Well-posedness in any dimension for Hamiltonian flows with non BV force terms

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Abstract

We study existence and uniqueness for the classical dynamics of a particle in a force field in the phase space. Through an explicit control on the regularity of the trajectories, we show that this is well posed if the force belongs to the Sobolev space $H^{3/4}$.

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1 Introduction

This paper studies existence and uniqueness of a flow for the equation

$$\begin{cases} \partial_t X(t,x,v) = V(t,x,v), & X(0,x,v) = x, \\ \partial_t V(t,x,v) = F(X(t,x,v)), & V(0,x,v) = v, \end{cases} \tag{1.1}$$

where x and v are in the whole $\mathbb{R}^d$ and F is a given function from $\mathbb{R}^d$ to $\mathbb{R}^d$. Those are of course Newton's equations for a particle moving in a force field F. For many applications the force field is in fact a potential

$$F(x) = -\nabla\phi(x), \tag{1.2}$$

even though we will not use the additional Hamiltonian structure that this is providing.

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This is a particular case of a system of differential equations

$$\partial_t \Xi(t,\xi) = \Phi(\Xi), \tag{1.3}$$

with $\Xi = (X, V)$, $\xi = (x, v)$, $\Phi(\xi) = (v, F(x))$. Cauchy-Lipschitz' Theorem applies to (1.1) and gives maximal solutions if F is Lipschitz. Those solutions are in particular global in time if for instance $F \in L^\infty$. Moreover because of the particular structure of Eq. (1.1), this solution has the additional

Property 1 *For any $t \in \mathbb{R}$ the application*

$$(x,v) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto (X(t,x,v), V(t,x,v)) \in \mathbb{R}^d \times \mathbb{R}^d \tag{1.4}$$

is globally invertible and has Jacobian 1 at any $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. It also defines a semi-group

$$\begin{aligned} &\forall s,t \in \mathbb{R}, \qquad && X(t+s,x,v) = X(s, X(t,x,v), V(t,x,v)),\\ &\textit{and} && V(t+s,x,v) = V(s, X(t,x,v), V(t,x,v)). \end{aligned} \tag{1.5}$$

In many cases this Lipschitz regularity is too demanding and one would like to have a well posedness theory with a less stringent assumption on F. That is the aim of this paper. More precisely, we prove

Theorem 1.1 *Assume that $F \in H^{3/4} \cap L^\infty$. Then, there exists a solution to (1.1), satisfying Property 1. Moreover this solution is unique among all limits of solutions to any regularization of* (1.1)*.*

Many works have already studied the well posedness of Eq. (1.3) under weak conditions for Φ. The first one was essentially due to DiPerna and Lions [19], using the connection between (1.3) and the transport equation

$$\partial_t u + \Phi(\xi) \cdot \nabla_\xi u = 0. \tag{1.6}$$

The notion of renormalized solutions for Eq. (1.6) provided a well posedness theory for (1.3) under the conditions $\Phi \in W^{1,1}$ and $\mathrm{div}_\xi \Phi \in L^\infty$. This theory was generalized in [28], [27] and [24].

Using a slightly different notion of renormalization, Ambrosio [2] obtained well posedness with only $\Phi \in BV$ and $\mathrm{div}_\xi \Phi \in L^\infty$ (see also the papers by Colombini and Lerner [12], [13] for the BV case). The bounded divergence condition was then slightly relaxed by Ambrosio, De Lellis and Malỳ in [4] with then only $\Phi \in SBV$ (see also [17]).

Of course there is certainly a limit to how weakly Φ may be and still provide uniqueness, as shown by the counterexamples of Aizenman [1] and Bressan [10]. The example by De Pauw [18] even suggests that for the general setting (1.3), BV is probably close to optimal.

But as (1.1) is a very special case of (1.3), it should be easier to deal with. And for instance Bouchut [6] got existence and uniqueness to (1.1) with $F \in BV$ in a simpler way than [2]. Hauray [23] handled a slightly less than BV case (BV_{loc}).

In dimension $d = 1$ of physical space (dimension 2 in phase space), Bouchut and Desvillettes proved well posedness for Hamiltonian systems (thus including (1.1) as F is always a derivative in dimension 1) without any additional derivative for F (only continuity). This was extended to Hamiltonian systems in dimension 2 in phase space with only L^p coefficients in [22] and even to any system (non necessarily Hamiltonian) with bounded divergence and continuous coefficient by Colombini, Crippa and Rauch [11] (see also [14] for low dimensional settings and [9] with a very different goal in mind).

Unfortunately in large dimensions (more than 1 of physical space or 2 in the phase space), the Hamiltonian or bounded divergence structure does not help so much. To our knowledge, Th. 1.1 is the first result to require less than 1 derivative on the force field F in any dimension. Note that the comparison between $H^{3/4}$ and BV is not clear as obviously $BV \not\subset H^{3/4}$ and $H^{3/4} \not\subset BV$. Even if one considers the stronger assumption that the force field be in $L^\infty \cap BV$, that space contains by interpolation H^s for $s < 1/2$ and not $H^{3/4}$. As the proof of Th. 1.1 uses orthogonality arguments, we do not know how to work in spaces non based on L^2 norms ($W^{3/4,1}$ for example). Therefore strictly speaking Th. 1.1 is neither stronger nor weaker than previous results.

We have no idea whether this $H^{3/4}$ is optimal or in which sense. It is striking because it already appears in a question concerning the related Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0. \tag{1.7}$$

Note that this is the transport equation corresponding to Eq. (1.1), just as Eq. (1.6) corresponds to (1.3). As a kinetic equation, it has some regularization property namely that the average

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, \psi(v)\, dv, \quad \text{with } \psi \in C_c^\infty(\mathbb{R}^d),$$

is more regular than f. And precisely if $f \in L^2$ and $F \in L^\infty$ then $\rho \in H^{3/4}$; we refer to Golse, Lions, Perthame and Sentis [21] for this result, DiPerna, Lions, Meyer for a more general one [20] or [26] for a survey of averaging lemmas. Of course we do not know how to use this kind of result for the uniqueness of (1.7) or even what is the connection between the $H^{3/4}$ of averaging lemmas and the one found here. It *could* just be a scaling property of those equations.

Note in addition that the method chosen for the proof may in fact be itself a limitation. Indeed it relies on an explicit control on the trajectories : for instance, we show that $|X(t,x,v)-X^\delta(t,x,v)|$ and $|V(t,x,v)-V^\delta(t,x,v)|$ remain approximately of order $|\delta|$ if

$$X^\delta(t,x,v) = X(t,x+\delta_1,v+\delta_2), \quad V^\delta(t,x,v) = V(t,x+\delta_1,v+\delta_2).$$

However the example given in Section 3 demonstrates that such a control in not always possible: Even in $1d$ it requires at least $1/2$ derivative on the force term ($F \in W^{1/2,1}_{loc}$) whereas well posedness is known with essentially $F \in L^p$ (see the references above).

This kind of control is obviously connected with regularity properties of the flow (differentiability for instance), which were studied in [5] (see also [3]). The idea to prove them directly and then use them for well posedness is quite recent, first by Crippa and De Lellis in [16] with the introduction and subsequent bound on the functional

$$\int_\Omega \sup_r \int_{|\delta|\leq r} \log\left(1+\frac{|\Xi(t,\xi)-\Xi(t,\xi+\delta)|}{|\delta|}\right) d\delta\, dx. \tag{1.8}$$

This gave existence/uniqueness for Eq. (1.3) with $\Phi \in W^{1,p}_{loc}$ for any $p > 1$ and a weaker version of the bounded divergence condition. This was extended in [7] and [25].

We use here a modified version of (1.8) which takes the different roles of x and v into account. The way of bounding it is also quite different as we essentially try to integrate the oscillations of F along a trajectory.

The paper is organized as follows: The next section introduces the functional that is studied, states the bounds that are to be proved and briefly explains the relation with the well posedness result Th. 1.1. The section after that presents the example in $1d$ and the last and longer section the proof of the bound.

Notation

- $u \cdot v$ denotes the usual scalar product of $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$.

- S^{d-1} denotes the $d-1$-dimensional unit sphere in $\mathbb{R}^d$.
- $B(x,r)$ is the closed ball of $\mathbb{R}^d$ for the standard Euclidean norm with center $x \in \mathbb{R}^d$ and radius $r \geq 0$.
- C denotes a positive constant that may change from line to line.

2 Preliminary results

2.1 Reduction of the problem

In the sequel, we give estimates on the flow to Eq. (1.1) for initial values (x,v) in a compact subset $\Omega = \Omega_1 \times \Omega_2$ of $\mathbb{R}^{2d}$ and for time $t \in [0,T]$. Fix some $A > 0$ and consider any $F \in L^\infty$ with $\|F\|_{L^\infty} \leq A$. Then for any solution to Eq. (1.1)

$$\begin{aligned} &|V(t,x,v) - v| \leq \|F\|_{L^\infty} t \leq A\,t \\ \text{and} \quad &|X(t,x,v) - x| \leq vt + \|F\|_{L^\infty} t^2/2 \leq vt + A\,t^2/2. \end{aligned}$$

Therefore, for any $t \in [0,T]$ and for any (x,v) at a distance smaller than 1 from Ω, $(X(t,x,v), V(t,x,v)) \in \Omega' = \Omega'_1 \times \Omega'_2$ for some compact subset Ω' of $\mathbb{R}^{2d}$. Moreover Ω' depends only on Ω and A. Similarly, we introduce Ω'' a compact subset of $\mathbb{R}^{2d}$ such that the couple $(X(-t,x,v), V(-t,x,v))$ belongs to Ω'' for any $t \in [0,T]$ and any (x,v) at a distance smaller than 1 from Ω'.

For $T > 0$, define the quantity

$$\begin{aligned} Q_\delta(T) = \iint_\Omega \log \Bigg(1 + \frac{1}{|\delta|^2} \Bigg(&\sup_{0 \leq t \leq T} |X(t,x,v) - X^\delta(t,x,v)|^2 \\ &+ \int_0^T |V(t,x,v) - V^\delta(t,x,v)|^2 \, dt \Bigg)\Bigg) \, dx\, dv, \end{aligned}$$

where X, V and X^δ, V^δ are two solutions to (1.1), satisfying

$$\begin{aligned} &X(0,x,v) = x, \qquad V(0,x,v) = v, \\ &|X(0,x,v) - X^\delta(0,x,v)| \leq |\delta|, \qquad |V(0,x,v) - V^\delta(0,x,v)| \leq |\delta|. \end{aligned} \tag{2.1}$$

We prove the following result

Proposition 2.1 *Fix $T > 0$, any $A > 0$ and $\Omega \in \mathbb{R}^{2d}$ compact. Define Ω' and Ω'' as in Section 2.1. There exists a constant $C > 0$ depending only of diam(Ω'), $|\Omega''|$, T and A, such that, for any $a \in (0, 1/4)$, $F \in H^{3/4+a}$ with $\|F\|_{L^\infty} \le A$ and any solutions (X, V) and (X^δ, V^δ) to* (1.1) *satisfying Property 1 and* (2.1)*, one has for any $|\delta| < 1/e$,*

$$Q_\delta(T) \le C\left(1 + \left(\log\frac{1}{|\delta|}\right)^{\max\{1-2a,1/2\}}\right)\left(1 + \|F\|_{H^{3/4+a}(\Omega'')}\right).$$

As will appear in the proof, this result can be actually extended without difficulty to any $F \in L^\infty$ such that

$$\int_{\mathbb{R}^d} |k|^{3/2}|\alpha(k)|^2 f(k)\, dk < \infty$$

for some function $f \ge 1$ such that $f(k) \to +\infty$ when $|k| \to +\infty$, where $\alpha(k)$ is the Fourier transform of F. We restrict ourselves to Prop. 2.1 to simplify the presentation in the proof but this remark means that the following modified proposition holds

Proposition 2.2 *Fix $T > 0$, $A > 0$, $\Omega \in \mathbb{R}^{2d}$ compact and any $f \ge 1$ such that $f(k) \to +\infty$ when $|k| \to +\infty$. Define Ω' and Ω'' as in Section 2.1. There exists a continuous, decreasing function $\varepsilon(\delta)$ with $\varepsilon(0) = 0$ s.t. for any $F \in H^{3/4} \cap L^\infty$ with $\|F\|_{L^\infty} \le A$, for any solutions (X, V) and (X^δ, V^δ) to* (1.1) *satisfying Property 1 and* (2.1)*, one has for any $|\delta| < 1/e$,*

$$Q_\delta(T) \le |\log\delta|\, \varepsilon(\delta)\, \left(1 + \int_{\mathbb{R}^d} |k|^{3/2}\, |\alpha(k)|^2\, f(k)\, dk\right)^{1/2},$$

with α the Fourier transform of F.

2.2 From Prop. 2.1 or 2.2 to Th. 1.1

It is now well known how to pass from an estimate like the one provided by Prop. 2.1 to a well posedness theory (see [16] for example) and therefore we only briefly recall the main steps. Take any $F \in H^{3/4+a} \cap L^\infty$.

We start by the existence of a solution. For that define F_n a regularizing sequence of F. Denote X_n, V_n the solution to (1.1) with F_n instead of F and $(X_n, V_n)(t = 0) = (x, v)$. For any $\delta = (\delta_1,\ \delta_2)$ in $\mathbb{R}^{2d}$, put

$$(X_n^\delta,\ V_n^\delta)(t, x, v) = (X_n,\ V_n)(t, x + \delta_1, v + \delta_2).$$

The function F_n and the solutions (X,V), (X^δ,V^δ) satisfy to all the assumptions of Prop. 2.1, as $F_n\in W^{1,\infty}$, using Property 1. Since $F\in L^\infty\cap H^{3/4+a}$, one may choose F_n uniformly bounded in this space. The proposition then shows that

$$Q_{\delta,n}(T)=\iint_\Omega \log\left(1+\frac{1}{|\delta|^2}\left(\sup_{0\le t\le T}|X_n(t,x,v)-X_n^\delta(t,x,v)|^2\right.\right.$$
$$\left.\left.+\int_0^T|V_n(t,x,v)-V_n^\delta(t,x,v)|^2\,dt\right)\right)\,dx\,dv,$$

is uniformly bounded in n and δ by

$$C\left(1+\left(\log\frac{1}{|\delta|}\right)^{\max\{1-2a,1/2\}}\right).$$

By Rellich criterion, this proves that the sequence (X_n,V_n) is compact in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^{2d})$. Denote by (X,V) an extracted limit, one directly checks that (X,V) is a solution to (1.1) by compactness and satisfies (1.4), (1.5). Thus existence is proved.

For uniqueness, consider another solution (X^δ,V^δ) to (1.1), which is the limit of solutions to a regularized equation (such as the one given by F_n or by another regularizing sequence of F). Then with the same argument, (X^δ,V^δ) also satisfies (1.4) and (1.5). Moreover

$$X(0,x,v)-X^\delta(0,x,v)=x-x=0,\quad V(0,x,v)-V^\delta(0,x,v)=v-v=0,$$

so that (X,V) and (X^δ,V^δ) also verify (2.1) for any $\delta\neq 0$. Applying again Prop. 2.1 and letting δ go to 0, one concludes that $X=X^\delta$ and $V=V^\delta$.

Note from this sketch that one has uniqueness among all solutions to (1.1) satisfying (1.4) and (1.5) and not only those which are limit of a regularized problem. However not all solutions to (1.1) (pointwise) necessarily satisfy those two conditions so that the uniqueness among all solutions to (1.1) is unknown. Indeed in many cases, it is not true, as there is a hidden selection principle in (1.4) (see the discussion in [4], [15] or [17]).

Finally if $F\in H^{3/4}$ only, then one first applies the De La Vallée Poussin's lemma to find a function f s.t. $f(k)\to+\infty$ when $|k|\to+\infty$ and

$$\int_{\mathbb{R}^d}|k|^{3/2}\,f(k)\,|\alpha(k)|^2\,dk<+\infty. \tag{2.2}$$

One proceeds as before with a regularizing sequence F_n which now has to satisfy uniformly the previous estimate. Using Prop. 2.2 instead of Prop. 2.1, the rest of the proof is identical.

3 The question of optimality : An example

It is hard to know whether the condition $F \in H^{3/4}$ is optimal and in which sense (see the short discussion in the introduction). Instead the purpose of this section is to give a simple example showing that $F \in W^{1/2,1}$ is a necessary condition in order to use the method followed in this paper; namely a quantitative estimate on $X - X^\delta$ and $V - V^\delta$. More precisely, for any $\alpha < 1/2$, we are going to construct a sequence of force fields $(F_N)_{N\geq 1}$ uniformly bounded in $W^{\alpha,1} \cap L^\infty$ and a sequence $(\delta_N)_{N\geq 1}$ converging to 0 such that functionals like $Q_\delta(T)$ cannot be uniformly bounded in N.

This example is one dimensional (2 in phase space) where it is known that much less is required to have uniqueness of the flow (almost F a measure). So this indicates in a sense that the method itself is surely not optimal. Moreover what this should imply in higher dimensions is not clear...

Through all this section we use the notation $f = O(g)$ if there exists a constant C s.t.

$$|f| \leq C\,|g| \text{ a.e.}$$

In dimension 1 all F derive from a potential so take

$$\phi(x) = x + \frac{h(N\,x)}{N^{\alpha+1}},\ F = -\phi'(x)$$

with h a periodic and regular function (C^2 at least) with $h(0) = 0$.

As ϕ is regular, we know that the solution (X, V) with initial condition (x, v) and the shifted one (X^δ, V^δ) corresponding to the initial condition $(x, v + \delta)$ satisfy the conservation of energy or

$$V^2 + 2\phi(X) = v^2 + 2\phi(x), \quad |V^\delta|^2 + 2\phi(X^\delta) = |v + \delta|^2 + 2\phi(x).$$

As ϕ is defined up to a constant, we do not need to look at all the trajectories and may instead restrict ourselves to the one starting at x s.t. $v^2 + \phi(x) = 0$. By symmetry, we may assume $v > 0$ and excluding the negligible set of initial data with $v = 0$, we may even take $v > \delta$.

Let t_0 and t_0^δ be the first times when the trajectories stop increasing: $V(t_0) = 0$ and $V^\delta(t_0^\delta) = 0$. As both velocities are initially positive, they stay so until t_0 or t_0^δ. So for instance

$$\dot{X} = V = \sqrt{-2\phi(X)}.$$

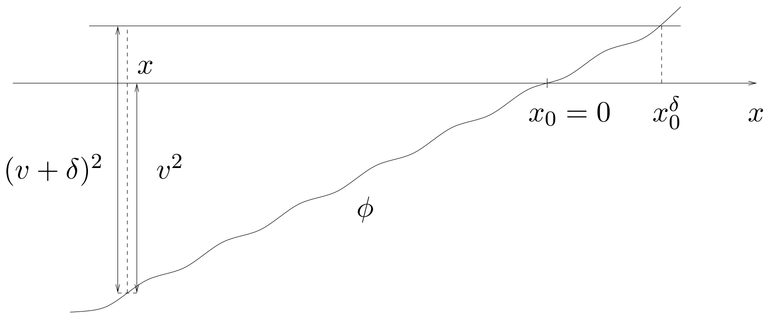

Figure 1: The potential ϕ and the construction of x_0 and x_0^δ

Hence t_0 is obtained by

$$t_0 = \int_0^{t_0} \frac{\dot{X}}{\sqrt{-2\phi(X)}}\,dt = \int_x^{x_0} \frac{dy}{\sqrt{-2\phi(y)}} \\ = \int_x^{x_0} \frac{dy}{\sqrt{-2y - 2h(Ny)\,N^{-1-\alpha}}},$$

if $x_0 = X(t_0)$. Of course by energy conservation $\phi(x_0) = 0$ and again as we are in dimension 1 this means that we may simply take $x_0 = 0$.

We have the equivalent formula for t_0^δ with x_0^δ (which we may not assume equal to 0). Put

$$C^\delta = |v+\delta|^2 + 2\phi(x) = \delta^2 + 2v\delta, \quad \eta = N(x_0^\delta - x_0) = N\,x_0^\delta$$

and note that $2\phi(x_0^\delta) - 2\phi(x_0) = C^\delta$, so that $|x_0^\delta - x_0| = |x_0^\delta| \leq C\delta$ since $\phi' \geq 1/2$ for N large enough. Then

$$t_0^\delta = \int_x^{x_0^\delta} \frac{dy}{\sqrt{C^\delta - 2\phi(y)}} = \int_{x-\eta/N}^{x_0} \frac{dy}{\sqrt{C^\delta - 2y - 2\eta/N - 2\,N^{-1-\alpha}\,h(N\,y+\eta)}} \\ = O(\delta) + \int_x^{x_0} \frac{dy}{\sqrt{-2y - 2\,N^{-1-\alpha}\,(h(N\,y+\eta) - h(\eta))}},$$

as the integral between x and $x - \eta/N$ is bounded by $O(\delta)$ (the integrand is bounded here) and

$$C^\delta = 2\phi(x_0^\delta) = 2x_0^\delta + \frac{2}{N^{1+\alpha}} h(N\,x_0^\delta) = 2\frac{\eta}{N} + \frac{2}{N^{1+\alpha}} h(\eta).$$

Note that as h is Lipschitz regular

$$\frac{|h(Nx+\eta)-h(\eta)|}{N^{1+\alpha}} = O\left(\frac{x}{N^\alpha}\right),$$
$$\frac{|h(Nx)|}{N^{1+\alpha}} = \frac{|h(Nx)-h(0)|}{N^{1+\alpha}} = O\left(\frac{x}{N^\alpha}\right).$$

So subtracting the two formula and making an asymptotic expansion

$$t_0 - t_0^\delta = O(\delta) + \int_x^{x_0} \frac{dy}{\sqrt{-2y}^3}\left(-\frac{2}{N^{1+\alpha}}(h(Ny)-h(Ny+\eta)+h(\eta)) \right.$$
$$\left. +O\left(\frac{h(Ny)}{N^{1+\alpha}\sqrt{y}}\right)^2\right).$$

Making the change of variable $Ny = z$ in the dominant term in the integral, one finds

$$t_0 - t_0^\delta = O(\delta) - 2\int_{Nx}^0 N^{1/2-1-\alpha}\,\frac{h(z)-h(z+\eta)+h(\eta)}{\sqrt{-2z}^3}\,dz$$
$$+ O(N^{-3/2-2\alpha}).$$

Consequently as long as

$$A(\eta) = \int_{-\infty}^0 \frac{h(z)-h(z+\eta)-h(\eta)}{\sqrt{-2z}^3}\,dz$$

is of order 1 then $t_0 - t_0^\delta$ is of order $N^{-1/2-\alpha}$. Note that $A(\eta)$ is small when η is, but it is always possible to find functions h s.t. $A(\eta)$ is of order 1 at least for some η. One way to see this is by observing that

$$A'(\eta) = -\int_{-\infty}^0 \frac{h'(y+\eta)+h'(\eta)}{\sqrt{-2y}^3}\,dy$$

cannot vanish for all η and functions h. Taking h such that $A'(\eta) \geq 1$ for η in some non-trivial interval, we can assume that A is of order 1 for $\eta \in [\underline{\eta}, \bar{\eta}]$ for some $\underline{\eta} < \bar{\eta}$.

Coming back to the definition of η and x_0^δ, $\eta \in [\underline{\eta}, \bar{\eta}]$ is equivalent to

$$\delta^2 + 2v\delta \in \phi([\underline{\eta}/N, \bar{\eta}/N]). \tag{3.1}$$

Using the formula for ϕ and the fact that $\underline{\eta}$ and $\bar{\eta}$ are independent of N or δ, we find

$$\delta^2 + 2v\delta + O(N^{-1-\alpha}) \in [\underline{\eta}/N, \bar{\eta}/N].$$

So let us finally choose $\delta = 1/N$ and denote by $\mathcal{V}$ the space of initial velocities v s.t. (3.1) is satisfied for N large enough. In view of the previous computation, there exists $N_0 \geq 1$ and $\gamma > 0$ such that for all $v \in \mathcal{V}$ and all $N \geq N_0$,

$$\gamma N^{-1/2-\alpha} \leq |t_0 - t_0^\delta| = |t_0(v) - t_0^\delta(v)| \leq \gamma^{-1} N^{-1/2-\alpha}.$$

We consider now the rest of the trajectories after times t_0 and t_0^δ. To this aim, we denote by $Y(t,y)$ and $W(t,y)$ the solution of (1.1) with initial data $(y,0)$. By uniqueness

$$X(t) = Y(t-t_0, x_0) \quad \forall t \geq t_0 \quad \text{and} \quad X^\delta(t) = Y(t-t_0^\delta, x_0^\delta) \quad \forall t \geq t_0^\delta.$$

Obviously one cannot have $V(t)$ small for all times, as initially $v \in \mathcal{V}$ was not small, and, as the force field $\nabla\phi$ is bounded, V is Lipschitz in time. So in conclusion for any $v \in \mathcal{V}$, there exists a time interval $I \subset (t_0, +\infty)$ of length of order v where V is larger than $v/2$.

Moreover $x_0^\delta \in x_0 + [\underline{\eta}/N, \bar{\eta}/N] = [\underline{\eta}/N, \bar{\eta}/N]$. Now either there exists a time interval J of order v s.t.

$$\forall t \in J, \quad |Y(t,x_0) - Y(t,x_0^\delta)| \geq \gamma N^{-1/2-\alpha}\, v/4.$$

or if it is not the case then on a subset $\tilde{I}$ of I of size v, one has

$$|Y(t-t_0,x_0) - Y(t-t_0,x_0^\delta)| \leq \gamma N^{-1/2-\alpha}\, v/4.$$

Note that t_0 may be replaced by t_0^δ in the previous inequality by reducing the interval $\tilde{I}$ (while keeping its length of order 1) since $|t_0 - t_0^\delta| = O(N^{-1/2-\alpha}) = o(1)$. Therefore for $t \in \tilde{I}$

$$\begin{aligned} |X(t) - X^\delta(t)| &\geq |X(t) - X(t+t_0-t_0^\delta)| - |Y(t-t_0^\delta, x_0) - Y(t-t_0^\delta, x_0^\delta)| \\ &\geq |t_0 - t_0^\delta|\, v/2 - \gamma N^{-1/2-\alpha}\, v/4 \geq \gamma\, v\, N^{-1/2-\alpha}/4, \end{aligned}$$

as V is larger than $v/2$.

Consequently in both situations, we have two solutions, $(Y(t,x_0), W(t,x_0))$ and $(Y(t,x_0^\delta), W(t,x_0^\delta))$ or (X,V) and (X^δ, V^δ), distant of $1/N$ initially but distant of order $N^{-1/2-\alpha}$ on a time interval of order v. Since h is periodic this provides many initial conditions with such a property. The difficulty that the distance between x_0 and x_0^δ is not fixed can be overcome since we are in two-dimensional setting (another trajectory starting further than x_0^δ from x_0 cannot approach more $(Y(t,x_0), W(t,x_0))$ than $(Y(t,x_0^\delta), W(t,x_0^\delta))$

does). Therefore we may control a functionals like $Q_\delta(T)/(\log(1/\delta))^{1-a}$ with $a > 0$ uniformly in N only if

$$N^{-1/2-\alpha} = O(\delta).$$

Since $\delta = 1/N$, this requires

$$\alpha \geq 1/2,$$

or $F = -\nabla\phi$ in at least $W^{1/2,1}$ as claimed.

4 Control of $Q_\delta(T)$: Proof of Prop. 2.1

Recall the notation α for the Fourier transform of F. The assumption of Proposition 2.1 corresponds to the following bound:

$$\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a}|\alpha(k)|^2\, dk = \|F\|^2_{H^{3/4+a}(\Omega'')} < +\infty.$$

4.1 Decomposition of $Q_\delta(T)$

Let

$$\begin{aligned} A_\delta(t,x,v) =& |\delta|^2 + \sup_{0\leq s\leq t} |X(s,x,v) - X^\delta(s,x,v)|^2 \\ &+ \int_0^t |V(s,x,v) - V^\delta(s,x,v)|^2\, ds. \end{aligned}$$

From (1.1), we compute

$$\begin{aligned} &\frac{d}{dt}\log\Bigg(1 + \frac{1}{|\delta|^2}\Big(\sup_{0\leq s\leq t} |X(s,x,v) - X^\delta(s,x,v)|^2 \\ &\qquad\qquad + \int_0^t |V(s,x,v) - V^\delta(s,x,v)|^2\, ds\Big)\Bigg) \\ &= \frac{2}{A_\delta(t,x,v)}\left(\frac{d}{dt}\left(\sup_{0\leq s\leq t} |X(s,x,v) - X^\delta(s,x,v)|^2\right)\right. \\ &\left.(V(t,x,v) - V^\delta(t,x,v))\int_0^t (F(X(s,x,v)) - F(X^\delta(s,x,v)))\, ds\right) \end{aligned}$$

Since, for any $f \in BV$,

$$\frac{d}{dt}\left(\max_{0\leq s\leq s} f(s)^2\right) \leq 2|f(s)f'(s)| \leq 4|f(s)|^2 + 4|f'(s)|^2,$$

we deduce from the previous computation that

$$
\begin{aligned}
Q_\delta(T) &\le 4 \iint_\Omega \int_0^T \frac{|X - X^\delta|^2 + |V - V^\delta|^2}{A_\delta(t,x,v)} \, dt \, dx \, dv + \tilde{Q}_\delta(T) \\
&\le 4|\Omega|(1+T) + \tilde{Q}_\delta(T)
\end{aligned}
$$

where,

$$
\begin{aligned}
\tilde{Q}_\delta(T) = -2 \int_0^T \iint_\Omega & \frac{V(t,x,v) - V^\delta(t,x,v)}{A_\delta(t,x,v)} \cdot \\
& \int_0^t \int_{\mathbb{R}^d} \alpha(k) \left(e^{ik\cdot X(s,x,v)} - e^{ik\cdot X^\delta(s,x,v)} \right) dk \, ds \, dx \, dv \, dt.
\end{aligned}
$$

We introduce a C_b^∞ function $\chi : \mathbb{R}_+ \to [0,1]$ such that $\chi(x) = 0$ if $x \le 1$ and $\chi(x) = 1$ if $x \ge 2$. Writing X_t (resp. V_t) for $X(t,x,v)$ (resp. $V(t,x,v)$) and X_t^δ (resp. V_t^δ) for $X^\delta(t,x,v)$ (resp. $V^\delta(t,x,v)$), and introducing

$$
\tilde{\alpha}(k) = \begin{cases} \alpha(k) & \text{if } |k| \ge (\log 1/|\delta|)^2 \\ 0 & \text{otherwise,} \end{cases}
$$

we may write

$$
\tilde{Q}_\delta(T) = \tilde{Q}_\delta^{(1)}(T) + \tilde{Q}_\delta^{(2)}(T) + \tilde{Q}_\delta^{(3)}(T) + \tilde{Q}^{(4)}(T),
$$

where

$$
\begin{aligned}
\tilde{Q}_\delta^{(1)}(T) = -2 \int_0^T \iint_\Omega \int_0^t \chi \left(\frac{|X_s - X_s^\delta|}{|\delta|^{4/3}} \right) & \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \\
\int_{\mathbb{R}^d} \tilde{\alpha}(k) & \left(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta} \right) dk \, ds \, dx \, dv \, dt,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{Q}_\delta^{(2)}(T) = -2 \int_0^T \iint_\Omega \int_0^t \chi \left(\frac{|X_s - X_s^\delta|}{|\delta|^{4/3}} \right) & \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \\
\int_{\mathbb{R}^d} (\alpha(k) - \tilde{\alpha}(k)) & \left(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta} \right) dk \, ds \, dx \, dv \, dt,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{Q}_\delta^{(3)}(T) = -2 \int_0^T \iint_\Omega \int_0^t \left(1 - \chi \left(\frac{|X_s - X_s^\delta|}{|\delta|^{4/3}} \right) \right) & \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \\
\int_{\{|k| \le |\delta|^{-4/3}\}} \alpha(k) & \left(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta} \right) dk \, ds \, dx \, dv \, dt,
\end{aligned}
$$

and

$$\tilde{Q}_\delta^{(4)}(T) = -2\int_0^T \iint_\Omega \int_0^t \left(1 - \chi\left(\frac{|X_s - X_s^\delta|}{|\delta|^{4/3}}\right)\right) \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \int_{\{|k|>|\delta|^{-4/3}\}} \alpha(k)\left(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta}\right) dk\, ds\, dx\, dv\, dt.$$

The proof is based on a control each of these terms. As proved in Subsection 4.2, the fourth term can be bounded with elementary computations. In Subsection 4.3, the second and third terms are bounded using standard results on maximal functions. Finally, the control of $\tilde{Q}_\delta^{(1)}(T)$ requires a more precise version of the maximal inequality, detailed in Subsection 4.4.

4.2 Control of $\tilde{Q}_\delta^{(4)}(T)$

Let us first state and prove a result that is used repeatedly in the sequel.

Lemma 4.1 *There exists a constant C such that, for $|\delta|$ small enough,*

$$\int_s^T \frac{|V_t - V_t^\delta|}{\sqrt{A_\delta(t,x,v)}}\, dt \le C(\log 1/|\delta|)^{1/2}.$$

Proof Using Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_s^T \frac{|V_t - V_t^\delta|}{\sqrt{A_\delta(t,x,v)}}\, dt &\le \int_s^T \frac{|V_t - V_t^\delta|}{\left(|\delta|^2 + \int_0^t |V_r - V_r^\delta|^2\, dr\right)^{1/2}}\, dt \\
&\le \sqrt{T}\left(\int_s^T \frac{|V_t - V_t^\delta|^2}{|\delta|^2 + \int_0^t |V_r - V_r^\delta|^2\, dr}\, dt\right)^{1/2} \\
&= \sqrt{T}\left(\log\left(\frac{|\delta|^2 + \int_0^T |V_r - V_r^\delta|^2\, dr}{|\delta|^2 + \int_0^s |V_r - V_r^\delta|^2\, dr}\right)\right)^{1/2} \\
&\le C\sqrt{T}\,(\log 1/|\delta|)^{1/2}
\end{aligned}$$

for $|\delta|$ small enough. □

Let us define the function

$$\tilde{F}(x) = \int_{\{|k|>|\delta|^{-4/3}\}} \alpha(k) e^{ik\cdot x}\, dx.$$

Since $\sqrt{A_\delta(t,x,v)} \geq |\delta|$, we have

$$\begin{aligned}|\tilde{Q}_\delta^{(4)}(T)| \leq C \int_0^T \iint_\Omega (|\tilde{F}(X_s)| + |\tilde{F}(X_s^\delta)|) \\ \times \int_s^T \frac{|V_t - V_t^\delta|}{|\delta|\sqrt{A_\delta(t,x,v)}}\, dt\, dx\, dv\, ds. \\ \leq C(\log 1/|\delta|)^{1/2}|\delta|^{-1} \int_0^T \iint_{\Omega'} |\tilde{F}(x)|\, dx\, dv\, ds \\ \leq C(\log 1/|\delta|)^{1/2}|\delta|^{-1} \left(\int_{\Omega_1'} |\tilde{F}(x)|^2\, dx \right)^{1/2},\end{aligned}$$

where the second line follows from Lemma 4.1 and from Property 1 applied to the change of variables $(x,v) = (X_s, V_s)$ and $(x,v) = (X_s^\delta, V_s^\delta)$. Then, it follows from Plancherel's identity that

$$\begin{aligned}|\tilde{Q}_\delta^{(4)}(T)| \leq C(\log 1/|\delta|)^{1/2}|\delta|^{-1} \left(\int_{\{|k|>|\delta|^{-4/3}\}} |\alpha(k)|^2\, dk \right)^{1/2} \\ \leq C(\log 1/|\delta|)^{1/2}|\delta|^{4a/3} \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2\, dk \right)^{1/2}.\end{aligned}$$

4.3 Control of $\tilde{Q}_\delta^{(2)}(T)$ and $\tilde{Q}_\delta^{(3)}(T)$

We recall that the maximal function Mf of $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, is defined by

$$Mf(x) = \sup_{r>0} \frac{C_d}{r^d} \int_{B(x,r)} f(z)\, dz, \qquad \forall x \in \mathbb{R}^d.$$

We are going to use the following classical results (see [29]). First, there exists a constant C such that, for all $x, y \in \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d)$,

$$|f(x) - f(y)| \leq C\,|x-y|(M|\nabla f|(x) + M|\nabla f|(y)). \tag{4.1}$$

Second, for all $1 < p < \infty$, the operator M is a linear continuous application from $L^p(\mathbb{R}^d)$ to itself.

We begin with the control of $\tilde{Q}_\delta^{(3)}(T)$. Let

$$\hat{F}(x) = \int_{\{|k|\leq|\delta|^{-4/3}\}} \alpha(k) e^{ik\cdot x}\, dx.$$

It follows from the previous inequality that

$$\left| \int_{\{|k|\le|\delta|^{-4/3}\}} \alpha(k)(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta})\, dk \right| = |\hat F(X_s) - \hat F(X_s^\delta)|$$
$$\le |X_s - X_s^\delta| \big(M|\nabla \hat F|(X_s) + M|\nabla \hat F|(X_s^\delta)\big).$$

Therefore, since $1-\chi(x) = 0$ if $|x|\ge 2$, following the same steps as for the contro, of $\tilde Q_\delta^{(4)}(T)$,

$$\begin{aligned} |\tilde Q_\delta^{(3)}(T)| &\le C \int_0^T \iint_\Omega \int_s^T \frac{|V_t - V_t^\delta|}{|\delta|\sqrt{A_\delta(t,x,v)}}\, |\delta|^{4/3} \\ &\qquad \big(M|\nabla \hat F|(X_s) + M|\nabla \hat F|(X_s^\delta)\big)\, dt\, dx\, dv\, ds. \\ &\le C(\log 1/|\delta|)^{1/2}|\delta|^{1/3} \left(\int_{\Omega_1'} (M|\nabla \hat F|(x))^2\, dx \right)^{1/2} \\ &\le C(\log 1/|\delta|)^{1/2}|\delta|^{1/3} \left(\int_{\Omega_1'} |\nabla \hat F|^2(x) \right)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} |\tilde Q_\delta^{(3)}(T)| &\le C(\log 1/|\delta|)^{1/2}|\delta|^{1/3} \left(\int_{\{|k|\le|\delta|^{-4/3}\}} |k|^2 |\alpha(k)|^2\, dk \right)^{1/2} \\ &\le C(\log 1/|\delta|)^{1/2}|\delta|^{4a/3} \left(\int_{\mathbb{R}^d} |k|^{\frac32 + 2a} |\alpha(k)|^2\, dk \right)^{1/2}. \end{aligned}$$

The control of $\tilde Q_\delta^{(2)}(T)$ follows from a similar computation: introducing $F_0(x) = \int_{\{k<(\log 1/|\delta|)^2\}} \alpha(k) e^{ik\cdot x}\, dx$, we obtain

$$\begin{aligned} |\tilde Q_\delta^{(2)}(T)| \le C \int_0^T \iint_\Omega \int_s^T & \frac{|V_t - V_t^\delta|}{\sqrt{A_\delta(t,x,v)}} \frac{|X_s - X_s^\delta|}{\sqrt{A_\delta(t,x,v)}} \\ & \big(M|\nabla F_0|(X_s) + M|\nabla F_0|(X_s^\delta)\big)\, dt\, dx\, dv\, dt. \end{aligned}$$

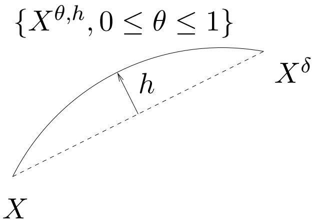

Figure 2: The graph of $\theta \mapsto X^{\theta,h}$

Since $|X_s - X_s^\delta| \le \sqrt{A_\delta(t,x,v)}$ for all $s \le t$

$$\begin{aligned}
|\tilde{Q}_\delta^{(2)}(T)| &\le C(\log 1/|\delta|)^{1/2} \int_0^T \left(\iint_{\Omega'} \big(M|\nabla F_0|(x)\big)^2 \, dx \, dv \right)^{1/2} ds \\
&\le C(\log 1/|\delta|)^{1/2} \left(\int_{\{|k|<(\log 1/|\delta|)^2\}} |k|^2 |\alpha(k)|^2 \, dk \right)^{1/2} \\
&\le C(\log 1/|\delta|)^{1-2a} \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2 \, dk \right)^{1/2}.
\end{aligned}$$

4.4 Control of $\tilde{Q}_\delta^{(1)}(T)$

The inequality (4.1) is insufficient to control $\tilde{Q}_\delta^{(1)}(T)$. Our estimate relies on a more precise version of this inequality, detailed below.

4.4.1 Definition of $X_s^{\theta,h}$

For any $\theta \in [0,1]$ and $h \in \mathbb{R}^d$, we define

$$X^{\theta,h}(t,x,v) = \theta X(t,x,v) + (1-\theta) X^\delta(t,x,v) + (1-(2\theta-1)^2) h,$$

and we write for simplicity $X_t^{\theta,h}$ for $X^{\theta,h}(t,x,v)$. Then, for any fixed $h \in \mathbb{R}^d$, by differentiation in θ

$$\begin{aligned}
&\int_{\mathbb{R}^d} \tilde{\alpha}(k) \left(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta} \right) dk \\
&\qquad = \int_{\mathbb{R}^d} \tilde{\alpha}(k) \int_0^1 e^{ik\cdot X_s^{\theta,h}} k \cdot (X_s - X_s^\delta + 4(1-2\theta)h) \, d\theta \, dk. \quad (4.2)
\end{aligned}$$

For any $x, y \in \mathbb{R}^d$, we introduce the hyperplane orthogonal to $x - y$

$$H(x,y) = \{h \in \mathbb{R}^d : h \cdot (x-y) = 0\}.$$

If $x = y$, we define for example $H(x, y) = H(0, e_1)$, where $e_1 = (1, 0, \ldots, 0)$. Fix a C_b^∞ function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(x) = 0$ for $x \notin [-1, 1]$ and $\int_{H(0,e_1)} \psi(|h|)\, dh = 1$. By invariance of $|h|$ with respect to rotations, we also have

$$\int_{H(x,y)} \psi(|h|)\, dh = 1$$

for all $x, y \in \mathbb{R}^d$.

Since the left-hand side of (4.2) does not depend on h, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \tilde{\alpha}(k) \left(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta} \right) dk \\
&\qquad = \frac{1}{|X - X^\delta|^{d-1}} \int_{H(X_s, X_s^\delta)} \psi\left(\frac{|h|}{|X - X^\delta|} \right) \int_{\mathbb{R}^d} \tilde{\alpha}(k) \\
&\qquad\qquad \int_0^1 e^{ik\cdot X_s^{\theta,h}} k \cdot (X_s - X_s^\delta + 4(1 - 2\theta)h)\, d\theta\, dk\, dh
\end{aligned}$$

in the case where $X_s \neq X_s^\delta$. If $X_s = X_s^\delta$, the previous quantity is 0.

Let $\rho : [0, 1] \to \mathbb{R}_+$ be a C_b^∞ function such that $\rho(x) = 1$ for $0 \le x \le 1/4$, $\rho(x) = 0$ for $3/4 \le x \le 1$ and $\rho(x) + \rho(1 - x) = 1$ for $0 \le x \le 1$. Then, one has

$$\int_{\mathbb{R}^d} \tilde{\alpha}(k) \left(e^{ik\cdot X_s} - e^{ik\cdot X_s^\delta} \right) dk = B_\delta(s, x, v) + C_\delta(s, x, v),$$

where

$$\begin{aligned}
B_\delta(s, x, v) &= \frac{1}{|X_s - X_s^\delta|^{d-1}} \int_{H(X_s, X_s^\delta)} \psi\left(\frac{|h|}{|X_s - X_s^\delta|} \right) \int_{\mathbb{R}^d} \tilde{\alpha}(k) \\
&\qquad \int_0^1 \rho(\theta) e^{ik\cdot X_s^{\theta,h}} k \cdot (X_s - X_s^\delta + 4(1 - 2\theta)h)\, d\theta\, dk\, dh
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
C_\delta(s, x, v) &= \frac{1}{|X_s - X_s^\delta|^{d-1}} \int_{H(X_s, X_s^\delta)} \psi\left(\frac{|h|}{|X_s - X_s^\delta|} \right) \int_{\mathbb{R}^d} \tilde{\alpha}(k) \\
&\qquad \int_0^1 \rho(1 - \theta) e^{ik\cdot X_s^{\theta,h}} k \cdot (X_s - X_s^\delta + 4(1 - 2\theta)h)\, d\theta\, dk\, dh.
\end{aligned} \tag{4.4}$$

We focus on $B_\delta(s, x, v)$ as by symmetry between X and X^δ, C_δ is dealt with in exactly the same manner.

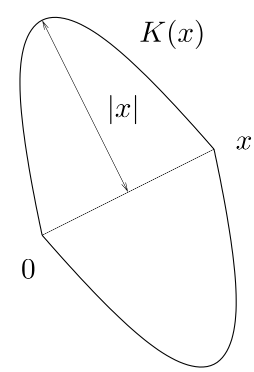

Figure 3: The set $K(x)$

4.4.2 Change of variable $z = X_s^{\theta,h}$

For any $x \in \mathbb{R}^d$, we introduce

$$K(x) = \{y \in \mathbb{R}^d : \exists \theta \in [0,1],\ h \in H(x,0) \ \text{ s.t. } |h| \le |x| \\ \text{and } y = \theta(x + 4(1-\theta)h)\}. \quad (4.5)$$

Observing that

$$\theta = \frac{y}{|x|} \cdot \frac{x}{|x|},$$

this set may also be defined as

$$K(x) = \left\{ y \in \mathbb{R}^d : \frac{y}{|x|} \cdot \frac{x}{|x|} \in [0,1] \right. \\ \left. \text{and } \left| \frac{y}{|x|} - \left(\frac{y}{|x|} \cdot \frac{x}{|x|} \right) \frac{x}{|x|} \right| \le 4 \frac{y}{|x|} \cdot \frac{x}{|x|} \left(1 - \frac{y}{|x|} \cdot \frac{x}{|x|} \right) \right\}.$$

Note that, for any $y \in K(x)$, taking θ and h as in (4.5), we have $|y|^2 = \theta^2(|x|^2 + 16(1-\theta^2)|h|^2) \le 17\theta^2|x|^2$. Therefore, denoting by (x,y) the angle between the vectors x and y,

$$\cos(x,y) = \frac{x}{|x|} \cdot \frac{y}{|y|} = \frac{\theta|x|}{|y|} \ge 17^{-1/2}. \quad (4.6)$$

For fixed $x, y \in \mathbb{R}^d$, we now introduce the application

$$F_{x,y} : [0,1] \times \{h \in H(x,y) : |h| \le |y-x|\} \to K(x-y) \\ (\theta, h) \mapsto \theta(x - y + 4(1-\theta)h).$$

It is elementary to check that $F_{x,y}$ is a bijection when $x \neq y$, with inverse

$$F_{x,y}^{-1}(z) = \left(\frac{z}{|x-y|} \cdot \frac{x-y}{|x-y|} \,,\, \frac{z - \left(\frac{z}{|x-y|} \cdot \frac{x-y}{|x-y|}\right)(x-y)}{4\,\frac{z}{|x-y|} \cdot \frac{x-y}{|x-y|} \left(1 - \frac{z}{|x-y|} \cdot \frac{x-y}{|x-y|}\right)} \right)$$

for $z \in K(x-y)$. Moreover, $F_{x,y}$ is differentiable and its differential, written in a basis of $\mathbb{R}^d$ with first vector $(x-y)/|x-y|$, is

$$\nabla F_{x,y}(\theta, h) = \begin{pmatrix} |x-y| & 4(1-2\theta)h \\ 0 & 4\theta(1-\theta)\mathrm{Id} \end{pmatrix}.$$

Therefore, the Jacobian of $F_{x,y}$ at (θ, h) is $(4\theta(1-\theta))^{d-1}|x-y|$.

Making the change of variable $z = F_{X_s, X_s^\delta}(\theta, h)$ in (4.3), we can now compute

$$\begin{aligned}
&B_\delta(s,x,v) \\
&= \int_{\mathbb{R}^d} \tilde{\alpha}(k) \int_0^1 \int_{H(X_s, X_s^\delta)} \frac{\rho(\theta)\psi\left(\frac{|h|}{|X_s - X_s^\delta|}\right)}{|X_s - X_s^\delta|^d (4\theta(1-\theta))^{d-1}}\, e^{ik\cdot X_s^{\theta,h}} \\
&\qquad k \cdot (X_s - X_s^\delta + 4(1-\theta)h - 4\theta h)\, (4\theta(1-\theta))^{d-1} |X_s - X_s^\delta|\, dh\, d\theta\, dk \\
&= B_\delta^1(s,x,v) - B_\delta^2(s,x,v), \qquad (4.7)
\end{aligned}$$

with

$$B_\delta^1(s,x,v) = \int_{\mathbb{R}^d} \tilde{\alpha}(k) \int_{\mathbb{R}^d} \frac{k\cdot z}{|z|^d} \psi^{(1)}\left(\frac{z}{|z|}, \frac{X_s - X_s^\delta}{|X_s - X_s^\delta|}, \frac{|z|}{|X_s - X_s^\delta|}\right) e^{ik\cdot(X_s^\delta + z)}\, dz\, dk,$$

and

$$B_\delta^2(s,x,v) = -\int_{\mathbb{R}^d} \tilde{\alpha}(k) \int_{\mathbb{R}^d} \frac{k}{|z|^{d-1}} \cdot \psi^{(2)}\left(\frac{z}{|z|}, \frac{X_s - X_s^\delta}{|X_s - X_s^\delta|}, \frac{|z|}{|X_s - X_s^\delta|}\right) e^{ik\cdot(X_s^\delta + z)}\, dz\, dk.$$

We defined, for $(a,b,c) \in S^{d-1} \times S^{d-1} \times (\mathbb{R} \setminus \{0\})$,

$$\psi^{(1)}(a,b,c) = \frac{\tilde{\rho}((a\cdot b)c)\psi\left(\frac{|a-(a\cdot b)b|}{4(a\cdot b)(1-(a\cdot b)c)}\right)}{4^{d-1}(a\cdot b)^d (1-(a\cdot b)c)^{d-1}}$$

and

$$\psi^{(2)}(a,b,c) = \frac{\tilde{\rho}((a\cdot b)c)\psi\left(\frac{|a-(a\cdot b)b|}{4(a\cdot b)(1-(a\cdot b)c)}\right)}{4^{d-1}(a\cdot b)^{d-1}(1-(a\cdot b)c)^d}\, c(a-(a\cdot b)b),$$

where $\tilde{\rho}(x) = \rho(x)$ if $x \in [0,1]$, and $\tilde{\rho}(x) = 0$ otherwise.

It follows from (4.6) and from the definition of ρ that these two functions have support in

$$\{(u,v) \in (S^{d-1})^2 : \cos(u,v) \geq 17^{-1/2}\} \times [0, 3/4].$$

Moreover, they belong to $C_b^{0,\infty,\infty}(S^{d-1}, S^{d-1}, \mathbb{R}\setminus\{0\})$. Indeed, since $\tilde{\rho}(x) = 0$ for $x \geq 3/4$, the terms $(1-(a\cdot b)c)$ in the denominators do not cause any regularity problem. Moreover, since $\psi(x) = 0$ for $x \notin [-1,1]$ and

$$\frac{|a-(a\cdot b)b|}{|a\cdot b|} \geq \frac{1}{|a\cdot b|} - 1$$

for all $a, b \in S^{d-1}$, the terms $a\cdot b$ in the denominators do not cause any worry either. Finally, since $\tilde{\rho} \in C_b^\infty(\mathbb{R}\setminus\{0\})$, the discontinuity of $\tilde{\rho}$ at 0 can only cause a problem in the neighborhood of points such that $a\cdot b = 0$ (c being nonzero). Therefore, the previous observation also solves this difficulty.

4.4.3 Decomposition of $B_\delta^1(s,x,v)$: integration by parts

Writing $\psi_t^{(1)}$ for

$$\psi^{(1)}\left(\frac{z}{|z|}, \frac{X_t - X_t^\delta}{|X_t - X_t^\delta|}, \frac{|z|}{|X_t - X_t^\delta|}\right), \tag{4.8}$$

we decompose $B_\delta^1(s,x,v)$

$$\begin{aligned} B_\delta^1(s,x,v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{\alpha}(k) \frac{e^{ik\cdot(X_s^\delta+z)}\, k\cdot z}{|z|^d} \psi_s^{(1)} \frac{i\frac{k}{|k|}\cdot V_s^\delta}{|k|^{-1/2} + i\frac{k}{|k|}\cdot V_s^\delta}\, dk\, dz \\ &+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{\alpha}(k) \frac{e^{ik\cdot(X_s^\delta+z)}\, k\cdot z}{|z|^d} \psi_s^{(1)} \frac{|k|^{-1/2}}{|k|^{-1/2} + i\frac{k}{|k|}\cdot V_s^\delta}\, dk\, dz \\ &=: B_\delta^{11}(s,x,v) + B_\delta^{12}(s,x,v). \end{aligned}$$

Now, let us write χ_s for

$$\chi\left(\frac{|X_s - X_s^\delta|}{|\delta|^{4/3}}\right), \tag{4.9}$$

and let us define similarly as in (4.8) and (4.9) the notation $\nabla_2\psi_s^{(1)}$, $\nabla_3\psi_s^{(1)}$ and χ_s'. Note that the term $i\frac{k}{|k|}\cdot V_s^\delta\ e^{ik\cdot(X_s^\delta+z)}$ is exactly the time derivative of $\frac{1}{|k|}\,e^{ik\cdot(X_s^\delta+z)}$. So integrating by parts in time, we obtain

$$\int_0^t \chi_s B_\delta^{11}(s,x,v)ds = \mathrm{I}(t,x,v)-\mathrm{II}(t,x,v)-\mathrm{III}(t,x,v)-\mathrm{IV}(t,x,v)-\mathrm{V}(t,x,v),$$

with

$$\mathrm{I}(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde\alpha(k)\,\frac{k\cdot z}{|k|\,|z|^d}\,\frac{e^{ik\cdot(X_t^\delta+z)}\chi_t\psi_t^{(1)}}{|k|^{-1/2}+i\frac{k}{|k|}\cdot V_t^\delta}\,dk\,dz,$$

$$\mathrm{II}(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde\alpha(k)\,\frac{k\cdot z}{|k|\,|z|^d}\,\frac{e^{ik\cdot(x+\delta_1+z)}\chi_0\psi_0^{(1)}}{|k|^{-1/2}+i\frac{k}{|k|}\cdot(v+\delta_2)}\,dk\,dz,$$

$$\begin{aligned}\mathrm{III}(t,x,v) = \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde\alpha(k)\,\frac{k\cdot z}{|k|\,|z|^d}\,&\frac{e^{ik\cdot(X_s^\delta+z)}\psi_s^{(1)}\chi_s'}{|k|^{-1/2}+i\frac{k}{|k|}\cdot V_s^\delta}\\ &\frac{(X_s-X_s^\delta)\cdot(V_s-V_s^\delta)}{|\delta|^{4/3}|X_s-X_s^\delta|}\,dk\,dz\,ds,\end{aligned}$$

correspondingly

$$\begin{aligned}\mathrm{IV}(t,x,v) = &\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde\alpha(k)\,\frac{k\cdot z}{|k|\,|z|^d}\,\frac{e^{ik\cdot(X_s^\delta+z)}\chi_s}{|k|^{-1/2}+i\frac{k}{|k|}\cdot V_s^\delta}\\ &\Bigg[-\nabla_3\psi_s^{(1)}\frac{|z|}{|X_s-X_s^\delta|^3}(X_s-X_s^\delta)\cdot(V_s-V_s^\delta)\\ &+\nabla_2\psi_s^{(1)}\cdot\left(\frac{V_s-V_s^\delta}{|X_s-X_s^\delta|}-\frac{X_s-X_s^\delta}{|X_s-X_s^\delta|^3}(X_s-X_s^\delta)\cdot(V_s-V_s^\delta)\right)\Bigg]\,dk\,dz\,ds,\end{aligned}$$

and

$$\mathrm{V}(t,x,v) = \int_0^t\int_{\mathbb{R}^{2d}} \tilde\alpha(k)\,\frac{k\cdot z}{|k|\,|z|^d}\,\frac{e^{ik\cdot(X_s^\delta+z)}\chi_s\psi_s^{(1)}}{\left(|k|^{-1/2}+i\frac{k}{|k|}\cdot V_s^\delta\right)^2}\,i\frac{k}{|k|}\cdot F(X_s^\delta)\,dk\,dz\,ds.$$

Let us define

$$\mathrm{I}(T) = \int_0^T\iint_\Omega \frac{V_t-V_t^\delta}{A_\delta(t,x,v)}\cdot\mathrm{I}(t,x,v)\,dx\,dv\,dt,$$

and $\mathrm{II}(T)$, $\mathrm{III}(T)$, $\mathrm{IV}(T)$ and $\mathrm{V}(T)$ similarly.

We are going to bound each of these terms. The last one gives the good order of $\int_{\mathbb{R}^d}|k|^{3/2+a}|\alpha(k)|^2dk$. The others are bounded by integrals involving lower powers of $|k|$.

4.4.4 Upper bound for $|\mathrm{V}(T)|$

First, we make the change of variables $z' = z + X_s^\delta$, followed by the change of variable $(x', v') = (X_s^\delta, V_s^\delta)$ in the integral defining $\mathrm{V}(T)$. When $(x, v) \in \Omega$, the variable (x', v') belongs to the set $\Omega_s = \{(X^\delta(s, x, v), V^\delta(s, x, v)), (x, v) \in \Omega\}$. Note also that $X(-s, x', v') = x + \delta_1$ and $V(-s, x', v') = v + \delta_2$.

Writing for convenience x, v, z instead of x', v', z', it follows from these changes of variables and from Property 1 that

$$\mathrm{V}(T) = \int_0^T \int_0^t \iint_{\Omega_s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\chi}_s \tilde{\psi}_s^{(1)} \frac{\tilde{V}_{t,s}^\delta - V_{t-s}}{|\delta|^2 + \sup_{0\le r\le t} |\tilde{X}_{r,s}^\delta - X_{r-s}|^2 + \int_0^t |\tilde{V}_{r,s}^\delta - V_{r-s}|^2\, dr} \cdot \tilde{\alpha}(k) \frac{k\cdot(z-x)}{|k|\,|z-x|^d}\, e^{ik\cdot z} \frac{i\frac{k}{|k|}\cdot F(x)}{\left(|k|^{-1/2} + i\frac{k}{|k|}\cdot v\right)^2}\, dk\, dz\, dx\, dv\, ds\, dt,$$

where

$$\begin{aligned} \tilde{X}_{t,s}^\delta &= X^\delta(t, X(-s, x, v), V(-s, x, v)), \\ \tilde{V}_{t,s}^\delta &= V^\delta(t, X(-s, x, v), V(-s, x, v)), \\ \tilde{\psi}_s^{(1)} &= \psi^{(1)}\left(\frac{z-x}{|z-x|}, \frac{\tilde{X}_{s,s}^\delta - x}{|\tilde{X}_{s,s}^\delta - x|}, \frac{|z-x|}{|\tilde{X}_{s,s}^\delta - x|}\right). \end{aligned}$$

and

$$\tilde{\chi}_s = \chi\left(\frac{|\tilde{X}_{s,s}^\delta - x|}{|\delta|^{4/3}}\right).$$

Writing the tensor (remember that $\alpha(k) \in \mathbb{R}^d$)

$$G_{\mathrm{V}}(v, z) = \int_{\mathbb{R}^d} \frac{k\otimes k}{|k|^2} \otimes \frac{\tilde{\alpha}(k)\, e^{ik\cdot z}}{\left(|k|^{-1/2} + i\frac{k}{|k|}\cdot v\right)^2}\, dk,$$

and reminding that $\Omega_s \subset \Omega'$ for all $s \in [0, T]$, we have

$$|\mathrm{V}(T)| \le C \int_0^T \iint_{\Omega'} \int_0^t \int_{\mathbb{R}^d} \frac{\tilde{\chi}_s \tilde{\psi}_s^{(1)}\, |F(x)|\, \|G_{\mathrm{V}}(v, z)\|}{|z-x|^{d-1}\left(|\delta| + |\tilde{X}_{s,s}^\delta - x|\right)} \frac{|\tilde{V}_{t,s}^\delta - V_{t-s}|}{\left(|\delta|^2 + \int_0^t |\tilde{V}_{r,s}^\delta - V_{r-s}|^2\, dr\right)^{1/2}}\, dz\, ds\, dx\, dv\, dt,$$

where $\|a\|^2 = \sum_{i,j,k=1}^d a_{ijk}^2$ for any tensor with three entries $a = (a_{ijk})$ with $1 \leq i, j, k \leq d$. So

$$|\mathrm{V}(T)| \leq C \int_0^T \iint_{\Omega'} \int_{\mathbb{R}^d} \frac{\tilde{\psi}_s^{(1)} \, \|G_{\mathrm{V}}(v,z)\|}{|z-x|^{d-1} \left(|\delta| + |\tilde{X}_{s,s}^\delta - x| \right)} \int_s^T \frac{|\tilde{V}_{t,s}^\delta - V_{t-s}|}{\left(|\delta|^2 + \int_0^t |\tilde{V}_{r,s}^\delta - V_{r-s}|^2 \, dr \right)^{1/2}} \, dt \, dz \, dx \, dv \, ds.$$

Now, on the one hand, following the same computation as in Lemma 4.1, the integral with respect to t can be upper bounded by $C(\log 1/|\delta|)^{1/2}$ for $|\delta|$ small enough. On the other hand,

$$\iint_{\Omega'} \int_{\mathbb{R}^d} \frac{\tilde{\psi}_s^{(1)} \, \|G_{\mathrm{V}}(v,z)\|}{|z-x|^{d-1} \left(|\delta| + |\tilde{X}_{s,s}^\delta - x| \right)} \, dz \, dx \, dv$$
$$\leq \left(\iint_{\Omega'} \int_{\mathbb{R}^d} \frac{\tilde{\psi}_s^{(1)}}{|z-x|^{d-1} \left(|\delta| + |\tilde{X}_{s,s}^\delta - x| \right)} \, dz \, dx \, dv \right)^{1/2} \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \int_{\Omega_1} \frac{\tilde{\psi}_s^{(1)} \, \|G_{\mathrm{V}}(v,z)\|^2}{|z-x|^{d-1} \left(|\delta| + |\tilde{X}_{s,s}^\delta - x| \right)} \, dx \, dz \, dv \right)^{1/2},$$

and this last term is bounded by

$$C \left(\iint_{\Omega'} \frac{1}{|\tilde{X}_{s,s}^\delta - x|} \int_{x + K(\tilde{X}_{s,s}^\delta - x)} \frac{dz}{|z-x|^{d-1}} \, dx \, dv \right)^{1/2} \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{V}}(v,z)\|^2 \int_{\Omega_1'} \frac{dx}{|z-x|^{d-1} \left(|\delta| + |z-x| \right)} \, dz \, dv \right)^{1/2}$$
$$\leq C(\log 1/|\delta|)^{1/2} \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{V}}(v,z)\|^2 \, dz \, dv \right)^{1/2}, \quad (4.10)$$

where we have used that, for any $z \in x + K(\tilde{X}_{s,s}^\delta - x)$, $|z-x| \leq |\tilde{X}_{s,s}^\delta - x|$, and where the last inequality can be obtained by a spherical coordinate change of variable centered at x in the variable z in the first term, and centered at z in the variable x in the second term.

Now,

$$\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{V}}(v,z)\|^2 \, dz \, dv$$
$$= \int_{\Omega_2'} \int_{\mathbb{R}^d} \sum_{i,j,n=1}^{d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{k_i l_i k_j l_j}{|k|^2 |l|^2} \frac{\tilde{\alpha}_n(k) \overline{\tilde{\alpha}_n(l)} e^{iz\cdot(k-l)}}{\left(|k|^{-1/2} + i\frac{k}{|k|} \cdot v\right)^2}$$
$$\left(|l|^{-1/2} - i\frac{l}{|l|} \cdot v\right)^{-2} \, dl \, dk \, dz \, dv,$$

and integrating first in z and l, this is equal to

$$\int_{\Omega_2'} \sum_{i,j}^{d} \int_{\mathbb{R}^d} \frac{k_i^2 k_j^2}{|k|^4} \frac{|\tilde{\alpha}(k)|^2}{\left||k|^{-1/2} + i\frac{k}{|k|} \cdot v\right|^4} \, dk \, dv.$$

Therefore

$$\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{V}}(v,z)\|^2 \, dz \, dv \leq C \int_{\mathbb{R}^d} \int_{\Omega_2'} \frac{|\tilde{\alpha}(k)|^2}{\left(\frac{1}{|k|} + \left(\frac{k\cdot v}{|k|}\right)^2\right)^2} \, dv \, dk$$
$$\leq C \int_{\mathbb{R}^d} |k|^2 |\tilde{\alpha}(k)|^2 \int_{-\infty}^{+\infty} \frac{dv_1}{(1+|k|v_1^2)^2} \, dk,$$

where we write the vector v as $(v_1, \ldots, v_d)$ in an orthonormal basis of $\mathbb{R}^d$ with first vector $k/|k|$. In conclusion

$$\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{V}}(v,z)\|^2 \, dz \, dv \leq C \int_{\{|k|>(\log 1/|\delta|)^2\}} |k|^{3/2} |\alpha(k)|^2 \, dk$$
$$\leq C(\log 1/|\delta|)^{-4a} \int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2 \, dk, \quad (4.11)$$

Combining this inequality with (4.10), we finally get

$$|\mathrm{V}(T)| \leq C(\log 1/|\delta|)^{1-2a} \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2 \, dk\right)^{1/2}.$$

4.4.5 Upper bound for $|\mathrm{IV}(T)|$

Applying to $\mathrm{IV}(T)$ the same change of variable as we did for $\mathrm{V}(T)$, we have

$$|\mathrm{IV}(T)| \leq C \int_0^T \iiint_{\Omega'} \int_0^t \int_{\mathbb{R}^d} \frac{|\tilde{V}^\delta_{s,s} - v|\, |\tilde{V}^\delta_{t,s} - V_{t-s}|}{|\delta|^2 + \int_0^t |\tilde{V}^\delta_{r,s} - V_{r-s}|^2\, dr} \frac{\tilde{\chi}_s\, |\hat{\psi}^{(1)}_s|\, \|G_{\mathrm{IV}}(v,z)\|}{|z-x|^{d-1}|\tilde{X}^\delta_{s,s} - x|}\, dz\, ds\, dx\, dv\, dt$$

where $\|a\|^2 = \sum_{i,j=1}^d a_{ij}^2$ for any matrix $a = (a_{ij})_{1\leq i,j\leq d}$,

$$G_{\mathrm{IV}}(v,z) = \int_{\mathbb{R}^d} \frac{k}{|k|} \otimes \frac{\tilde{\alpha}(k)\, e^{ik\cdot z}}{|k|^{-1/2} + i\frac{k}{|k|}\cdot v}\, dk$$

and

$$\begin{aligned}\hat{\psi}^{(1)}_s = &-\nabla_3\psi^{(1)}\left(\frac{z-x}{|z-x|}, \frac{\tilde{X}^\delta_{s,s}-x}{|\tilde{X}^\delta_{s,s}-x|}, \frac{|z-x|}{|\tilde{X}^\delta_{s,s}-x|}\right)\frac{|z|\,(\tilde{X}^\delta_{s,s}-x)}{|\tilde{X}^\delta_{s,s}-x|^2}\\ &-\left(\nabla_2\psi^{(1)}\left(\frac{z-x}{|z-x|}, \frac{\tilde{X}^\delta_{s,s}-x}{|\tilde{X}^\delta_{s,s}-x|}, \frac{|z-x|}{|\tilde{X}^\delta_{s,s}-x|}\right)\cdot\frac{\tilde{X}^\delta_{s,s}-x}{|\tilde{X}^\delta_{s,s}-x|}\right)\frac{\tilde{X}^\delta_{s,s}-x}{|\tilde{X}^\delta_{s,s}-x|}\\ &+\nabla_2\psi^{(1)}\left(\frac{z-x}{|z-x|}, \frac{\tilde{X}^\delta_{s,s}-x}{|\tilde{X}^\delta_{s,s}-x|}, \frac{|z-x|}{|\tilde{X}^\delta_{s,s}-x|}\right).\end{aligned}$$

Note that, because of the properties of $\psi^{(1)}$ obtained in Section 4.4.2,

$$|\hat{\psi}^{(1)}_s| \leq C\mathbb{I}_{\{z-x\in K(\tilde{X}^\delta_{s,s}-x)\}}$$

for some constant C.

Then, following a similar computation as the one leading to (4.10),

$$\begin{aligned}|\mathrm{IV}(T)| \leq C &\left(\int_{\Omega'_2}\int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \int_{\Omega'_1} \frac{dx\, dz\, dv}{|z-x|^{d-1}\left(|\delta|^{4/3} + |z-x|\right)}\right)^{1/2}\\ &\left(\iiint_{\Omega'}\int_0^T\int_0^t \frac{|\tilde{V}^\delta_{s,s}-v|^2\, |\tilde{V}^\delta_{t,s}-V_{t-s}|^2}{|\delta|^4 + \left(\int_0^t |\tilde{V}^\delta_{r,s}-V_{r-s}|^2\, dr\right)^2}\right.\\ &\left.\frac{1}{|\tilde{X}^\delta_{s,s}-x|}\int_{x+K(\tilde{X}^\delta_{s,s}-x)} \frac{dz}{|z-x|^{d-1}}\, ds\, dt\, dx\, dv\right)^{1/2}.\end{aligned}$$

Hence

$$|\mathrm{IV}(T)| \le C(\log 1/|\delta|)^{1/2} \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \right)^{1/2}$$

$$\left(\iint_{\Omega'} \int_0^T \int_0^t \frac{|\tilde{V}^\delta_{s,s} - v|^2 \, |\tilde{V}^\delta_{t,s} - V_{t-s}|^2}{|\delta|^4 + \left(\int_0^t |\tilde{V}^\delta_{r,s} - V_{r-s}|^2 \, dr \right)^2} \right)^{1/2} \tag{4.12}$$

where we have used that $|\tilde{X}^\delta_{s,s} - x| \ge |\delta|^{4/3}$ and $|\tilde{X}^\delta_{s,s} - x| \ge |z - x|$ when $\tilde{\chi}_s \, |\hat{\psi}^{(1)}_s| \ne 0$.

Now, making the change of variable $(x', v') = (X^\delta(-s, x.v), V^\delta(-s, x, v))$ and denoting (x', v') as (x, v) for convenience, we have

$$\begin{aligned} &\iint_{\Omega'} \int_0^T \int_0^t \frac{|\tilde{V}^\delta_{s,s} - v|^2 \, |\tilde{V}^\delta_{t,s} - V_{t-s}|^2}{|\delta|^4 + \left(\int_0^t |\tilde{V}^\delta_{r,s} - V_{r-s}|^2 \, dr \right)^2} \\ &\qquad \le \iint_{\Omega''} \int_0^T \frac{|V_t - V^\delta_t|^2 \int_0^t |V_s - V^\delta_s|^2 \, ds}{|\delta|^4 + \left(\int_0^t |V_s - V^\delta_s|^2 \, ds \right)^2} \, dt \, dx \, dv \\ &\qquad = \frac{1}{2} \iint_{\Omega''} \log \left(\frac{|\delta|^4 + \left(\int_0^T |V_s - V^\delta_s|^2 \, ds \right)^2}{|\delta|^4} \right) dx \, dv \\ &\qquad \le C \log(1/|\delta|). \end{aligned}$$

Next, similarly as in the computation leading to (4.11), we have

$$\begin{aligned} \int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \, dz \, dv &\le C \int_{\mathbb{R}^d} \int_{\Omega_2'} \frac{|\tilde{\alpha}(k)|^2}{\frac{1}{|k|} + \left(\frac{k \cdot v}{|k|} \right)^2} \, dv \, dk \\ &\le C \int_{\mathbb{R}^d} |k| \, |\tilde{\alpha}(k)|^2 \int_{-\infty}^{+\infty} \frac{dv_1}{1 + |k| v_1^2} \, dk \\ &\le C \int_{\mathbb{R}^d} |k|^{1/2} |\tilde{\alpha}(k)|^2 \, dk \\ &\le C(\log 1/|\delta|)^{-2-4a} \int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2 \, dk. \end{aligned}$$

The combination of these inequalities finally yields

$$|\mathrm{IV}(T)| \le C \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2 \, dk \right)^{1/2}$$

if $|\delta| < 1/e$.

4.4.6 Upper bound for $|\mathrm{III}(T)|$

As before, we compute

$$|\mathrm{III}(T)| \le C \int_0^T \iint_{\Omega'} \int_0^t \int_{\mathbb{R}^d} \frac{|\tilde{V}^\delta_{s,s} - v|\, |\tilde{V}^\delta_{t,s} - V_{t-s}|}{|\delta|^2 + \int_0^t |\tilde{V}^\delta_{r,s} - V_{r-s}|^2\, dr} \frac{\tilde{\psi}^{(1)}_s\, |\tilde{\chi}'_s|\, \|G_{\mathrm{IV}}(v,z)\|}{|\delta|^{4/3}|z-x|^{d-1}|}\, dz\, ds\, dx\, dv\, dt.$$

Then, proceeding as in (4.12),

$$|\mathrm{III}(T)| \le \frac{C}{|\delta|^{4/3}} \left(\int_{\Omega'_2} \int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \int_{B(z,2|\delta|^{4/3})} \frac{dx}{|z-x|^{d-1}}\, dz\, dv \right)^{1/2}$$
$$\left(\iint_{\Omega'} \int_0^T \int_0^t \frac{|\tilde{V}^\delta_{s,s} - v|^2\, |\tilde{V}^\delta_{t,s} - V_{t-s}|^2}{|\delta|^4 + \left(\int_0^t |\tilde{V}^\delta_{r,s} - V_{r-s}|^2\, dr \right)^2} \right.$$
$$\left. \mathbb{I}_{\{|\tilde{X}^\delta_{s,s} - x| \le 2|\delta|^{4/3}\}} \int_{x + K(\tilde{X}^\delta_{s,s} - x)} \frac{dz}{|z-x|^{d-1}}\, ds\, dt\, dx\, dv \right)^{1/2},$$

so that

$$|\mathrm{III}(T)| \le C(\log 1/|\delta|)^{1/2} \left(\int_{\Omega'_2} \int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \right)^{1/2}$$

where we have used that $|z-x| \le |\tilde{X}^\delta_{s,s} - x| \le 2|\delta|^{4/3}$ when $\tilde{\psi}^{(1)}_s\, |\tilde{\chi}'_s| \neq 0$.

Finally,

$$|\mathrm{III}(T)| \le C \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2\, dk \right)^{1/2}$$

if $|\delta| < 1/e$.

4.4.7 Upper bound for $|\mathrm{I}(T)|$ and $|\mathrm{II}(T)|$

We only detail the computation of a bound for $|\mathrm{I}(T)|$. The case of $|\mathrm{II}(T)|$ is very similar and is left to the reader.

We compute as before

$$|\mathrm{I}(T)| \leq C \int_0^T \iint_{\Omega'} \int_{\mathbb{R}^d} \frac{|\tilde{V}_{t,t}^\delta - v|}{\left(|\delta|^2 + \int_0^t |\tilde{V}_{r,t}^\delta - V_{r-t}|^2 \, dr\right)^{1/2}} \frac{\tilde{\chi}_s \, \tilde{\psi}_s^{(1)} \, \|G_{\mathrm{IV}}(v,z)\|}{|z-x|^{d-1}\big(|\delta| + |\tilde{X}_{t,t}^\delta - x|\big)} \, dz \, dx \, dv \, dt.$$

Next, the computation is very similar to (4.12):

$$|\mathrm{I}(T)| \leq C \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \int_{\Omega_1'} \frac{dx}{|z-x|^{d-1}(|\delta| + |z-x|)} \, dz \, dv \right)^{1/2} \left(\iint_{\Omega'} \int_0^T \frac{|\tilde{V}_{t,t}^\delta - v|^2}{|\delta|^2 + \int_0^t |\tilde{V}_{r,t}^\delta - V_{r-t}|^2 \, dr} \frac{1}{|\tilde{X}_{t,t}^\delta - x|} \int_{x+K(\tilde{X}_{t,t}^\delta - x)} \frac{dz}{|z-x|^{d-1}} \, dt \, dx \, dv \right)^{1/2}.$$

Proceeding as before

$$|\mathrm{I}(T)| \leq C (\log 1/|\delta|)^{1/2} \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \right)^{1/2} \left(\iint_{\Omega''} \int_0^T \frac{|V_t - V_t^\delta|^2 \, dt}{|\delta|^2 + \int_0^t |V_r - V_r^\delta|^2 \, dr} \, dx \, dv \right)^{1/2},$$

so that eventually

$$\begin{aligned} |\mathrm{I}(T)| &\leq C \log(1/|\delta|) \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{\mathrm{IV}}(v,z)\|^2 \right)^{1/2} \\ &\leq C \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2 \, dk \right)^{1/2}. \end{aligned}$$

This completes the proof that

$$|B_\delta^{11}(T)| \leq C \left(1 + (\log |1/|\delta|)^{1-2a}\right) \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\alpha(k)|^2 \, dk \right)^{1/2},$$

where

$$B_\delta^{11}(T) := \int_0^T \iint_\Omega \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \int_0^t \chi_s \, B_\delta^{11}(s,x,v) \, ds \, dx \, dv \, dt.$$

4.4.8 Upper bound for $|B_\delta^{12}(T)|$

Let us define

$$B_\delta^{12}(T) := \int_0^T \iint_\Omega \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \int_0^t \chi_s \, B_\delta^{12}(s,x,v) \, ds \, dx \, dv \, dt.$$

As will appear below, this term is very similar to $\mathrm{V}(T)$.

We apply the same method as before, *without integrating by parts in time*:

$$|B_\delta^{12}(T)| \leq C \int_0^T \iint_{\Omega'} \int_0^t \int_{\mathbb{R}^d} \frac{|\tilde{V}_{t,s}^\delta - V_{t-s}|}{\left(|\delta|^2 + \int_0^t |\tilde{V}_{r,s}^\delta - V_{r-s}|^2 \, dr\right)^{1/2}} \frac{\tilde{\psi}_s^{(1)} \, \|G_{12}(v,z)\|}{|z-x|^{d-1}(|\delta| + |\tilde{X}_{s,s}^\delta - x|)} \, dz \, ds \, dx \, dv \, dt$$

where

$$G_{12}(v,z) = \int_{\mathbb{R}^d} \frac{k}{|k|^{1/2}} \otimes \frac{\tilde{\alpha}(k) \, e^{ik\cdot z}}{|k|^{-1/2} + i \frac{k}{|k|} \cdot v} \, dk.$$

Again,

$$\begin{aligned}
|B_\delta^{12}(T)| &\leq C \log(1/|\delta|) \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \|G_{12}(v,z)\|^2 \, dz \, dv \right)^{1/2} \\
&\leq C \log(1/|\delta|) \left(\int_{\Omega_2'} \int_{\mathbb{R}^d} \frac{|k| \, |\tilde{\alpha}(k)|^2}{\frac{1}{|k|} + \left(\frac{k\cdot v}{|k|}\right)^2} \, dk \, dv \right)^{1/2} \\
&\leq C \log(1/|\delta|) \left(\int_{\mathbb{R}^d} |k|^{3/2} |\tilde{\alpha}(k)|^2 \, dk \right)^{1/2} \\
&\leq C (\log 1/|\delta|)^{1-2a} \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\tilde{\alpha}(k)|^2 \, dk \right)^{1/2}.
\end{aligned}$$

4.4.9 Conclusion

Combining all the previous inequalities, we obtain that

$$|B_\delta^1(T)| \leq C \left(1 + (\log 1/|\delta|)^{1-2a}\right) \left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\tilde{\alpha}(k)|^2 \, dk \right)^{1/2}$$

where

$$B_\delta^1(T) := \int_0^T \iint_\Omega \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \int_0^t \chi_s\, B_\delta^1(s,x,v)\, ds\, dx\, dv\, dt.$$

Now, we observe from (4.7) that $B^2(s,x,v)$ has exactly the same structure as $B^1(s,x,v)$: a singularity of order $d-1$ in z, a function $\psi^{(2)}$ that has all the required regularity, and a term $e^{ik\cdot(X_s^\delta+z)}$. It is then easy to see that this term can be treated by exactly the same method as $B^1(s,x,v)$. We leave the details to the reader.

From this follows that

$$|B_\delta(T)| \le C\left(1 + (\log 1/|\delta|)^{1-2a}\right)\left(\int_{\mathbb{R}^d} |k|^{\frac{3}{2}+2a} |\tilde{\alpha}(k)|^2\, dk\right)^{1/2}$$

where

$$B_\delta(T) := \int_0^T \iint_\Omega \frac{V_t - V_t^\delta}{A_\delta(t,x,v)} \cdot \int_0^t \chi_s\, B_\delta(s,x,v)\, ds\, dx\, dv\, dt.$$

Finally, the term $C_\delta(s,x,v)$ of (4.4) can be bounded exactly as $B_\delta(s,x,v)$ by simply exchanging the roles of X_s and X_s^δ. Therefore, the proof of Proposition 2.1 is completed.

References

[1] M. Aizenman, On vector fields as generators of flows: A counterexample to Nelson's conjecture. *Ann. Math.* (2) **107** (1978), pp. 287–296.

[2] L. Ambrosio, Transport equation and Cauchy problem for *BV* vector fields. *Invent. Math.* **158**, 227–260 (2004).

[3] L. Ambrosio, G. Crippa, Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields. Lecture notes of the Unione Matematica Italiana, Springer Verlag, to appear.

[4] L. Ambrosio, C. De Lellis, J. Malý, On the chain rule for the divergence of vector fields: applications, partial results, open problems, Perspectives in nonlinear partial differential equations, 31–67, *Contemp. Math.*, **446**, Amer. Math. Soc., Providence, RI, 2007.

[5] L. Ambrosio, M. Lecumberry, S. Maniglia, Lipschitz regularity and approximate differentiability of the DiPerna-Lions flow. *Rend. Sem. Mat. Univ. Padova* **114** (2005), 29–50.

[6] F. Bouchut, Renormalized solutions to the Vlasov equation with coefficients of bounded variation. *Arch. Ration. Mech. Anal.* **157** (2001), pp. 75–90.

[7] F. Bouchut, G. Crippa, Uniqueness, renormalization, and smooth approximations for linear transport equations. *SIAM J. Math. Anal.* **38** (2006), no. 4, 1316–1328.

[8] F. Bouchut, L. Desvillettes, On two-dimensional Hamiltonian transport equations with continuous coefficients. *Diff. Int. Eq.* (8) **14** (2001), 1015–1024.

[9] F. Bouchut, F. James, One dimensional transport equation with discontinuous coefficients. *Nonlinear Anal.* **32** (1998), 891–933.

[10] A. Bressan, An ill posed Cauchy problem for a hyperbolic system in two space dimensions. *Rend. Sem. Mat. Univ. Padova* **110** (2003), 103–117.

[11] F. Colombini, G. Crippa, J. Rauch, A note on two-dimensional transport with bounded divergence. *Comm. Partial Differential Equations* **31** (2006), 1109–1115.

[12] F. Colombini, N. Lerner, Uniqueness of continuous solutions for BV vector fields. *Duke Math. J.* **111** (2002), 357–384.

[13] F. Colombini, N. Lerner, Uniqueness of L^∞ solutions for a class of conormal BV vector fields. Geometric analysis of PDE and several complex variables, 133–156, *Contemp. Math.* **368**, Amer. Math. Soc., Providence, RI, 2005.

[14] F. Colombini, J. Rauch, Uniqueness in the Cauchy problem for transport in $\mathbb{R}^2$ and $\mathbb{R}^{1+2}$. *J. Differential Equations* **211** (2005), no. 1, 162–167.

[15] G. Crippa, The ordinary differential equation with non-Lipschitz vector fields. *Boll. Unione Mat. Ital.* (9) **1** (2008), no. 2, 333–348.

[16] G. Crippa, C. DeLellis, Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math.* **616** (2008), 15–46.

[17] C. De Lellis, Notes on hyperbolic systems of conservation laws and transport equations. Handbook of differential equations, Evolutionary equations, Vol. 3 (2007).

[18] N. De Pauw, Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan. *C.R. Math. Sci. Acad. Paris* **337** (2003), 249–252.

[19] R.J. DiPerna, P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** (1989), 511–547.

[20] R. DiPerna, P.L. Lions and Y. Meyer, L^p regularity of velocity averages. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **8** (1991), 271–287.

[21] F. Golse, P.L. Lions, B. Perthame and R. Sentis, Regularity of the moments of the solution of a transport equation. *J. Funct. Anal.*, **26** (1988), 110-125.

[22] M. Hauray, On two-dimensional Hamiltonian transport equations with $\mathbb{L}^p_{loc}$ coefficients. *Ann. IHP. Anal. Non Lin.* (4) **20** (2003), 625–644.

[23] M. Hauray, On Liouville transport equation with force field in $BV_{\rm loc}$. *Comm. Partial Differential Equations* **29** (2004), no. 1-2, 207–217.

[24] M. Hauray, C. Le Bris, P.L. Lions, Deux remarques sur les flots généralisés d'équations différentielles ordinaires. *C. R. Math. Acad. Sci. Paris* **344** (2007), no. 12, 759–764.

[25] P.E. Jabin, Differential Equations with singular fields. Preprint.

[26] P.E. Jabin, Averaging Lemmas and Dispersion Estimates for kinetic equations. To appear *Riv. Mat. Univ. Parma.*

[27] C. Le Bris, P.L. Lions, Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications. *Ann. Mat. Pura Appl.* **183** (2004), 97–130.

[28] P.L. Lions, Sur les équations différentielles ordinaires et les équations de transport. *C. R. Acad. Sci., Paris, Sr. I, Math.* **326** (1998), 833–838.

[29] E.M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. *Princeton University Press, Princeton, NJ*, 1993.