i,j}\cdot|x_j - x_i|^2\, dt \le C\, N^3 + C\, N\int_0^T D(t)\, dt.\end{aligned}$$

For $k \notin \mathcal{C}_1$, we fix $\varepsilon > 0$ and introduce $I_\varepsilon$ from proposition 2.4. Thus, if $t \notin I_\varepsilon$, then $\phi_{i,k} = \phi_{j,k} = 0$ (since $k \not\sim i$ and $k \not\sim j$). Therefore,

$$\begin{aligned} I_2 &\leq \int_{I_\varepsilon} \sum_{i,j \in \mathcal{C}_1,\, k \notin \mathcal{C}_1} |\phi_{i,k} - \phi_{j,k}| \cdot |x_j - x_i| \, dt \\ &\leq C\, N^3 |I_\varepsilon| \leq C\, N^3. \end{aligned}$$

Regrouping $I_1$ and $I_2$, we deduce that:

$$\sum_{i \in \mathcal{C}_1,\, j \in \mathcal{C}_1} \int_0^T a_{i,j} |x_j - x_i| \, dt \leq C\, N \int_0^T D(t)\, dt + C\, N^3. \qquad (3.9)$$

Combining (3.5) and (3.9) concludes the proof. □

## 3.3 Convergence in the multi-dimensional case

From the convergence of the cluster centers $\{y_k\}_k$, it is now easy to conclude and prove that the solution $\{x_i\}_i$ converges to a stationary state.
**Proof** *(Th. 1)*. Consider $x_i \in \mathcal{C}_k$ and let $y_k$ be the center of the cluster $\mathcal{C}_k$ and $N_k$ the number of elements in $\mathcal{C}_k$.

$$|x_i - y_k| = \left| x_i - \frac{1}{N_k} \sum_{j \in \mathcal{C}_k} x_j \right| \leq \frac{1}{N_k} \sum_{j \in \mathcal{C}_k} |x_i - x_j|.$$

For any $j \in \mathcal{C}_k$, since $i \sim j$, we have $|x_i(t) - x_j(t)| \stackrel{t\to\infty}{\longrightarrow} 0$. Therefore, $|x_i(t) - y_k(t)| \stackrel{t\to\infty}{\longrightarrow} 0$.

Thanks to the proposition 3.2, we know that $y_k(t)$ converges. Therefore, $x_i(t)$ converges as well, which concludes the proof of Th. 1. □

**Remark 3.3** *The main difficulty in the proof was the lack of symmetry for the coefficients* $a_{i,j} = \frac{\phi_{ij}}{\sum_j \phi_{ij}}$. *For instance, it is much easier to prove the convergence of the symmetric dynamics:*

$$\dot{x}_i = \frac{1}{N} \sum_j \phi_{ij}(x_j - x_i)\,, \quad \phi_{ij} = \phi(|x_j - x_i|^2).$$

*Using the symmetry, we obtain*

$$\frac{d}{dt} \sum_i |x_i|^2 = -\frac{1}{N} \sum_{ij} \phi_{ij} |x_j - x_i|^2.$$

*Thus, the result of the Prop. 3.1 comes without additional assumptions on $\phi$. The convergence of the cluster centers $y_k$ is also proven easily since:*

$$\sum_{i,j\in\mathcal{C}_k} \phi_{ij}(x_j - x_i) = 0.$$

*As a consequence, the internal dynamics within the cluster $\mathcal{C}_k$ has no effect on the cluster center $y_k$. Hence, it is not necessary to use the equality* (3.6).

**Remark 3.4** *Given the convergence provided by Th. 1, we may try to find explicit rates of convergence. Unfortunately this is not possible in general.*

*First of all if $\phi(1-) = 0$ then examples are obvious. Take for instance two agents $x_1$ and $x_2$ and fix the initial positions s.t. $x_2(0) - x_1(0) = 1 - \varepsilon$. Then since $\phi$ vanishes around 1, the time before which $|x_1(t) - x_2(t)| \leq \frac{1}{2}$ is obviously diverging to $+\infty$ as $\varepsilon$ goes to 0. Thus the speed of convergence depends intrinsically on the initial distribution.*

*When $\phi(1-) > 0$, in dimension 1, the previous behavior is excluded and rates of convergence should be available. However this still cannot be true in dimension strictly larger than 1. For instance in dimension 2, take $\phi = \mathbb{I}_{[0,\ 1]}$ and consider 3 agents with $x_1(0) = (-\frac{1}{2}, 0)$, $x_2(0) = (0, \frac{1}{2})$, $x_3 = (0, 1-\varepsilon)$. It is easy to see that all three will converge to the same limit $x_\infty \approx (0, 1/3)$. But the dynamics already exhibits a very interesting meta-stability phenomena. On a first time interval $[0,\ t_1]$, both $|x_1 - x_3|$ and $|x_2 - x_3|$ are strictly larger than 1 so there is only an interaction between $x_1$ and $x_2$. Therefore $x_1(t) = (-\alpha(t), 0)$ and $x_2(t) = (\alpha(t), 0)$ with $\alpha(t) \to 0$ (exponentially fast here, i.e. $\alpha(t) \leq e^{-Ct}$). One does not have an interaction with $x_3$ until $\alpha(t_1) = 2\varepsilon - \varepsilon^2$. After time $t_1$, we again have exponential convergence toward the real steady state. But one sees that $t_1 \sim |\log \varepsilon|$ and hence the first (meta-stable) phase may last as long as desired.*

*In higher dimensions, such examples abound where agents are first disconnected (do not interact) and then after an arbitrary long time reconnect again; this may even happen several times. In view of those possible complex phenomena, even the simple convergence provided by Th. 1 looks remarkable.*

# 4 Discrete dynamics: Proof of Theorem 3

The proof of Theorem 3 is very similar to the continuous case (1.1). Many of the properties of the continuous dynamics remain valid in the discrete setting: the convex hull is decaying (Prop. 2.1) and the stationary states are cluster formations (Cor. 2.5). However, we need additional assumptions to show that the system (1.5) is dissipative. For instance, we require that the interaction function $\phi$ is non increasing. This additional assumption on $\phi$ ensures that its primitive $\Phi$ is concave-down which gives us a crucial inequality.

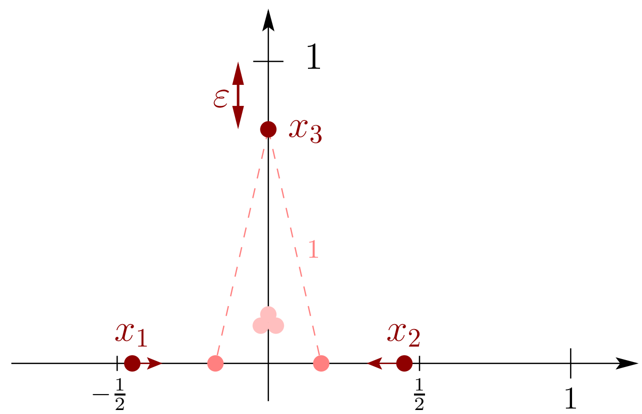


Figure 7: An initial configuration presenting meta-stability phenomena. First, $x_1$ and $x_2$ are converging toward $(0, 0)$. Then, when $x_3$ is at a distance less than 1 from $x_1$ and $x_2$, all the agents converge toward a unique point close to $(0, \frac{1}{3})$. The smaller $\varepsilon$ is, the longer it will take for the system to converge.

## 4.1 Discrete dissipative systems

We start by extending the proposition 2.2 proving that the functional $E$ (2.2) is also decreasing for the discrete dynamics when $\phi$ is non increasing.

**Proposition 4.1** *Let $\{x_i^n\}_i$ be the solution of the discrete consensus model (1.5) with $\phi$ non increasing. Then $E^n = E(\{x_i^n\}_i)$ satisfies:*

$$E^{n+1} - E^n \quad \leq \quad -4 \sum_{ij} \phi_{ij}^n |\Delta x_i^n|^2 \tag{4.1}$$

*with $\Delta x_i^n = x_i^{n+1} - x_i^n$.*

**Proof.** Using that $\Phi$ is concave down, we have:

$$\begin{aligned} E^{n+1} - E^n &= \left( \sum_{ij} \Phi(|x_j^{n+1} - x_i^{n+1}|^2) - \Phi(|x_j^n - x_i^n|^2) \right) \\ &\leq \sum_{ij} \Phi'(|x_j^n - x_i^n|^2) \Big( |x_j^{n+1} - x_i^{n+1}|^2 - |x_j^n - x_i^n|^2 \Big) \end{aligned}$$

The equality $|u|^2 - |v|^2 = \langle u - v, u + v \rangle$ leads to:

$$\begin{aligned} E^{n+1} - E^n &\leq \sum_{ij} \phi_{ij}^n \langle \Delta x_j^n - \Delta x_i^n \,,\, x_j^{n+1} - x_i^{n+1} + x_j^n - x_i^n \rangle \\ &= -2 \sum_{ij} \phi_{ij}^n \langle \Delta x_i^n \,,\, x_j^{n+1} - x_i^{n+1} + x_j^n - x_i^n \rangle, \end{aligned}$$

since $\phi_{ij} = \phi_{ji}$. Writing $x_j^{n+1} = x_j^n + \Delta x_j^n$, we deduce:

$$E^{n+1} - E^n \quad \leq \quad -2\sum_{ij} \phi_{ij}^n \langle \Delta x_i^n \,,\, \Delta x_j^n - \Delta x_i^n \,+\, 2(x_j^n - x_i^n)\rangle.$$

Finally, using the equality $\sum_j \phi_{ij}^n \Delta x_i^n = \sum_j \phi_{ij}^n (x_j^n - x_i^n)$, we conclude that:

$$E^{n+1} - E^n \quad \leq \quad -2\sum_{ij} \phi_{ij}^n \langle \Delta x_i^n \,,\, \Delta x_j^n + \Delta x_i^n\rangle = -4\sum_{ij} \phi_{ij}^n |\Delta x_i^n|^2.$$

□

As in the continuous case, one can deduce several inequalities from the decay of $E$. For instance, we find that:

$$E^{n+1} - E^n \leq -\frac{\left(\sum_{ij} \phi_{ij}^n |x_j^n - x_i^n|^2\right)^2}{\sum_{ij} \phi_{ij}^n |x_i^n|^2}. \tag{4.2}$$

This inequality implies that the discrete dynamics also generates clusters. We can extend the Prop. 2.4 and therefore regroup the agents $\{x_i^n\}_i$ by clusters $\mathcal{C}_k$.

**Proposition 4.2** *Let $\{x_i^n\}_i$ be the solution of the consensus model (1.5) with $\phi$ non-increasing. For any $\varepsilon > 0$, there exists a time $N_\varepsilon > 0$ such that for any $i,\ j$*

$$|x_j^n - x_i^n| \in [0,\ \varepsilon] \cup [1-\varepsilon,\ \infty) \quad \textit{for any } n \geq N_\varepsilon. \tag{4.3}$$

*Moreover, if $\phi(1-) > \phi(1+)$, the set:*

$$I_\varepsilon = \{n \geq 0 \;/\; \textit{ there is } i,\, j \textit{ satisfying } |x_j^n - x_i^n| \in [\varepsilon, 1)\} \tag{4.4}$$

*is finite.*

**Remark 4.3** *As in the continuous case, the decay of the functional $E$ combined with a convexity argument is sufficient to prove the convergence of the discrete dynamics* (1.5).

## 4.2 Convergence for the discrete dynamics

One difficulty remains to prove the convergence of the discrete dynamics. We have to extend the proposition 3.1 and in the discrete setting we cannot use the integration by parts. Instead, we have to use wisely the Abel's lemma.

**Proposition 4.4** *Assume that $\phi$ is non-increasing, concave down and satisfies (1.4). Then the solution $\{x_i^n\}_i$ satisfies:*

$$\sum_{n\geq 0} \sum_{i,j} \phi_{ij}^n |x_j^n - x_i^n|^2 \leq C\, N^3. \tag{4.5}$$

**Proof.** Denoting $D_n = \frac{1}{2}\sum_{ij}\phi_{ij}^n|x_j^n - x_i^n|^2$, we have:

$$D_n = \sum_{ij}\phi_{ij}^n\langle x_i^n, x_j^n - x_i^n\rangle = \sum_{ij}\phi_{ij}^n\langle x_i^n, \Delta x_i^n\rangle.$$

Writing $x_i^n = \frac{x_i^n - x_i^{n+1}}{2} + \frac{x_i^n + x_i^{n+1}}{2}$, we deduce:

$$\begin{aligned} D_n &= -\frac{1}{2}\sum_{ij}\phi_{ij}^n|\Delta x_i^n|^2 + \frac{1}{2}\sum_{ij}\phi_{ij}^n\langle x_i^n + x_i^{n+1}, x_i^{n+1} - x_i^n\rangle \\ &= \frac{1}{2}A_n + \frac{1}{2}B_n. \end{aligned}$$

Thanks to (4.1), the sum $\sum_{n\geq 0} A_n$ is bounded. For the sum of $B_n$, we use the Abel's formula:

$$\begin{aligned} \sum_{n=0}^{M} B_n &= \sum_{n=0}^{M}\sum_{ij}\phi_{ij}^n\Big(|x_i^{n+1}|^2 - |x_i^n|^2\Big) \\ &= \sum_{ij}\Big[\phi_{ij}^{M+1}|x_i^{M+1}|^2 - \phi_{ij}^0|x_i^0|^2\Big] + \sum_{n=0}^{M}\sum_{ij}\Big(\phi_{ij}^{n+1} - \phi_{ij}^n\Big)|x_i^{n+1}|^2. \end{aligned}$$

Since $\phi$ is concave-down, we deduce:

$$\sum_{n=0}^{M}\Big(\phi_{ij}^{n+1} - \phi_{ij}^n\Big) \leq \sum_{n=0}^{M}\phi'(|x_j^n - x_i^n|^2)\Big(|x_j^{n+1} - x_i^{n+1}|^2 - |x_j^n - x_i^n|^2\Big).$$

Using the equality $|u|^2 - |v|^2 = \langle u - v, u + v\rangle$ and $x_i^{n+1} = x_i^n + \Delta x_i^n$, we can write:

$$|x_j^{n+1} - x_i^{n+1}|^2 - |x_j^n - x_i^n|^2 = \langle \Delta x_j^n - \Delta x_i^n, 2(x_j^n - x_i^n) + \Delta x_j^n - \Delta x_i^n\rangle.$$

Thus,

$$\begin{aligned} \sum_{n=0}^{M}|x_i^{n+1}|^2\Big(\phi_{ij}^{n+1} - \phi_{ij}^n\Big) &\leq \sum_{n=0}^{M}|x_i^{n+1}|^2\,\phi'(|x_j^n - x_i^n|^2)\Big(|\Delta x_j^n - \Delta x_i^n|^2 \\ &\qquad + 2|x_j^n - x_i^n||\Delta x_j^n - \Delta x_i^n|\Big) \\ &\leq C + 2\sum_{n=0}^{M}|x_i^{n+1}|^2\,\phi'(|x_j^n - x_i^n|^2)|x_j^n - x_i^n||\Delta x_j^n - \Delta x_i^n|, \end{aligned}$$

since $\sum_{n=0}^{M} |x_i^{n+1}|^2 \, \phi'(|x_j^n - x_i^n|^2) |\Delta x_j^n - \Delta x_i^n|^2$ is bounded. Using Cauchy-Schwartz inequality, we deduce:

$$\begin{aligned}
\sum_{n=0}^{M} |x_i^{n+1}|^2 \left(\phi_{ij}^{n+1} - \phi_{ij}^{n}\right) &\leq C + C \left( \sum_{n=0}^{M} \left(\phi'(|x_j^n - x_i^n|^2)\right)^2 |x_j^n - x_i^n|^2 \right)^{\frac{1}{2}} \\
&\qquad \left( \sum_{n=0}^{M} |\Delta x_j^n - \Delta x_i^n|^2 \right)^{\frac{1}{2}} \\
&\leq C + C \left( \sum_{n=0}^{M} \phi_{ij}^n |x_j^n - x_i^n|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

using the assumption on $\phi$ (1.4). As a consequence, we have:

$$\begin{aligned}
\sum_{n=0}^{M} B_n &\leq C + C \sum_{ij} \left( \sum_{n=0}^{M} \phi_{ij}^n |x_j^n - x_i^n|^2 \right)^{\frac{1}{2}} \\
&\leq C + C\sqrt{N} \left( \sum_{ij} \sum_{n=0}^{M} \phi_{ij}^n |x_j^n - x_i^n|^2 \right)^{\frac{1}{2}} = C + C\sqrt{N} \left( \sum_{n=0}^{M} D_n \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{M} D_n \leq C + C\sqrt{N} \left( \sum_{n=0}^{M} D_n \right)^{\frac{1}{2}}.$$

As in the proof of proposition 3.1, we conclude that $\sum_{n=0}^{M} D_n$ is bounded uniformly in $M$. $\square$

To conclude, we study the center $y_k$ (3.3) of each cluster $\mathcal{C}_k$:

$$y_k^n = \frac{1}{N_k} \sum_{i \in \mathcal{C}_k} x_i^n. \tag{4.6}$$

We give an analogue of the proposition 3.2 in the discrete case.

**Proposition 4.5** *Assume $\phi$ satisfies (1.4). Then the center $y_k^n$ of each cluster $\mathcal{C}_k$ verifies*

$$\sum_{n=0}^{M} |\Delta y_i^n| \leq C\, N^3 + C\, N \sum_{n=0}^{M} D_n \tag{4.7}$$

*with $D_n = \frac{1}{2} \sum_{i,j} \phi_{ij}^n |x_j^n - x_i^n|^2$.*

*Therefore, by Prop. 4.4, if $\phi$ is non-increasing and concave-down, $y_k^n$ converges for any $k$.*

The proof of this proposition is similar to the continuous case (Prop. 3.2). We only have to notice that:

$$\Delta y_1^n = \frac{1}{N_1}\sum_{i\in\mathcal{C}_1}\Delta x_i^n = \frac{1}{N_1}\sum_{i\in\mathcal{C}_1}\sum_j a_{ij}^n(x_j^n - x_i^n),$$

with $a_{ij}^n = \frac{\phi_{ij}^n}{\sum_j \phi_{ij}^n}$.

From the proposition 4.5, we conclude that the solution $\{x_i^n\}_i$ converges since:

$$|x_i^n - y_k^n| \le \frac{1}{N_k}\sum_{j\in\mathcal{C}_k}|x_i^n - x_j^n| \stackrel{n\to\infty}{\longrightarrow} 0.$$

Thus, we have proven the Th. 3.

**Remark 4.6** *We have discussed the convergence of the discretization of the continuous dynamics with $\Delta t = 1$. The result of the section can be extended easily with $0 < \Delta t < 1$. The discretization gives to the following equations:*

$$x_i^{n+1} = x_i^n + \Delta t \frac{\sum_j \phi_{ij}^n x_j^n}{\sum_j \phi_{ij}^n}. \tag{4.8}$$

*Similarly, the functional $E$ is also decaying thanks to the following equality:*

$$\sum_j \phi_{ij}^n \Delta x_i^n = \Delta t \sum_j \phi_{ij}^n x_j^n.$$

*Thus, we prove similarly the emergence of clusters and the convergence of the dynamics. Those results explain the numerical results obtain in [19].*

# 5 Conclusion

In this work, we have analyzed the asymptotic behavior of a model of consensus. We have rigorously established the emergence of clusters and deduced the convergence in time of the dynamics. The proof is based on energy dissipation but requires additional techniques since the dynamics is asymmetric without any invariant. One of the key element is to study the evolution of the center of the clusters $y_k$ for which we can obtain a better control of the time derivative. Following a similar approach, we have also established the convergence of a discrete version of the consensus model. Several questions remain open concerning the dynamics. One could try to improve the results by removing some assumptions made on the interaction function $\phi$ (e.g. condition (1.4)). Similarly, we have assumed in the discrete case that the function $\phi$ was non-increasing and concave. Thus, one could ask whether it is possible to remove those constraints.

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