Large time asymptotics for a modified coagulation model

Juan Calvo* & Pierre-Emmanuel Jabin†

Abstract

We study the large time asymptotics of a simplified two species model for particles (typically molecules or cells). The particles can be in two states: A "free" state where they simply move with a given velocity or a aggregated state where they do not move anymore. We show that depending only on the strength of the interaction between particles, either all of them eventually coagulate or some may escape.

1 Introduction

The aim of this paper is to propose a suitably modified coagulation kinetic model, taking distinct states for the particles or cells (for applications to biology) into account.

The models representing caogulation phenomena can be classified according to the chosen scale of description. Microscopic descriptions try to represent the evolution of a finite set of individual particles, the Smoluchowski–type models are typical examples in this context, see [16] and references therein. Mean-field (mesoscopic) models are concerned with the evolution of the number of particles of each possible size, and not that of the individual particles; these descriptions are valid when the number of particles is sufficiently high. Mesoscopic models may or may not include the spatial distribution of the particles, [2, 20]. On the other hand, macroscopic models describe the evolution of some macroscopic quantities, which represent some kind of average of the microscopic properties of the system (such as the mean cluster size), [21, 23].

*Departamento de Matemática Aplicada. Avda. Fuentenueva s/n. Facultad de Ciencias, Universidad de Granada. 18071 Granada, Spain (juancalvo@ugr.es).

†Lab. Dieudionné, Univ. Nice-Sophia Antipolis, Parc Valrose. F-06108 Nice Cedex 02, and EPI Tosca, Inria Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia Antipolis, France, (Pierre-Emmanuel.JABIN@unice.fr).

We study in this article a kinetic model. Those approaches to the phenomenon of coagulation and fragmentation, take into account the effects of the movement and trajectories of the particles; see for example [3, 4, 14, 18, 19] for other studies of kinetic models for coagulation or fragmentation.

Many physical phenomena consist of a great number of small particles that can stick together in some way to form aggregates or new particles of larger size. At the same time big particles could split into smaller ones. This occurs in multiphase fluids, in many examples of phase change, in the behavior of aerosols with liquid or solid particles suspended in a gas and in crystallization in colloids, among other examples in this context, for example [1, 12, 17, 25, 27, 30].

The sticking together, coagulation, aggregation or adhesion into a cluster, of particles, whether they are cells, lipid droplets, proteins, etc., is of fundamental importance in biological and biotechnological processes. This is the primary motivation for the model that we study.

For example, in animals, small cells called platelets cluster at the site of an injury to the skin or blood vessels. Also, during the development of an embryo, space between aggregated cells decreases and cell-to-cell contact increases. Other example of this process can be found in cell aggregation by Chemotaxis or in flocculation of sticky phytoplankton cells into rapidly sinking aggregates, which has been invoked as a mechanism explaining mass sedimentation of phytoplankton blooms in the ocean. In the biological explanation of this context appear surface ligands that mediate cell-to-cell adhesion or any molecule involved in cellular adhesive phenomena such as in liver cell adhesion molecule and neural cell adhesion molecule. Experimental observations show that cell aggregation in suspension promoted cell survival and proliferation, in particular it has been demonstrated a correlation between tumor cell aggregation in suspension and cell growth.

The interaction forces between particles ultimately determine the stability and rheological properties of any co system, and in many biological cases the principal adsorbed component that mediates these interaction forces is protein.

Coagulation and aggregation phenomena have been the object of many studies in the recent years, both in physics and mathematics [12, 22]. For applications in physics, one typically assumes that the coagulation of two particles preserves the total mass and total momentum. However there are cases where the last is not true. The prime example that we have in mind is the dynamics of some cells in biology; for instance endothelial cells, but it is a very common phenomenon in biology. Those cells may move freely when they are alone. However they may also join with other cells of the same kind and then may not move any more. In particular endothelial cells compose blood vessels, once they are combined with other endothelial cells, and hence do not move [15, 29].

The aim of this paper is to propose a suitably modified coagulation model, taking distinct states for the particles or cells into account. A first state corresponds to free particles that, consequently, have velocities v and will require to consider the density in the phase space. A second state represents the coagulated particles that are fixed and thus only have a density in the physical space.

We introduce the two corresponding densities

$$f : [0,T] \times \mathbb{R}^d_x \times \mathbb{R}^d_v \to \mathbb{R} \quad \text{representing free particles}$$

$$\rho : [0,T] \times \mathbb{R}^d_x \to \mathbb{R} \quad \text{representing coagulated or stuck particles.}$$

It of course remains to explain how those quantities evolve in time. For free particles, we assume that each one moves with its velocity that does not change (as long as it remains free). This is a simplification and more realistic models should consider how this velocity may change (influence of chemoattractants, stochastic jumps...). As we focus mainly on the coagulation phenomenon, this assumption is however reasonable.

Free particles may move freely, interact one with another and coagulate. They may also meet already coagulated particles, interact and combine with them.

For stuck particles, the situation is simpler. They do not move and may therefore only interact with free particles that occupy the same position in space.

We consequently consider the following system of equations:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = -f(t,x,v) \int_{\mathbb{R}^d} \alpha(v,v') f(t,x,v')\, dv' - \beta(v)\rho(t,x) f(t,x,v) \quad (1)$$

$$\frac{\partial \rho}{\partial t} = \int_{\mathbb{R}^{2d}} \alpha(v,v') f(t,x,v') f(t,x,v)\, dv' dv + \rho(t,x) \int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\, dv \quad (2)$$

supplied with initial data $0 \le f^0(x,v) \in L^1(\mathbb{R}^{2d})$ and $0 \le \rho^0(x) \in L^1(\mathbb{R}^d_x)$.

The functions $\alpha(v,v')$ and $\beta(v)$ are collision or coagulation kernels and give the probability that two free particles with velocities v and v' will coagulate for α or one free particle with velocity v will coagulate with a stuck particle for β.

The collision kernels $\alpha(v,v')$, $\beta(v)$ are nonnegative. In most physical situations they behave polynomially; moreover by Galilean invariance, α

should essentially depend on the relative velocity of two particles $v-v'$. For these reasons we assume that they satisfy the following domination property: There exists a constant $C>0$ such that

$$\alpha(v,v') \le C|v-v'|^a,\ \beta(v) \le C|v|^a, \quad \text{for some } a \in \mathbb{R}. \tag{3}$$

Note nevertheless that some of the results that we shall present here can be generalized to abstract kernels (only integrability conditions and dependence on $v-v'$ assumed), even measure-valued kernels.

In some biological situations, coagulation between two cells touching eachother would always occur. This would correspond to $a=+\infty$ and would lead to a sort of sticky particles dynamics. Even in dimension 1, the analysis of such models is quite difficult (see for instance [6]) and especially so for the modified models that one would obtain in this case.

The main result in the paper is the characterization of the asymptotic behaviour depending on a. It is obvious from the equations that the mass of free particles may only decrease and the mass of coagulated particles only increase. Hence the main issue as $t\to+\infty$ is whether all free particles finally coagulate or if some of them remain free. We show that this depends only on the strength of the interactions (i.e. the value of a). The analysis is based on precise dispersion estimates for kinetic equations.

Those kind of dispersion properties constitute one of the fundamentals features in hyperbolic or transport equations which, in some sense, could be considered the counterpart to the diffusion effects in parabolic systems. Weak dispersion estimates allow to prove convergence to the equilibria, stability or even basic properties of existence in various situations, of which Strichartz–type estimates could be an example, see [7].

As a very representative example of hyperbolic transport equations we have the Vlasov–Poison system, whose behaviour depends in a crucial way on the character of the potential. In the plasma physival case all solutions exhibit a (strong or L^q-norm) dispersive character, see [26]. In the gravitational case the dynamics is more complex and rich: Its is possible to find stable (orbital stability) or unstable stationary spatial configurations, time-periodic solutions (breathers) and (fully or partially) dispersive solutions. The different notions of dispersion in this context have been studied in [11, 8] in terms of the macroscopic parameters associated to the initial conditions.

With respect to classical kinetic coagulation models, the existence and uniqueness theory is quite simple as a priori estimates are obtained in a standard way. It is nevertheless included for the sake of completeness.

We may summarize the results of the paper with the following

Theorem 1 *Assume that the integral kernels are non-negative, satisfy* (3) *and* $a+d>0$*. For any* $0 \le f^0 \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$*,* $0 \le \rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$ *and such that for some* $\eta > 0$

$$f^0(x,v) \le \frac{C}{1+|v|^{\max(a,0)+d+\eta}}, \text{ for a.e. } (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v, \tag{4}$$

there exists a unique weak solution of the system (1)–(2) *with initial data* $f(0,x,v) = f^0$ *and* $\rho(0,x) = \rho^0$*. Moreover, there exists a function* $g_\infty(x,v)$ *such that*

$$\left\| f(t,x,v) - g_\infty\left(\frac{x}{t}, t(v-\frac{x}{t})\right) \right\| \to 0 \quad \text{as } t \to \infty$$

in the norm of $W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))$*. Furthermore,*

- *if* $a > 1-d$ *(or* $a > 1$ *if* $d = 1$*) and* f^0 *and* ρ^0 *are compactly supported in* x*, the amount of mass* $\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv$ *is bounded from below by a positive constant indepently of time.*

- *if* $-d < a \le 1-d$*, the amount of mass* $\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv$ *is strictly positive for all times but converges to zero as* t *goes to infinity.*

Existence and uniqueness are dealt with in the second section, where we also explain what we mean by weak solution. The traveling wave form of the solution is proved in the last section. Section 3 investigates the issue of vanishing free particles.

The nonnegativity of the kernels and the condition (3) will be assumed (for $C = 1$) for the rest of the paper, with no further mention.

2 Existence and Uniqueness

In this section we state our concept of solution and prove existence and uniqueness under certain decay assumptions for the initial data and for the integral kernels.

Definition 1 *A weak solution of the system (1)–(2) in the time interval* $[0,T]$ *is a pair of nonnegative funtions* $f \in L^\infty([0,T],\ L^1 \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v))$*,* $\rho \in L^\infty([0,T],\ L^1(\mathbb{R}^d_x))$ *with initial data* $0 \le f^0(x,v) \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$

and $0 \le \rho^0(x) \in L^1(\mathbb{R}^d_x)$ *and which satisfies the following weak formulation:*

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} \frac{\partial \varphi}{\partial t} f \, dtdxdv - \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} \varphi(0,x,v) f^0(x,v)\, dxdv \\
&-\int_0^T \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} v\cdot\nabla_x \varphi f \, dtdxdv \\
&\quad = -\int_0^T \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} \varphi f \left[\int_{\mathbb{R}_{v'}} \alpha(v,v') f(t,x,v')\, dv' + \beta(v)\rho(t,x)\right] dtdxdv
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^d_x} \frac{\partial \psi}{\partial t}\rho \, dxdt - \int_{\mathbb{R}^d_x} \psi(0,x)\rho(0,x)\, dx \\
&\quad = \int_0^T \int_{\mathbb{R}^d_x} \psi(t,x)\rho(t,x) \int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\, dv\, dtdx \\
&\quad + \int_0^T \int_{\mathbb{R}^d_x} \psi(t,x) \int_{\mathbb{R}^d_v}\int_{\mathbb{R}_{v'}} \alpha(v,v') f(t,x,v) f(t,x,v')\, dv' dvdxdt,
\end{aligned}
$$

for every $\varphi \in \mathcal{D}([0,T[\times\mathbb{R}^d_x\times\mathbb{R}^d_v)$ *and every* $\psi \in \mathcal{D}([0,T[\times\mathbb{R}^d_x)$.

A-priori estimates will show that for initial data in an appropriate class the property $\partial_t f \in L^\infty([0,T], W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v)))$ holds, so that the statement $f(0,x,v) = f^0(x,v)$ makes sense. To give a meaning for $\rho(0,x) = \rho^0(x)$ is easier, because this holds in $L^1(\mathbb{R}^d_x)$.

We need to introduce some extra notation:

Definition 2 *The density function associated with the population f is given by*

$$\rho_f(t,x) = \int_{\mathbb{R}^d_v} f(t,x,v)\ dv.$$

Definition 3 *The total mass of the system is represented by the quantity*

$$M = \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} f(t,x,v)\, dxdv + \int_{\mathbb{R}^d_x} \rho(t,x)\, dx$$

We explain here some conventions that will be used in the paper. Hereafter C will stand for a generic positive constant, whose value may change from line to line. The notation $C(a,b,\ldots)$ indicates that the constant C depends explicitly on the specified quantities. We will use $B(r)$ to refer to a ball centered at the origin with radius r. The space on which this ball is considered will either be clear from the context or indicated by a proper subscript. $B(r)^c$ denotes the complement of such a ball in its corresponding space. We use $|A|$ to represent the Lebesgue measure of a set A.

The following stability result essentially implies the existence result in Theorem 1 (as the approximation of our system does not pose any problem)

Theorem 2 *Consider $a > -d$. Assume that $f^0 \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $\rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$ and that for some $\epsilon > 0$ the following bound is verified:*

$$f^0(x,v) \leq \frac{C}{1+|v|^{\max\{a,0\}+d+\epsilon}} \quad a.e.\ (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v. \tag{5}$$

Then, any sequence $\{(f_n,\ \rho_n)\}$ of smooth solutions to (1)-(2) converges weakly in any $L^p([0,T] \times \mathbb{R}^d_x \times \mathbb{R}^d_v),\ 1 \leq p \leq \infty$, to a weak solution $(f,\ \rho)$ of (1)–(2).

Remark 1 *The assumptions are quite reasonable from the point of view of applications to physics or biology. They imply that $(1+|v|^\epsilon)f^0 \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v) \cap L^1(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x))$. The way in which the existence result is stated is not completely sharp. This can be seen tracking carefully the proof. Besides this we can use a more technical proof and sharpen the result even more, lowering the integral regularity which is required for the initial data. Generally speaking the less we ask for f^0, the more we have to demand from ρ^0, and vice versa. However the corresponding assumptions are not easy to state; and we prefer to restrict to this non optimal form of the result.*

Proof. First we outline the proof for the case $a \geq 0$, then we explain the modifications that are needed for the case $a < 0$.

Note that introducing the characteristics curves for (1), which are straight lines indeed, we get a representation of $f(t,x,v)$ as

$$f(t,x,v) = f^0(x-vt,v)\, m(t,x,v),$$

with $m \in [0,\ 1]$ a damping factor. This shows that $f(t)$ is nonnegative if f^0 is, and we also get some a priori estimates as a consequence. These are gathered here.

Lemma 1 *For any $0 < t < T$ and any solution of (1)-(2), the following estimates hold:*

1. *$f(t,x,v) \leq f^0(x-vt,v)$.*

2. $\|f(t)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \le \|f^0\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}^d_v)}$, $1 \le p \le \infty$.

3. $\|\rho_f(t)\|_{L^p(\mathbb{R}^d_x)} \le \|f^0\|_{L^1(\mathbb{R}^d_v, L^p(\mathbb{R}^d_x))}$, $1 \le p \le \infty$.

4. $\displaystyle\int_{\mathbb{R}^{2d}} \alpha(v,v')f(t,x,v')f(t,x,v)\,dv'dv \in L^\infty([0,T]\times\mathbb{R}^d_x)$.

5. $\displaystyle\int_{\mathbb{R}^d_v} \beta(v)f(t,x,v)\,dv \in L^\infty([0,T]\times\mathbb{R}^d_x)$.

Proof. Estimate *2* follows from *1*; estimate *3* follows from *1* and Minkowsky's inequality. To prove *4*, we recall that for a given $a > 0$ there exists a constant $C = 2^{\max\{0,\ a-1\}} > 0$ such that $|v-v'|^a \le C(|v|^a + |v'|^a)$. Using this fact,

$$\begin{aligned}
&\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v)f(t,x,v')\,dvdv' \\
&\le 2C\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |v|^a f(t,x,v)f(t,x,v')\,dvdv' \\
&\le C\|\rho_f\|_\infty \int_{\mathbb{R}^d_v} |v|^a f^0(x-vt,v)\,dv,
\end{aligned}$$

where we used *3* to ensure the finiteness of $\|\rho_f\|_\infty$. Using then (5) we conclude the proof of *4*. The one for *5* is similar. □

Next, we integrate the equation (2) and knowing that $f(t)$ is nonnegative we infer that $\rho(t)$ has also this property if it does initially. At this stage it is then meaningful to introduce the total mass of the system M, which is conserved during the evolution for classical solutions and therefore trivially non-increasing in the general case.

The conservation of mass shows that $\rho(t) \in L^1(\mathbb{R}^d_x)$ uniformly in time. If we prove that $\rho(t)$ is bounded in some $L^p(\mathbb{R}^d_x)$ space we can show in the usual way the convergence of all the linear terms involved in the weak formulation. Indeed, we can get an estimate for ρ in $L^\infty([0,T]\times\mathbb{R}^d_x)$, as (2) is readily integrated and then estimates *4* and *5* of Lemma 1 allow to deduce it.

The last point to prove the stability result is to show the convergence of the product terms. For the sake of completeness we recall here an useful result which can be found in [10]. Here $\mathcal{T}$ denotes the transport operator, the lhs of (1).

Lemma 2 *Suppose that* $\{g_n\} \subset L^1(]0,T[, L^1_{loc}(\mathbb{R}^d_x\times\mathbb{R}^d_v))$ *is weakly relatively compact, and that* $\{\mathcal{T}g_n\}$ *is weakly relatively compact in* $L^1_{loc}(]0,T[\times\mathbb{R}^d_x\times$

$\mathbb{R}^d_v$)*. Then, if* $\{\psi_n\}$ *is a bounded sequence in* $L^\infty(]0,T[, L^\infty_{loc}(\mathbb{R}^d_x \times \mathbb{R}^d_v))$ *that converges a.e., then* $\int g_n \psi_n \, dv$ *is strongly compact in* $L^1(]0,T[, L^1_{loc}(\mathbb{R}^d_x))$*.*

We describe here how to deal with one of the product terms. We can estimate the corresponding difference

$$\left| \int_0^T \int_{\mathbb{R}^d_x} \psi(t,x) \int_{\mathbb{R}^d_v \times \mathbb{R}^d_{v'}} \alpha(v,v') \left(f(t,x,v) f(t,x,v') - f_n(t,x,v) f_n(t,x,v') \right) \right|$$

using terms like

$$\left| \int_0^T \int_{\mathbb{R}^d_x} \psi(t,x) \int_{(B_v(R) \times B_{v'}(R))^c} \alpha(v,v') f_n(v) f_n(v') \; dv dv' dx dt \right|$$

and

$$\int_0^T \int_{\mathbb{R}^d_x} |\psi(t,x)| \; \mathrm{sup}_{v \in B_v(R)} |f_n(v)|$$

$$\times \int_{B_v(R)} \left| \int_{B_{v'}(R)} \alpha(v,v') f_n(v') \; dv' - \int_{B_{v'}(R)} \alpha(v,v') f(v') \; dv' \right| \; dv \; dx dt,$$

where for the sake of simplicity we have omitted the dependence on x as this does not cause confusion. The aim is to make these quantities less than any given $\epsilon > 0$. To control terms as the first one, we use a parameter $R > 0$ and estimate as follows:

$$\begin{aligned}
&\int_{(B_v(R) \times B_{v'}(R))^c} \alpha(v,v') f_n(v) f_n(v') \; dv dv' \\
&\leq \int_{(B_v(R) \times B_{v'}(R))^c} \alpha(v,v') \frac{(|v|^2 + |v'|^2)^{r/2}}{(|v|^2 + |v'|^2)^{r/2}} f_n(v) f_n(v') \; dv dv' \\
&\leq \frac{1}{R^r} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} \alpha(v,v') \left[\sqrt{|v|^2 + |v'|^2} \right]^r f_n(v) f_n(v') \; dv dv' \leq \frac{C}{R^r}.
\end{aligned}$$

This works thanks to (5), for $r > 0$ suitably small. Then, we choose $R^r = \frac{2}{\epsilon} C \|\psi\|_1$ and we can force this type of terms to be smaller than $\epsilon/2$.

It remains to show that the terms involving velocity averages over a compact set can also be made as small as wanted. To use the averaging results it suffices to show that $\mathcal{T} f_n$ is uniformly in $L^{1+\epsilon}([0,T], L^1_{loc}(\mathbb{R}^d_x \times \mathbb{R}^d_v))$ for some $\epsilon > 0$ and in this way we avoid concentration phenomena. To do so, notice that we already know that $f_n, \rho_n \in L^1 \cap L^\infty$. The integral

$\int_{\mathbb{R}^d_{v'}} \alpha(v,v') f(t,x,v')\, dv'$ is then bounded a.e. (t,x). Finally, $\beta(v) f(t,x,v)$ belongs to $L^\infty([0,T]\times\mathbb{R}^d_x, L^p(\mathbb{R}^d_v))$ for any $p>1$, as thanks to (5) we get

$$(\beta(v) f_n(t,x,v))^p \leq \left(\frac{C|v|^a}{1+|v|^{a+d+\epsilon}}\right)^p \leq \left(\frac{C}{1+|v|^{d+\epsilon}}\right)^p.$$

The rest of the product terms can be handled with slight variations of the arguments sketched above.

The case $a<0$: to proceed we introduce for the remaining of the section the notation

$$q := d/|a|,$$

which comes from the fact that $|\cdot|^{-|a|} \in L^q_w(\mathbb{R}^d)$. Basic facts about weak Lebesgue spaces can be found in [24] or [5]. The main differences with the previous case are the following:

- The estimates *4*-*5* of Lemma 1 are proved in a different way. For *4* we use the Hardy-Littlewood-Sobolev inequality [28], combined with suitable spatial regularity. This would require

$$f^0 \in L^{\frac{2q}{2q-1}}(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x)), \tag{6}$$

  which by interpolation is always true as thanks to (5) our initial datum f^0 belongs to $L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)\cap L^1(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x))$. Note that $\frac{2q}{2q-1} = \frac{2d}{2d+a}$. The estimate *5* is dealt away combining Hölder's inequality with a layer-cake-type argument. More precisely:

  Lemma 3 *Let* $g\in L^1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$*, with* $p>q'$*. Define*

$$e(p,q) = \frac{(p')^2}{q-p'(1-p')}.$$

  Then, for any $\lambda\in\mathbb{R}^d$*, we have*

$$\int_{\mathbb{R}^d} \frac{|g(x)|}{|x-\lambda|^{|a|}}\, dx \leq C(p,q)\|g\|_1^{1-e(p,q)}\|g\|_p^{e(p,q)}.$$

  The use of this result to obtain the estimate *5* requires $f^0 \in (L^1\cap L^{\frac{d}{d+a}+\delta})(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x))$ for some $\delta>0$ suitably close to zero. This is again implied by (5) and $f^0\in L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)$. Note that the hypothesis (6) is contained in this one and these in turn are implied by the assumptions in Theorem 2.

- To have the convergence of the product terms the procedure is different. First of all, we use Lemma 2 or a similar result (the ones in [13] for instance) to prove the convergence for a regularized kernel. Secondly we show that the integral against the difference between the regularized kernel and the non regularized one tends to 0 as the parameter of regularization tends to 0 and this uniformly in n. This is easily implied by the uniform bounds on f_n.

□

2.1 Uniqueness

We have uniqueness in the class of weak solutions that can be approximated by classical solutions in $L^\infty(0,T,L^1(\mathbb{R}^d_x\times\mathbb{R}^d_v)\times L^1(\mathbb{R}^d_x))$, thanks to the following result.

Proposition 1 *Any weak solution of (1)–(2) which is limit of classical solutions and satisfies the assumptions of Theorem 2 is unique.*

Proof. We consider two solutions (f_1,ρ_1), (f_2,ρ_2). Since those are the limits of classical solutions, justifying the computations performed below is easy: Consider them as classical solutions and simply pass to the limit at the end. Note however that given the simple form of the system, it is likely that this could be done for any weak solution. As this is only an accessory point in the paper, we do not investigate it fully.

Let us introduce the functions $g=f_1-f_2$ and $h=\rho_1-\rho_2$. We will conclude uniqueness with a Grönwall argument applied to the integral of $|g|+|h|$. First we compute an equation for g. For simplicity we will omit the integration domains as this will cause no confusion.

$$\begin{aligned}\partial_t g + v\cdot\nabla_x g = &-g(t,x,v)\int \alpha(v,v')f_1(t,x,v')\,dv'\\ &-f_2(t,x,v)\int \alpha(v,v')g(t,x,v')\,dv'\\ &-h(t,x)\beta(v)f_2(t,x,v)-\beta(v)g(t,x,v)\rho_1(t,x).\end{aligned}$$

We use this to obtain an equation for $\int|g|\;dxdv$. To do so, an equation for $\frac{d}{dt}\int\phi_n(g)\;dxdv$ is computed first, being ϕ_n a suitable smooth approximation of the sign function —a.e. convergent and matching ± 1 outside a compact set containing the origin—, and then we pass to the limit on that equation.

Thus, we get

$$\frac{d}{dt}\int |g(t,x,v)|\,dxdv = -\int\int\int |g(t,x,v)|\alpha(v,v')f_1(t,x,v')\,dv'dvdx$$
$$-\int\int\int f_2(t,x,v)\alpha(v,v')g(t,x,v')\text{sign}\,[g(t,x,v)]\,dv'dvdx$$
$$-\int\int \beta(v)\rho_1(t,x)|g(t,x,v)|\,dxdv$$
$$-\int\int h(t,x)\beta(v)f_2(t,x,v)\text{sign}\,[g(t,x,v)]\,dxdv.$$

Then we compute an equation for h,

$$\partial_t h = \int\int \alpha(v,v')f_1(t,x,v')g(t,x,v)\,dvdv'$$
$$+\int\int \alpha(v,v')f_2(t,x,v)g(t,x,v')\,dvdv'$$
$$+\left(\int \beta(v)f_2(t,x,v)\,dv\right)h(t,x)+\rho_1(t,x)\left(\int \beta(v)g(t,x,v)\,dv\right),$$

so that, doing as before, we find

$$\frac{d}{dt}\int |h(t,x)|\,dx = \int\int\int \alpha(v,v')f_1(t,x,v')g(t,x,v)\text{sign}[h(t,x)]\,dxdvdv'$$
$$+\int\int\int \alpha(v,v')f_2(t,x,v)g(t,x,v')\text{sign}[h(t,x)]\,dxdvdv'$$
$$+\int\int |h(t,x)|\beta(v)f_2(t,x,v)\,dxdv$$
$$+\int\int \beta(v)g(t,x,v)\rho_1(t,x)\text{sign}[h(t,x)]\,dxdv.$$

Adding both we get to

$$\frac{d}{dt}\left[\int\int |g(t,x,v)|\,dxdv+\int |h(t,x)|\,dx\right]$$
$$\leq 2\int\int\int f_2(t,x,v)\alpha(v,v')|g(t,x,v')|\,dxdvdv'$$
$$+2\int\int |h(t,x)|\beta(v)f_2(t,x,v)\,dxdv.$$

In case that $a<0$ we use Lemma 3 to show that

$$\int\int |h(t,x)|\beta(v)f_2(t,x,v)\,dxdv \leq C\int |h(t,x)|\,dx$$

and likewise

$$\int\int\int f_2(t,x,v)\alpha(v,v')|g(t,x,v')|\,dxdvdv' \le C\int\int |g(t,x,v')|\;dxdv'.$$

Whereas if $a>0$ we achieve the same inequality for the β-integral using the compact support in velocities. The same can be done for the α-integral if the velocity supports of the solutions under consideration are compact, so that $\alpha(v,v')$ can be majorized independently of v', which is the case. □

3 Large time behaviour

The aim of this section is to investigate the behaviour of the solution for large times. From the very form of the equations of the model we can see that f will lose mass progressively, which in principle will be transferred to the population ρ. The issue that we address here is the following: does the species f eventually vanish completely, thus transferring all its mass to ρ, or some of this mass is going to be lost to infinity? Under some decay assumptions on the initial data we will show that this is not so for the range $a\in]1-d,+\infty]$, while the total transfer of mass is achieved in infinite time for powers $a\in]-d,1-d]$.

We use the notation

$$M(t)=\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} f(t,x,v)\,dxdv$$

for the mass carried by the species f. We will typically need some compactness assumptions on the initial data. Compactness may be assumed in space or in velocity and we introduce the following set of notations and possible assumptions:

$$\text{supp }\rho_f^0\subset B(R)\text{ for some }R<\infty. \tag{7}$$

$$\cup_{x\in\mathbb{R}^d_x}\text{supp}_v\ f^0\subset B(V)\text{ for }V:=\operatorname{ess\,sup}\{|v|/f^0(x,v)>0\}<\infty. \tag{8}$$

$$\text{supp }\rho^0\subset B(\tilde R)\text{ for some }\tilde R<\infty. \tag{9}$$

3.1 The non-vanishing case: bounded velocity supports.

We start by the simplest case with bounded compact support in space and velocity. This will be extended in the next subsection but for the sake of a better understanding we present the main arguments (some dispersive inequalities) in this simplified setting.

This section is devoted to the proof of the following statement:

Proposition 2 *For $a+d>1$, assume that $\rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$, $f^0 \in (L^1 \cap L^\infty)(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ and verifies the hypotheses (7)–(9). Then, for each weak solution of (1)–(2) associated to this class of initial data and such that it can be approximated by smooth solutions in $L^\infty_t(0,\infty, L^1(\mathbb{R}^{2d}) \times L^1(\mathbb{R}^d_x))$, there exists a constant $C = C(f^0, \rho^0) > 0$ such that the total mass associated to f satisfies*

$$M(t) \geq C \quad \forall t \geq 0.$$

The upshot is that a certain amount of particles starting with velocities high enough do not get trapped. The proof of the above statement is done through a series of lemmas.

The following result yields some basic useful information:

Lemma 4 *Assume that $|supp\ \rho^0_f| < +\infty$. Then, the measure of the set $\{v \in \mathbb{R}^d,\ f(t,x,v) > 0\}$ decreases in time like t^{-d}, pointwise in x. This implies*

$$\rho_f(t,x) \leq \frac{|supp\ \rho^0_f| \|f^0\|_\infty}{t^d}. \tag{10}$$

Furthermore, for each pair $z = (v,x,t)$ and $z' = (v',x,t)$ such that $f(z) > 0$ and $f(z') > 0$, we have $|v - v'| \leq \frac{2R}{t}$.

Proof. Indeed, a particle starting from a position x_0 with a velocity v reaches the position x at time t if and only if $x = x_0 + tv$. Thus, we have that

$$v = \frac{x - x_0}{t} \in \frac{x - \text{supp}\ \rho^0_f}{t},$$

the latter being a set of measure $|\text{supp}\ \rho^0_f| t^{-d}$. □

The estimate (10) shows that all the local mass associated to f will eventually vanish; the question is how much of this is going to be transferred to ρ and how much is going to be lost to infinity. As a common framework to deal with this problem, the assumptions of Proposition 2 will be implicitly taken for granted in all the statements to follow in this section.

To proceed we introduce particular fractions of mass M_ϵ, which account for the contribution of particles with non-vanishing velocities. These masses are going to be non-vanishing for large times if this is true for short times. We do not take care of the remaining part of the initial mass. Define accordingly

$$M_\epsilon(t) = \int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} f(t,x,v)\,dxdv.$$

It can be readily shown that this function satisfies the following equation:

$$\begin{aligned} \frac{dM_\epsilon}{dt} &= -\int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} \int_{\mathbb{R}^d_{v'}} \alpha(v,v') f(t,x,v) f(t,x,v')\,dxdvdv' \\ &\quad - \int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} \beta(v)\rho(t,x) f(t,x,v)\,dxdv = -I - II. \end{aligned} \tag{11}$$

The basic estimate for M_ϵ is the following.

Lemma 5 *The function M_ϵ satisfies*

$$\frac{dM_\epsilon(t)}{dt} \geq -\frac{C}{t^{a+d}} M_\epsilon(t) - C\|\rho\|_{L^\infty(\Omega)} M_\epsilon(t),$$

with $\Omega = \{x \in \mathbb{R}^d / \exists |v| > \epsilon,\ s.t.\ f(t,x,v) > 0\}$. Furthermore, this function does not vanish in finite time.

Proof. To deal with the integral I in (11), note that we only have to estimate it in the following set of velocities: $\{v, v' \in \mathbb{R}^d$ such that $v - v' \in \frac{2}{t}\text{supp } \rho^0_f\}$. We can use that $a + d > 0$ to write

$$\int_{\{v'/v-v' \in \frac{2}{t}\text{supp } \rho^0_f\}} |v-v'|^a \, dv' \leq |\mathbb{S}^{d-1}| \int_0^{\frac{2R}{t}} r^{a+d-1}\, dr = \frac{|\mathbb{S}^{d-1}|}{a+d} \frac{(2R)^{a+d}}{t^{a+d}}$$

and then we get the estimate

$$I \leq \int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} \int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v')\,dxdvdv'$$

$$\leq \|f_0\|_\infty \int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} \left(\int_{\{v'/v-v' \in \frac{2}{t}\text{supp } \rho^0_f\}} |v-v'|^a \, dv' \right) f(t,x,v)\,dvdx$$

$$\leq \frac{\|f^0\|_\infty}{a+d} |\mathbb{S}^{d-1}| \frac{(2R)^{a+d}}{t^{a+d}} M_\epsilon(t).$$

To treat the integral II we notice that the integral with respect to x is actually computed over the set Ω. Then, if $a \geq 0$

$$II \leq V^a \|\rho(t)\|_{L^\infty(\Omega)} M_\epsilon(t)$$

and if $a < 0$,

$$II \leq \epsilon^{-|a|} \|\rho(t)\|_{L^\infty(\Omega)} M_\epsilon(t),$$

which in both cases concludes the proof of the differential inequality.

Finally, the later claim follows from the rough estimates

$$\frac{dM_\epsilon}{dt} \geq -(2V)^a \|\rho_f\|_\infty(t) M_\epsilon(t) - V^a \|\rho\|_\infty(t) M_\epsilon(t)$$

if $a \geq 0$ and

$$\frac{dM_\epsilon}{dt} \geq -C(f^0) M_\epsilon(t) - (\epsilon)^{-|a|} \|\rho\|_\infty(t) M_\epsilon(t)$$

if $a < 0$, where Lemma 3 has been used. □

In order to control the factor $\|\rho\|_{L^\infty(\Omega)}$, we also need to estimate the terms appearing in the rhs of (2).

Lemma 6 *The function $\rho(t,x)$ satisfies the following inequality:*

$$\frac{\partial \rho}{\partial t}(t,x) \leq \frac{C}{t^{a+2d}} + \frac{C}{t^{a^*}} \rho(t,x),$$

with $a^* = \min\{d, a+d\}$.

Proof. The fact that $a + d > 0$ allows us to estimate

$$\int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} |v - v'|^a f(t,x,v) f(t,x,v') \, dv' dv$$

$$\leq \|f^0\|_\infty \int_{\mathbb{R}^d_v} \int_{\{v'/v - v' \in \frac{2}{t} \mathrm{supp}\ \rho_f^0\}} |v - v'|^a \, dv' f(t,x,v) \, dv$$

$$\leq \|f^0\|_\infty |\mathbb{S}^{d-1}| \int_0^{\frac{2R}{t}} r^{a+d-1} \, dr \int_{\mathbb{R}^d_v} f(t,x,v) \, dv$$

$$\leq \frac{|\mathbb{S}^{d-1}|}{a+d} \frac{(2R)^{a+d}}{t^{a+d}} \|f^0\|_\infty \frac{|\mathrm{supp}\ \rho_f^0| \, \|f_0\|_\infty}{t^d} = \|f^0\|_\infty^2 \frac{|\mathrm{supp}\ \rho_f^0| (2R)^{a+d}}{a+d} \frac{|\mathbb{S}^{d-1}|}{t^{a+2d}},$$

so that

$$\int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} \alpha(v,v') f(t,x,v) f(t,x,v') \, dv' dv \leq \frac{C}{t^{a+2d}}. \tag{12}$$

Next, if $a \geq 0$ we find

$$\int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\ dv \leq V^a \int_{\mathbb{R}^d_v} f(t,x,v)\ dv \leq \frac{C}{t^d},$$

whereas if $a < 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^d_v} |v|^a f(t,x,v)\ dv &= \int_{|v|\leq 1/t} |v|^a f(t,x,v)\ dv + \int_{|v|>1/t} |v|^a f(t,x,v)\ dv \\ &\leq |\mathbb{S}^{d-1}| \|f^0\|_\infty \int_0^{1/t} r^{a+d-1}\ dr + t^{|a|} \int_{\mathbb{R}^d_v} f(t,x,v)\ dv \\ &\leq \frac{|\mathbb{S}^{d-1}| \|f^0\|_\infty}{(a+d) t^{a+d}} + \frac{C}{t^{a+d}},\end{aligned}$$

where in both cases we have used (10). Summing up,

$$\int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\ dv \leq \frac{C}{t^{a^*}}. \tag{13}$$

□

Now we go to the core of our method of proof.

Lemma 7 *Given $t_0 > 2R/V$, if $V > \epsilon \geq \frac{2R}{t_0}$ the sets $\{(x,v)/|x| \leq R + V t_0\}$ and supp $f(2Vt_0/\epsilon, \cdot, \cdot) \cap \{(x,v)/|v| > \epsilon\}$ are disjoint.*

Proof. Any pair $(x,v) \in \{|x| \leq R + V t_0\} \cap \text{supp } f(2Vt_0/\epsilon, \cdot, \cdot)$ satisfies the relation $2Vt_0|v|/\epsilon - R \leq R + Vt_0$, and then $|v| \leq \frac{R}{t_0} + \frac{\epsilon}{2}$. □

In the next result we obtain some control over the size of the support of $\rho(t)$. This result is the principal technical difference between the case that we are considering here and the non-compactly supported one.

Lemma 8 *Whenever $t > \overline{\tau} = \max\{\frac{\tilde{R}-R}{V}, 0\}$, we have that supp $\rho(t) \subset B(R+Vt)$.*

Proof. Integrating (2) we deduce that

$$\text{supp } \rho(t) \subset \text{supp } \rho_0 \cup \left(\cup_{\tau \leq t} \text{ supp } \rho_f(\tau)\right).$$

□

Integration of the inequality for ρ, given by Lemma 6, yields the estimate

$$\rho(t,x) \leq \rho(t_0,x) \exp \left\{\int_{t_0}^t \frac{C}{\tau^{a^*}}\ d\tau\right\} + \int_{t_0}^t \frac{C}{\tau^{a+2d}} \exp \left\{\int_\tau^t \frac{C\ ds}{s^{a^*}}\right\}\ d\tau.$$

If we consider it in the range $t_0 > \overline{\tau}$ and $|x| > R + Vt_0$ we get rid of the first term. From now on we set

$$t_0 := \epsilon t/(2V) > \overline{\tau}$$

(the range of t is restricted accordingly) and thus

$$\rho(t,x) \leq \int_{\epsilon t/(2V)}^{t} \frac{C}{\tau^{a+2d}} \exp\left\{ \int_{\tau}^{t} \frac{C\ ds}{s^{a^*}} \right\} d\tau,$$

so that

$$\rho(t,x) \leq \frac{C(2^{a+2d-1}-1)}{a+2d-1} \frac{1}{t^{a+2d-1}} \exp\left\{ \frac{C(2^{a^*-1}-1)}{a^*-1} \frac{1}{t^{a^*-1}} \right\}.$$

We shall substitute this estimate into the inequality granted by Lemma 5, with Lemma 7 assuring that Ω does not include the region $|x| \leq R + Vt_0$. We are left with

$$\frac{dM_\epsilon}{dt}(t) \geq -\frac{C}{t^{a+d}} M_\epsilon(t) - \frac{C}{t^{a+2d-1}} \exp\left\{ \frac{C}{t^{a^*-1}} \right\} M_\epsilon(t).$$

After integration in time,

$$M_\epsilon(t) \geq M_\epsilon(t_0) \exp\left\{ - \int_{t_0}^{t} \frac{C}{\tau^{a+d}} + \frac{Ce^{\tau^{1-a^*}}}{\tau^{a+2d-1}}\ d\tau \right\}.$$

If we show that the above integral is convergent we can perform the limit $t \to \infty$ to obtain that M_ϵ does not vanish. Simply note that for τ big enough

$$\frac{C}{\tau^{a+d}} + \frac{Ce^{\tau^{1-a^*}}}{\tau^{a+2d-1}} \leq \frac{C}{\tau^{a+d}}.$$

So that, as $a + d - 1 > 0$,

$$\exp\left\{ - \int_{t_0}^{t} \frac{C}{\tau^{a+d}} + \frac{Ce^{\tau^{1-a^*}}}{\tau^{a+2d-1}}\ d\tau \right\} \geq \exp\left\{ C\left(\frac{1}{t^{a+d-1}} - \frac{1}{t_0^{a+d-1}} \right) \right\}.$$

Meaning that

$$M_\epsilon(\infty) \geq M_\epsilon(t_0) \exp\left\{ -\frac{C}{t_0^{a+d-1}} \right\},$$

or that the total mass, which is larger than M_ϵ, may not vanish.

Finally notice that the restrictions concerning the time for the above arguments to be valid are:

- $t \geq \frac{4RV}{\epsilon^2}$ to assure the applicability of Lemma 7.
- $t > 2\frac{\overline{\tau}V}{\epsilon}$ to be able to control the growth of the supports of both species in an easy way.

It is always possible to work in this range as no mass may vanish in finite time.

3.2 Non-vanishing case: unbounded velocity supports.

Here we extend the result of the previous section in order to allow unbounded velocities. As a consequence compactness in velocity will be replaced by a more precise decay assumption.

Only the case $d > 1$ will be considered for the moment; we defer the special case $d = 1$ to the next subsection.

Proposition 3 *Assume $a + d > 1$ and that for some $\eta > 0$ we have*

$$f^0(x, v) \leq \frac{C}{1 + |v|^{\max(a,0)+d+\eta}} \quad a.e.\ (x, v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v. \tag{14}$$

We also assume that the compact support conditions (7) and (9) hold. Then, for any weak solution of (1)–(2) given by Theorem 2, there exists a constant $C = C(f^0, \rho^0) > 0$ such that

$$M(t) \geq C \quad \forall t \geq 0.$$

Note that the assumptions imply that $f^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ and that $f^0 \in L^1(\mathbb{R}^d_v,\ L^\infty(\mathbb{R}^d_x))$.

The rest of the section is devoted to prove this statement. The assumptions of Theorem 3 are implicitly taken for granted in all the intermediate lemmas.

It is still easy to check that no mass may vanish in finite time.

Lemma 9 *The functions M_ϵ do not vanish in finite time.*

Proof. Set for any $k > 1$

$$M_{\epsilon,k\epsilon}(t) := \int_{\mathbb{R}^d_x} \int_{k\epsilon > |v| > \epsilon} f(t, x, v)\, dxdv.$$

Choose a number $k > 1$ such that $M_{\epsilon,k\epsilon}(0) > 0$. Then the same proof as in Lemma 5 with $V = k\epsilon$ ensures that the function $M_{\epsilon,k\epsilon}(t)$ does not vanish in finite time. Thus, being $M_\epsilon(t) \geq M_{\epsilon,k\epsilon}(t)$ the statement follows. □

Let us turn to the crucial estimates in large times. It first goes along the same lines as for the case with full compact support.

Lemma 10 *The function M_ϵ satisfies*

$$\frac{d\,M_\epsilon}{dt} \geq -\frac{C}{t^{a+d}}\,M_\epsilon - C\,M_\epsilon(t) \sup_{|x|\geq t\epsilon/2} (1+|x|)^{\max(a,0)}\,\rho(t,x)$$

for $t > 2R/\epsilon$.

Proof. Let us start with the integral II in (11). Consider first the case $a \geq 0$. As f^0 is compactly supported in x, whenever $f > 0$ then $v \in B(x/t, R/t)$ and the following chain of estimates

$$\int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \beta(v)\,\rho(t,x)\,f(t,x,v)\,dx\,dv \leq \int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} |v|^a\,\rho(t,x)\,f(t,x,v)\,dx\,dv$$
$$\leq \frac{C}{t^a} \int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} (R+|x|)^a\,\rho(t,x)\,f(t,x,v)\,dx\,dv$$

holds. Since $|v| \geq \epsilon$ any particle with such speed issuing form a point x_0 occupies at time t a position x that verifies $|x - x_0| \geq \epsilon t$. If $t > 2R/\epsilon$, then $|x| \geq \frac{\epsilon t}{2}$. Thus, we have

$$\int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \beta(v)\,\rho(t,x)\,f(t,x,v)\,dx\,dv \leq C\,M_\epsilon(t) \sup_{|x|\geq t\epsilon/2} (1+|x|)^a\,\rho(t,x).$$

In the case $a < 0$, one simply has

$$\int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \beta(v)\,\rho(t,x)\,f(t,x,v)\,dx\,dv$$
$$\leq \int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} |v|^a\,\rho(t,x)\,f(t,x,v)\,dx\,dv$$
$$\leq \epsilon^a \int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \rho(t,x)\,f(t,x,v)\,dx\,dv$$
$$\leq \epsilon^a\,M_\epsilon(t) \sup_{|x|\geq t\epsilon/2} \rho(t,x).$$

Combining both, one gets in every case

$$\int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \beta(v)\,\rho(t,x)\,f(t,x,v)\,dx\,dv$$
$$\leq C\,M_\epsilon(t) \sup_{|x|>t\epsilon/2} (1+|x|)^{\max(a,0)}\,\rho(t,x).$$

The control of I follows the line of the case with compact support in velocity. Again if $f(t,x,v)>0$ and $f(t,x,v')>0$ then $|x-vt|\le R$ and $|x-v't|\le R$ so that $|v-v'|\le 2R/t$ and

$$I \le C\int_{\mathbb{R}^d_x}\int_{|v|>\epsilon} f(t,x,v)\int_{v'\in B(v,2R/t)} |v-v'|^a\,dv'\,dv\,dx \le \frac{C}{t^{d+a}}\,M_\epsilon.$$

□

So now we have to exhibit some decay for the x moment of ρ that appears in the lemma. We start with a technical result that will prove useful in the sequel.

Lemma 11 *The estimate*

$$\int_{\mathbb{R}^d_v} |v|^a f(t,x,v)\ dv \le \frac{C}{t^d}.$$

is verified for a.e. $|x|\ge \epsilon\, t/2$ *and for* $t>4R/\epsilon$.

Proof. Note that since f^0 is compactly supported in x, in case that $f(t,x,v)>0$ then $|x-vt|\le R$. So that $v\in B(x/t,R/t)$ holds under these circumstances. In particular, if $|x|\ge \epsilon\, t/2$ then $|v|>\epsilon/4$ for t large enough ($t>4R/\epsilon$ indeed).

That means that in the case $a<0$ we will have $|v|>\epsilon/2+|v|/2$ and so $|v|^a<(\epsilon/2+|v|/2)^a$. Obviously if $a>0$ then $|v|^a\le(\epsilon+|v|)^a$. Consequently,

$$\begin{aligned}\int_{\mathbb{R}^d_v} |v|^a f(t,x,v)\ dv &\le \int_{B(x/t,R/t)} |v|^a f^0(x-vt,v)\ dv\\ &\le\ C\ \sup_{v\in\mathbb{R}^d}(\epsilon+|v|)^a\, f^0(x-vt,v)\ \int_{B(x/t,R/t)} dv \le \frac{C}{t^d}.\end{aligned}$$

□

This holds also when $a>0$, as $|v|^a\le(\epsilon+|v|)^a$ in this case, being the supremum finite thanks to (14).

Next we bound ρ in terms of the quadratic terms of the equation.

Lemma 12 *The following inequality*

$$\rho(t,x) \le C\int_0^t\int_{\mathbb{R}^d_{v'}\times\mathbb{R}^d_v} \alpha(v,v')f(\tau,x,v)f(\tau,x,v')\ dvdv' \tag{15}$$

holds for any $|x|>\epsilon\, t/2$ *and* $t>\frac{2R}{\epsilon}$.

Proof. If $|x| > \epsilon\, t/2$ then we know that x is not in the initial support of ρ^0 for $t > \frac{2R}{\epsilon}$ and the integration of (2) gives

$$\rho(t,x) = \int_0^t \int_{\mathbb{R}^d_{v'} \times \mathbb{R}^d_v} \alpha(v,v') f(\tau,x,v) f(\tau,x,v')\ dvdv' \\ \times \exp\left\{ \int_\tau^t \int_{\mathbb{R}^d_v} \beta(v) f(s,x,v)\ dvds \right\}\ d\tau.$$

A direct application of Lemma 11 yields the estimate

$$\int_{\mathbb{R}^d_v} \beta(v) f(s,x,v)\ dv \le \int_{\mathbb{R}^d_v} |v|^a\, f(s,x,v)\ dv \le \frac{C}{s^d}.$$

So finally we have

$$\int_\tau^t \int_{\mathbb{R}^d_v} \beta(v) f(s,x,v)\ dvds \le C$$

for $\tau \ge 1$. To control the integration between 0 and τ, one simply uses that the integral $\int_{\mathbb{R}^d_v} |v|^a f(s)\, dv$ is bounded for any value of s. This is due to (14) and the fact that $a > -d$. Therefore the lemma is proved. $\square$

Lemma 13 *The estimate*

$$\int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v')\, dvdv' \le \frac{C(R)}{|x|^k t^{a+2d-k}} \\ \times \left\| (1+|v|)^{k/2} f^0(x-vt,v) \right\|^2_{L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)}$$

holds true for any $k \ge 0$.

Proof. Use the bound $f \le f^0(x-vt,v)$ and the compact spatial support of f^0 to get $|x| \le |x-vt| + |v|\, t \le R + |v|\, t$ and hence

$$|x|^k \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v')\, dvdv' \\ \le C \int_{|v-v'| \le C/t} |v-v'|^a\, (R+|v|\, t)^{k/2}\, f(t,x,v)(R+|v'|\, t)^{k/2}\, f(t,x,v')\, dvdv'.$$

This is in turn dominated by

$$
\begin{aligned}
&C\,t^k \int_{|v-v'|\le C/t} |v-v'|^a\,(1+|v|^{k/2})\,f(t,x,v)(1+|v'|^{k/2})\,f(t,x,v')\,dvdv' \\
\le& C\,t^k\,\|(1+|v|^{k/2})\,f(t)\|_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \int_{\mathbb{R}^d_v} dv(1+|v|^{k/2})\,f(t,x,v) \\
&\times \int_{|v-v'|\le \frac{C}{t}} |v-v'|^a\,dv'.
\end{aligned}
$$

Using Lemma 11 we finally arrive to

$$
\begin{aligned}
|x|^k &\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v)f(t,x,v')\,dvdv' \\
&\le \frac{Ct^k}{t^{a+d}}\,\|(1+|v|^{k/2})\,f(t)\|_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \int_{\mathbb{R}^d_v}(1+|v|^{k/2})\,f(t,x,v)\,dv \\
&\le \frac{C}{t^{a+2d-k}}\,\|(1+|v|^{k/2})\,f(t)\|^2_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)}
\end{aligned}
$$

□

Combining Lemmas 12 and 13, we find that

$$
\rho(t,x) \le C\int_0^1\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} |v-v'|^a f(\tau,x,v)\,f(\tau,x,v')\,dv\,dv' + \int_1^t \frac{C\,d\tau}{|x|^k\,\tau^{a+2d-k}}
$$

holds for $|x| > \epsilon\,t/2$ and t large enough. The first term is easy to bound: we use again Lemma 13, with the choice $k = 2d+a$. By means of (14) and taking into account that $a > 1-d$, there exists some $\delta > 0$ in order to have

$$
\begin{aligned}
&\int_0^1\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} |v-v'|^a f(\tau,x,v)\,f(\tau,x,v')\,dv\,dv' \\
&\qquad \le \frac{C}{|x|^{2d+a}}\,\|(1+|v|)^{d+a/2}\,f^0\|_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \le \frac{C}{|x|^{d-1+\delta}}.
\end{aligned}
$$

When computing the supremum over $|x| > \epsilon\,t/2$ we will have the inequality

$$
(1+|x|)^{\max(a,0)}\rho(t,x) \le \frac{C}{|x|^{k-\max(a,0)}} + \frac{C}{|x|^{d-1}} \le \frac{C}{|t|^{k-\max(a,0)}} + \frac{C}{|t|^{d-1+\delta}}
$$

as long as $a+2d-k > 1$, for some $\delta > 0$. Inserting this in Lemma 10 gives

$$
\frac{d\,M_\epsilon}{dt} \ge -\frac{C}{t^{a+d}}\,M_\epsilon - \frac{C}{t^{d-1+\delta}+t^{k-\max(a,0)}}\,M_\epsilon, \tag{16}
$$

which, as before, shows that M_ϵ does not vanish for large times. Indeed the only constraint on $k-a$ is $k-a<2d-1$ and one may therefore always have $k-a>1$. Since we also have $d-1+\delta>1$, the second coefficient is always integrable in time. This is enough to conclude, as Lemma 9 assures that M_ϵ does not vanish in finite time.

3.2.1 The case $d=1$

The proof in the previous section can cover also the case $d=1$ when some minor changes are introduced, which we indicate here briefly. We can prove the following result:

Proposition 4 *Assume that $a>1$ and (14) for $d=1$, together with the compact support hypotheses (7) and (9). Then, for any weak solution of (1)–(2) constructed in Theorem 2, there exists a constant $C=C(f^0,\rho^0)>0$ such that*

$$M(t)\geq C \quad \forall t\geq 0.$$

We describe below a brief sketch of the modifications required for the proof given in the previous section to work in the present context.

The proof of Lemma 10 can be modified to give, in this case $d=1$, the following result.

Lemma 14 *The function M_ϵ satisfies*

$$\frac{d\,M_\epsilon}{dt}\geq -\frac{C}{t^{a+1}}\,M_\epsilon - C\,M_\epsilon(t)\,t^{-a}\sup_{|x|\geq\epsilon/2\,t}(R+|x|)^a\;\rho(t,x),$$

for $t>2R/\epsilon$.

We come back to (12), which we write as

$$\rho(t,x)\leq C\,t\int_0^1\int_{\mathbb{R}_{v'}\times\mathbb{R}_v}\alpha(v,v')f(\tau,x,v)f(\tau,x,v')\;dvdv'd\tau$$

$$+C\,t\int_1^t\int_{\mathbb{R}_{v'}\times\mathbb{R}_v}\alpha(v,v')f(\tau,x,v)f(\tau,x,v')\;dvdv'd\tau$$

for $|x|>\epsilon\,t/2$. When inserted into the inequality of Lemma 14, we find that we have to compute the supremum of the following quantity:

$$Ct\left(\frac{|x|}{t}\right)^a\int_0^1\int_{\mathbb{R}_{v'}\times\mathbb{R}_v}\alpha(v,v')f(\tau,x,v)f(\tau,x,v')\;dvdv'd\tau$$

$$+Ct\left(\frac{|x|}{t}\right)^a \int_1^t \int_{\mathbb{R}_{v'}\times\mathbb{R}_v} \alpha(v,v')f(\tau,x,v)f(\tau,x,v')\; dvdv'd\tau.$$

That is, we are dealing with

$$C\left(\frac{|x|}{t}\right)^a t|x|^{-k_0}\int_0^1 \frac{d\tau}{\tau^{a+2-k_0}} + C\left(\frac{|x|}{t}\right)^a t|x|^{-k_\infty}\int_1^t \frac{d\tau}{\tau^{a+2-k_\infty}}$$

where Lemma 13 was applied twice. To conclude, we need $k_\infty < a+1$ and $k_0 > a+1$ to assure integrability while $k_\infty, k_0 > \max(a,2)$ for compensating the factors in front and getting an overall decay better than t^{-1}. These conditions are compatible only if $a > 1$. The choices $k_\infty = \max(a,2)+\delta$, $k_0 = a+1+\delta$ for $\delta > 0$ suitably close to zero conclude with the proof.

3.3 The vanishing case

In this section we study what happens in the complementary regime $-a \in [d-1, d[$. The main result is

Proposition 5 *Assume $-a \in [d-1,d[$ and that there exists some $k > 0$ such that*

$$\int_{\mathbb{R}_x^d}\int_{\mathbb{R}_v^d}(1+|x|^k+|v|^k)f^0(x,v)\; dxdv < +\infty.$$

Then, for each weak solution of (1)–(2) given by Theorem 2, we have that

$$\lim_{t\to\infty} M(t) = 0.$$

Remark 2 *This result shows that there occurs a total transfer of mass from f to ρ. Precise rates of convergence for $M(t)$ are given in the proof for the case of smooth solutions. But note that to assure that $M(t)$ does not vanish in finite time some extra decay assumptions are needed; for instance* (4) *would do — use the proof of Lemma 9—.*

Proof. Suppose first that f^0 satisfies (7) and (8). Then this implies that the support of $\rho_f(t)$ lies within $B(R+Vt)$. Using Jensen's inequality we get

$$\int_{\mathbb{R}_x^d}\rho_f(t,x)^2\, dx \geq \frac{\left(\displaystyle\int_{\mathbb{R}_x^d}\rho_f(t,x)\, dx\right)^2}{|\mathbb{S}^{d-1}|(R+Vt)^d}.$$

Recalling the differential inequality (11) we get

$$\begin{aligned}\frac{dM(t)}{dt} &\le -\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |v-v'|^{-|a|} f(t,x,v)f(t,x,v')\,dxdvdv' \\ &\le -\frac{t^{|a|}}{(\operatorname{diam}\operatorname{supp}\,\rho_f^0)^{|a|}}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} f(t,x,v')f(t,x,v)\,dxdvdv' \\ &= -Ct^{|a|}\int_{\mathbb{R}^d_x}\rho_f(x)^2\,dx,\end{aligned}$$

where we have used Lemma 4. Combining with the previous, we obtain the estimate

$$\frac{dM(t)}{dt} \le -C\frac{t^{|a|}M(t)^2}{(R+Vt)^d}.$$

This implies logarithmic decay of $M(t)$ in the case $d = 1+|a|$, and a power decay at the rate $t^{d-1-|a|}$ if $d < 1+|a| < d+1$. In both cases the mass finally vanishes.

For the general case, we introduce a parameter V and perform the following decomposition of the initial datum:

$$f^0 = g_V^0 + f_V^0 = g_V^0 + f^0(x,v)\chi_{\{|x|\le V\}}\chi_{\{|v|\le V\}}.$$

The evolution of the solution is decomposed accordingly:

$$f(t) = g_V(t) + f_V(t) = g_V(t) + f(t,x,v)\chi_{\{|x|\le V\}}\chi_{\{|v|\le V\}}.$$

Note that

$$\begin{aligned}\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} g_V^0\,dxdv &= \int_{\mathbb{R}^d_x\times\mathbb{R}^d_v}\frac{|x|^k+|v|^k}{|x|^k+|v|^k} g_V^0\,dxdv \\ &\le \frac{1}{V^k}\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v}(|x|^k+|v|^k)f^0\,dxdv \le \frac{C}{V^k}.\end{aligned}$$

The mass associated to $g_V(t)$ does not increase: This follows from the fact that the function $g_V(t)$ satisfies the equation

$$\partial_t g_V + v\cdot\nabla_x g_V = -g_V(t,x,v)\int_{\mathbb{R}^d_{v'}}\alpha(v,v')f(t,x,v')\,dv' - \beta(v)\rho(t,x)g_V(t,x,v).$$

This implies that

$$\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} g_V(t,x,v)\,dxdv \le \frac{C}{V^k}.$$

On the other hand, we can repeat with the function $f_V(t)$ what we did before, obtaining

$$M_{f_V}(t) \le \frac{C}{\frac{1}{M(f_V^0)} + \frac{\int_0^t \frac{\tau^{|a|}}{(1+\tau)^d}\, d\tau}{V^{|a|+d}}}.$$

To conclude we optimize in V. In case that $d < |a| + 1$ the mass decays like t to the power of $\frac{k(d-|a|-1)}{d+|a|+k}$. In the borderline case $d = |a| + 1$ the mass decays as $\log t$ to the power of $\frac{-k}{|a|+d+k}$. We recover the rates of the compactly supported case if we can allow infinite moments for the initial datum. $\square$

4 Self-similar solutions

The aim of this section is to show that the solution f can be approximated by a function of the self-similar variables $y = \frac{x}{t}$, $w = t(v - \frac{x}{t})$ for large times. More precisely,

Proposition 6 *Assume that* $a + d > 0$, $f^0 \in (L^1 \cap L^\infty)(\mathbb{R}_x^d \times \mathbb{R}_v^d)$ *and that the compact support hypothesis* (7) *is verified. If* $a > 0$ *assume also that*

$$f^0(x, v) \le \frac{C}{1 + |v|^a} \quad a.e.\ (x, v) \in \mathbb{R}_x^d \times \mathbb{R}_v^d. \tag{17}$$

Then, for each weak solution of (1)–(2), given by Theorem 2, there exists a function $g_\infty(x, v)$ *such that*

$$\|f(t, x, v) - g_\infty(x/t, t(v - x/t))\| \to 0, \quad as\ t \to \infty,$$

in the norm of $W^{-1,1}(\mathbb{R}_x^d, L^1(\mathbb{R}_v^d))$.

The rest of the section outlines a proof for this statement. To begin with, let us introduce the function g defined by $f(t, x, v) = g(t, \frac{x}{t}, t(v - \frac{x}{t}))$, or, in an equivalent way, $g(t, y, w) = f(t, ty, y + \frac{w}{t})$. We have to prove that g has a limit as $t \to +\infty$. The function g satisfies the following equation:

$$\begin{aligned} \frac{\partial g}{\partial t} + \frac{w}{t^2} \nabla_y g = &- g(t, y, w) \int_{\mathbb{R}_{w'}^d} \alpha\left(\frac{w}{t}, \frac{w'}{t}\right) \frac{g(t, y, w')}{t^d}\, dw' \\ &- \frac{1}{t^2} \bar{\rho}(t, y) g(t, y, w) \beta\left(y + \frac{w}{t}\right), \end{aligned} \tag{18}$$

with $\rho(t, x) = \frac{1}{t^2} \bar{\rho}(t, \frac{x}{t})$.

Now we can prove that $g(t)$ is a Cauchy sequence in the space $W_{x,v} := W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))$, that is, for the norm

$$\|g(t)\|_{W_{x,v}} = \sup_{\Delta} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} \varphi(x,v)\, g(t,x,v)\, dx\, dv,$$

with $\Delta = \{\varphi \in \mathcal{D}(\mathbb{R}^d_x \times \mathbb{R}^d_v) / |\varphi| \leq 1,\ |\nabla_x \varphi| \leq 1\}$.

Lemma 15 *For $0 < s < t$, the following estimate holds*

$$\|g(t) - g(s)\|_{W_{y,w}} \leq |s-t| \left[\frac{1}{ts} + \frac{1}{t^{a+d}} + \frac{1}{t^{a^*}}\right] C(R, d, a, f^0, M),$$

being $a^* = \min\{a+d, d\}$.

Proof. We compute

$$\begin{aligned}
&\int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} \varphi\, (g(t,x,v) - g(s,x,v))\, dxdv \\
&\qquad = \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi\, (g(t,y,w) - g(s,y,w))\, dydw \\
&\qquad = \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w) \left(g(t,y,w) - g(t, y + \frac{w}{s} - \frac{w}{t}, w)\right) dydw \\
&\qquad + \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w) \left(g(t, y + \frac{w}{s} - \frac{w}{t}, w) - g(s,y,w)\right) dydw = I + II
\end{aligned}$$

The first term is handled as follows:

$$\begin{aligned}
I &= \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} g(t,y,w) \left(\varphi(y,w) - \varphi(y - \frac{w}{s} + \frac{w}{t}, w)\right) dydw \\
&\leq \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \int g(t,y,w) |w| \frac{|s-t|}{|ts|}\, dydw \\
&= \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \frac{|s-t|}{|ts|} \int |w| f(t, ty, y + \frac{w}{t})\, dydw.
\end{aligned}$$

So, thanks to (7) we get

$$\begin{aligned}
I &\leq \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \frac{|s-t|}{ts} \int |tw - y| f(t,y,w)\, dydw \\
&\leq \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \frac{|s-t|}{ts} \int_{\mathbb{R}^d_r \times \mathbb{R}^d_w} |r| f^0(r,w)\, drdw \\
&\leq CM \frac{|s-t|}{ts}.
\end{aligned}$$

To deal with the second term, we introduce the function $\phi(\tau) = g(\tau, y + \frac{w}{s} - \frac{w}{\tau}, w)$. Notice that $\phi(t) = g(t, y + \frac{w}{s} - \frac{w}{t}, w)$ and $\phi(s) = g(s, y, w)$. Evaluating (18) at points of the form $(t, y + \frac{w}{s} - \frac{w}{t}, w)$, multiplying by $\varphi(y, w)$ and integrating we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w)\phi(t)\, dydw \\
&\quad = -\int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w)\phi(t) \int_{\mathbb{R}^d_{w'}} \frac{\alpha(w/t, w'/t)}{t^d} g(t, y + \frac{w}{s} - \frac{w}{t}, w')\, dydwdw' \\
&\quad\quad + \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w) \frac{1}{t^2}\phi(t)\overline{\rho}(t, y + \frac{w}{s} - \frac{w}{t})\beta(y + w/s)\, dydw = A + B.
\end{aligned}$$

Then we write

$$\begin{aligned}
II &= \left| \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w)\phi(t)\, dydw - \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w)\phi(s)\, dydw \right| \\
&\leq |t - s| \sup_{\theta \in [s,t]} \left| \left[\frac{\partial}{\partial t} \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w)\phi(t)\, dydw \right]_{t=\theta} \right|.
\end{aligned}$$

Thus if we bound $|A|$ and $|B|$ we are done. Recalling that $g(t, y, w) = f(t, ty, y + \frac{w}{t})$, we have

$$\begin{aligned}
|A| \leq & C\|\varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w \times \mathbb{R}'^d_w} dydwdw' g(t, y + \frac{w}{s} - \frac{w}{t}, w) \\
& \times g(t, y + \frac{w}{s} - \frac{w}{t}, w') \frac{|w - w'|^a}{t^{d+a}} \\
= & \|\varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w \times \mathbb{R}'^d_w} dydwdw' f(t, ty + \frac{tw}{s} - w, y + \frac{w}{s}) \\
& \times f(t, ty + \frac{tw}{s} - w, y + \frac{w'}{s}) \frac{|w - w'|^a}{t^{d+a}}.
\end{aligned}$$

To continue we change variables inside the integral by means of $r = ty + \frac{tw}{s} - w$, $z = y + \frac{w}{s}$ and $z' = y + \frac{w'}{s}$. In particular, $|w - w'| = s|z - z'|$. In the case $d = 1$, the Jacobian matrix of the mapping $(y, w, w') \mapsto (ty + \frac{tw}{s} - w, y + \frac{w}{s}, y + \frac{w'}{s})$ is

$$\begin{pmatrix} t & t/s - 1 & 0 \\ 1 & 1/s & 0 \\ 1 & 0 & 1/s \end{pmatrix}$$

In the general case each entry corresponds now to a diagonal block of size d and all the elements equal to the correponding one-dimensional entry. Then the inverse Jacobian reduces to s^d; to see this, transform the matrix in order to have the second and third blocks of the first column equal to zero. Thanks to (7) we can use Lemma 4 and performing along the same lines of (12) we deduce

$$\begin{aligned}
|A| &\le \|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \int_{\mathbb{R}^d_r\times\mathbb{R}^d_z\times\mathbb{R}^d_{z'}} f(t,r,z)f(t,r,z')\frac{1}{t^{d+a}}s^d|s(z-z')|^a\,drdzdz' \\
&\le \|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \int_{\mathbb{R}^d_r\times\mathbb{R}^d_z\times\mathbb{R}^d_{z'}} f(t,r,z)f(t,r,z')|z-z'|^a\,drdzdz' \\
&\le \frac{C(R,d,a,f^0)}{t^{a+d}} \int_{\mathbb{R}^d_r\times\mathbb{R}^d_z} f(t,r,z)\;drdz \le \frac{C(R,d,a,f^0)M}{t^{a+d}}.
\end{aligned}$$

On the other hand,

$$B = \int_{\mathbb{R}^d_y\times\mathbb{R}^d_w} \varphi(y,w)f(t,ty+\frac{tw}{s}-w,y+\frac{w}{s})\rho(t,ty+\frac{tw}{s}-w)\beta\left(y+\frac{w}{s}\right)\,dydw.$$

We change to the new variables $z=y+\frac{w}{s}$, $r=ty+\frac{tw}{s}-w$ inside the integral. The Jacobian of the mapping $(y,w)\mapsto(y+\frac{w}{s},ty+\frac{tw}{s}-w)$ is 1 (transform to have a zero block in the left lower corner). Then,

$$\begin{aligned}
|B| &\le C\|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \int_{\mathbb{R}^d_r\times\mathbb{R}^d_z} f(t,r,z)|z|^a\rho(t,r)\,drdz \\
&\le \|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \sup_r \left(\int_{\mathbb{R}^d_z} f(t,r,z)|z|^a\;dz\right)\int_{\mathbb{R}^d_r}\rho(t,r)\;dr.
\end{aligned}$$

If $a<0$ we estimate like in (13) to get

$$|B| \le \frac{CM}{t^{a^*}}.$$

In the case $a\ge 0$ we claim that

$$\int_{\mathbb{R}^d_z} f(t,r,z)|z|^a\;dz \le \frac{C}{t^a}$$

uniformly in r and then we get exactly the same type of estimate. □

The previous claim relies on the following technical result, which covers a slightly more general situation than needed: We assume compact support in x and the decay condition (17).

Lemma 16 *Whenever the condition*

$$f^0(x,v) \le \frac{C}{1+|x|^{d+\epsilon}+|v|^k} \quad a.e.\ (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v$$

is fulfilled for some $\epsilon > 0$ *and some* $k > 0$ *the following estimate is verified:*

$$\int_{\mathbb{R}^d_v} |v|^k f(t,x,v)\ dv \le \frac{C}{t^d}.$$

Proof. Just follow the chain of inequalities:

$$\begin{aligned}
&\int_{\mathbb{R}^d_v} |v|^k f(t,x,v)\ dv \le \int_{\mathbb{R}^d_v} |v|^k f^0(x-vt,v)\ dv \\
&\le \int_{\mathbb{R}^d_v} \sup_{\xi\in\mathbb{R}^d} |\xi|^k f^0(x-vt,\xi)\ dv \le \frac{1}{t^d}\int_{\mathbb{R}^d_x} \sup_{\xi\in\mathbb{R}^d} |\xi|^k f^0(x,\xi)\ dx \\
&\le \frac{1}{t^d}\int_{\mathbb{R}^d_x} \sup_{\xi\in\mathbb{R}^d} \frac{C|\xi|^k}{1+|\xi|^k+|x|^{d+\epsilon}}\ dx \le \frac{1}{t^d}\int_{\mathbb{R}^d_x} \frac{C\ dx}{1+|x|^{d+\epsilon}}.
\end{aligned}$$

□

As a consequence there exists a function $g(\infty, y, w)$ such that $g(t) \to g(\infty)$ in the norm of $W^{-1,1}(\mathbb{R}^d_y, L^1(\mathbb{R}^d_w))$ and that $\|g(\infty) - g(t)\|_{W_{y,w}} \le \frac{C}{t}$. But note that, being 1 the jacobian of the mapping $(x,v) \mapsto (x/t, t(v - x/t))$, we have that

$$\|g(\infty,y,w) - g(t,y,w)\|_{W_{y,w}} = \|f(t,x,v) - g(\infty, x/t, t(v-x/t))\|_{W_{x,v}}.$$

Thus we can get from the self-similar variables to the original ones. We have proved that

$$\|f(t,x,v) - g(\infty, x/t, t(v-x/t))\|_{W(x,v)} \to 0 \ \text{ as } t \to \infty.$$

Acknowledgements. J.C. has been partially supported by by Ministerio de Ciencia e Innovación (Spain) and FEDER, Project MTM2008-05271 and Junta de Andalucía Project P08-FQM-4267. The authors want to thank Juan Soler for pointing out useful references and comments. J.C. wants to thank Simone Calogero for many useful comments.

References

[1] M. Aizerman, T. A. Bak. Convergence to equilibrium in a system of reacting polymers. Comm. Math. Phys. 65: 203–320, 1979.

[2] D.J. Aldous. Deterministic and stochastic models for coalescence (aggregation, coagulation): a review of the mean–field theory for probabilists. Bernouilli 5: 3–48, 1999.

[3] A.A. Amsden, P.J. O'Rourke. The T.A.B. method for numerical calculation of spray droplet breakup. Los Alamos National Laboratory, Los Alamos, New Mexico 87545.

[4] C. Baranger. Modeling of oscillations, breakup and collisions for droplets: the establishment of kernels for the T.A.B. model. Math. Models Methods Appl. Sci. 14 no. 5: 775–794, 2004.

[5] P. Benilan, H. Brezis, M.G. Crandall. A semilinear equation in $L^1(\mathbb{R}^N)$. Ann. Scuola Norm. Sup. di Pisa 4: 523–555, 1975.

[6] F. Bouchut and F. James, Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness. Comm. Partial Differential Equations, 24 (1999), 2173–2189.

[7] N. Bournaveas, V. Calvez, S. Gutiérrez, B. Perthame, Global Existence for a Kinetic Model of Chemotaxis via Dispersion and Strichartz Estimates, Communications in Partial Differential Equations, Volume 33, 2008 , pages 79 - 95

[8] Calogero S., Calvo J., Sánchez O., Soler J., Dispersive behavior in Galactic Dynamics, to appear in Discrete and Continuous Dynamical Systems-Series B.

[9] J. Cañizo, S. Mischler. Uniqueness and regularity of scaling profiles for Smoluchowski's coagulation equation, preprint.

[10] C. Cercignani, R. Illner, M. Pulvirenti. The Mathematical Theory of Dilute Gases. Applied Mathemetical Sciences, 106, Springer-Verlag.

[11] J. Dolbeault, O. Sanchez and J. Soler, Asymptotic behavior for the Vlasov-Poisson system in the stellar dynamics case, Arch. Rat. Mech. Anal., 171 (2004), 301–327.

[12] R.L. Drake. A general mathematical survey of the coagulation equation. In G. M. Hidy and J. R. Brock, editors, Topics in Current Aerosol Research (Part 2), volume 3 of International Reviews in Aerosol Physics and Chemistry, 201376. Pergamon, 1972.

[13] R. DiPerna, P.L. Lions, Y. Meyers. L^p regularity of velocity averages. Ann. Inst. H. Poincaré Anal. Non Linéaire 8: 271–287, 1991.

[14] G. Dufour, M. Massot, Ph. Villedieu. Etude d'un modèle de fragmentation secondaire pour les brouillards de gouttelettes. C. R. Math. Acad. Sci. Paris 336(5): 447–452, 2003.

[15] O.R. Etingin, R.L. Silverstein, H.M. Friedman, D.P. Hajjar. Viral activation of the coagulation cascade: molecular interactions at the surface of infected endothelial cells. Cell. 18; 61(4): 657–62, 1990.

[16] M. Escobedo, S. Mischler. Dust and self-similarity for the Smoluchowski coagulation equation. Annales de l 'IHP. 23: 331–362, 2006.

[17] U. Gasser, E. Weeks, A. Schoeld, P.N. Pursey, D.A. Weitz. Real-space imaging of nucleation and growth in colloidal crystallization. Science 292 (258), 2001.

[18] P.E. Jabin, J. Soler. A kinetic description of particle fragmentation. Math. Mod. Meth. Appl. Sci. 16, 6: 933–948, 2006.

[19] P.E. Jabin, J. Soler. A coupled Boltzmann & Navier–Stokes fragmentation model induced by a fluid-particle-spring interaction. Submited.

[20] J.C. Lasheras, C. Eastwood, C. Martínez-Bazán, J.L. Montañés. A review of statistical models for the break-up of an immiscible uid immersed into a fully developed turbulent ow. Int. J. Multiphase Flow 28: 247278, 2002.

[21] P. Laurencot, S. Mischler. From the discrete to the continuous coagulation-fragmentation equations. Proc. Roy. Soc. Edinburgh Sect. A 132: 1219–1248, 2002.

[22] P. Laurencot, S. Mischler. On coalescence equations and related models. In P. Degond, L. Pareschi, and G. Russo, editors, Topics in Current Aerosol Research (Part 2), Modelling and Simulation in Science, Engineering and Technology. Birkhauser, 2004.

[23] P. Laurencot, S. Mischler. The continuous coagulation–fragmentation equations with diffusion. Arch. Rational Mech. Anal. 162: 45–99, 2002.

[24] E.H. Lieb, M. Loss. Analysis. Graduate Studies in Mathematics, 14, American Mathematical Society, 2001.

[25] E.M. Lifshitz, L.P. Pitaevskii. Physical Kinetics. Pergamon Press, New York, 1981.

[26] B. Perthame, Time decay, propagation of low moments and dispersive effects for kinetic equations, Commun. Part. Diff. Eqns., **21** (1996), 659–686.

[27] G. Russo, P. Smereka. Kinetic theory for bubbly flow I and II. SIAM J. Appl. Math. 56: 327–371, 1996.

[28] E.M. Stein. Singular Integrals and Differentiability Properties of Functions. Princeton University Press, 30, New Jersey, 1970.

[29] D. Stern, P. Nawroth, D. Handley, W. Kisiel. An endothelial cell-dependent pathway of coagulation. PNAS 82 (8): 2523–2527, 1985.

[30] F.A. Williams. Combustion Theory. Benjamin/Cummings, Menlo Park, CA, 1985.

[31] Gershon Wolansky, Dynamics of a system of sticking particles of finite size on the line, Nonlinearity 20 (2007) 21752189.