Large time concentrations for solutions to kinetic equations with energy dissipation.

Pierre-Emmanuel Jabin

Ecole Normale Suprieure
Dpartement de Mathmatiques et d'Informatique
45 rue d'Ulm
75230 Paris cedex 05, France

Abstract. We consider the solutions to a kinetic equation which kinetic energy converges to zero fast enough. We prove that they concentrate near the speed zero and converge towards a measure which is a product of a measure on the spacial coordinates and a Dirac mass on the speed coordinates. The difficult point here is that the full solution converges since we do not know any characterisation of the limit problem for the spatial density. We give two results of this kind, depending on the regularity of the solution, and on the assumptions. Finally we present an example of equation which describes the interactions of particles in a flow and where these theorems apply.

Rsum. Nous démontrons ici que si une solution d'une quelconque équation cinétique a une énergie cinétique qui tend vers zero, alors toute la masse se concentre autour des vitesses nulles. Plus prcisément la solution admet une limite en temps grand qui se décompose en un produit d'une mesure sur les coordonnées spatiales et d'une masse de Dirac sur les coordonnées en vitesse. Nous détaillerons deux théorèmes avec leurs conditions d'application selon la régularité de la solution considérée et nous donnerons un exemple d'équation auquel ces théorèmes peuvent s'appliquer.

Key-words. Kinetic equation. Long time asymptotic. Vlasov equation. Systems of particles in a fluid.
AMS Classification. 35B40, 35B45, 35Q35, 76D07.

Introduction

This paper is devoted to the study of the qualitative properties for large times of the solutions to some nonlinear kinetic equation with friction terms. The specificity of these equations is a a very fast decay of the kinetic energy. As a consequence the solution concentrates near the speed zero and the speed of particles decreases until the particles are frozen.

We will consider equations of the general form

$$(1)\quad \begin{cases} \dfrac{\partial}{\partial t} f + \mathrm{div}_x(vf) + \mathrm{div}_v\big(F[f]\big) \; = \; 0 \;, \quad t \geq 0,\; x \in \mathbb{R}^{\mathrm{d}},\; \mathrm{v} \in \mathbb{R}^{\mathrm{d}} \;, \\ f(t=0,x,v) \; = \; f^0(x,v) \geq 0 \;. \end{cases}$$

This kind of equations appears in the problem of particles in a Stokes flow in dimension three. We consider the dynamics of a great number of spherical particles moving in an incompressible fluid that satisfies the Stokes equation, which means we only retain the viscous effects. With some approximation which consists mainly in supposing that the global interaction can be decomposed into two particles interactions, we can obtain such an equation for the distribution function of the particles. In this case, with some reasonnable assumptions detailed in 3. we are able to guarantee an enough decay of the kinetic energy. Some mathematical problems of particles moving in a incompressible and perfect fluid have already been studied by G. Russo and P. Smereka (see [11]) and H. Herrero, B. Lucquin-Desreux and B. Perthame (see [7]) but in that case the long time behaviour is unclear.

In the case of the viscous suspension the term $F[f]$ can be expressed as $(A \star_x j - \lambda v)f$ for some matrix A, a non-negative constant λ. Hence (1) becomes in this situation

$$(2)\quad \begin{cases} \dfrac{\partial}{\partial t} f + \mathrm{div}_x(vf) + \mathrm{div}_v\big((A \star_x j - \lambda v)f\big) \; = \; 0 \;, \quad t \geq 0,\; x, v \in \mathbb{R}^3 \;, \\ j \; = \; \displaystyle\int vf dv \;, \\ f(t=0,x,v) \; = \; f^0(x,v) \geq 0 \;. \end{cases}$$

It turns out that in this case we can give a precise description of the long term behaviour : provided the kinetic energy of $f(t,.,.)$ converges to zero when t goes to infinity and provided this energy belongs to $L^{\frac{1}{2}}([0,\ \infty])$ then the solution f to equation (1) converges to a distribution $\bar{\rho}(x)\delta_0(v)$ in a weak sense (in the distributional sense or in the weak topology of Radon measures depending of the assumptions we make). One of the main difficulty we solve here is to prove that the full solution, and not only subsequences, converges in large time because we are unable to derive an equation for $\bar{\rho}$.

Another classical example for (1) is the Vlasov-Poisson equation with friction where, in dimension three,

$$\begin{cases} F[f] = (\nabla_x(\frac{4\pi}{|x|}) \star \rho - \lambda v) f \ , \\ \rho = \int_{\mathbb{R}^3} f dv \ . \end{cases}$$

When $\lambda = 0$ it is known that for this equation all energy becomes kinetic energy in large times (see [10]). However the situation is completely different for $\lambda > 0$ and we do not have any precise informations about the behaviour of the kinetic energy of the solution. Thus we are not sure that this energy will tend to zero in large times and the results presented here cannot apply.

Some results concerning the limit in large times of solutions of kinetic equations posed in the whole space have already been proved. F. Bouchut and J. Dolbeault have thus shown that the solution of a Vlasov-Fokker-Planck equation converges towards the equilibrium state (see [4]). A precise decay of solutions to the Boltzmann equation can be found in [2], and for renormalized solutions in [10]. D. Benedetto, E. Caglioti and M. Pulvirenti in [3] have also studied a simplified model whose solution converges to a Dirac mass. The main difference between these results and our own is that we are not able to precise exactly the limit which depends in general on the parameters of the equation and on the initial data.

The first part of this paper will be devoted to precise exactly the assumptions, especially on the initial data, and the kind of convergence we get. The theorems written in this first part will be proved in the second part. We will deal with equation (2) in the third part and give there some existence results that ensure that with, some conditions on A and f^0, we can apply our theorems to the solution of this equation.

1. Main results

We will always consider non-negative initial data and assume it satisfies the two following conditions

$$f^0(x,v) \ \in \ L^1 \cap L^\infty \ (\mathbb{R}^{2\mathrm{d}}) \ , \tag{3}$$

$$E^0 \ = \ \int_{\mathbb{R}^{2\mathrm{d}}} |v|^2 f^0 dv dx \ < +\infty \ . \tag{4}$$

We will completely ignore for the time being the question of existence of solutions to such a system and we therefore admit that there exists a non-negative solution, in the distributional sense, at least in $L^\infty([0\ T], L^1 \cap L^\infty(\mathbb{R}^{2\mathrm{d}}))$ for all T and that this solution satisfies the natural energy estimate

$$E(t) \ = \ \int_{\mathbb{R}^{2\mathrm{d}}} |v|^2 f(t,x,v) dv dx \ \le \ E^0 \ . \tag{5}$$

We consider operators F continuous from the space of functions in $L^1(\mathbb{R}^{2\mathrm{d}}, (1+|\mathrm{v}|^2)\mathrm{dxdv})$ and in $L^\infty(\mathbb{R}^{2\mathrm{d}})$ to some $L^p(\mathbb{R}^{2\mathrm{d}})$, $1 \le p \le \infty$. A natural additionnal condition on F is that for all $f \in L^1 \cap L^\infty(\mathbb{R}^{2\mathrm{d}})$ with finite energy, $vF[f]$ belongs to $L^1(\mathbb{R}^{2\mathrm{d}})$ with continuity, and that the non-linear term $F[f]$ contributes to a decay of the kinetic energy :

$$\int_{\mathbb{R}^{2\mathrm{d}}} vF[f](x) f(t,x,v) dv dx \le 0 \ , \ \forall t > 0 \ . \tag{6}$$

Here we do not need such a strong condition but only a consequence. Namely we will suppose that the kinetic energy vanishes fast enough, that means the square root of the kinetic energy is L^1 in time

$$E(t) \ \longrightarrow \ 0 \text{ as } t \to +\infty \ , \tag{7}$$

$$\sqrt{E(t)} \ = \ \left(\int_{\mathbb{R}^{2\mathrm{d}}} |v|^2 f(t,x,v) dv dx \right)^{\frac{1}{2}} \ \in \ L^1([0, \ +\infty]) \ . \tag{8}$$

It is nevertheless interesting to consider regular solutions : if we want to deal with *classical solutions* we will impose

$$f \ \in \ C^1 \left([0, \ T], L^1 \cap C^1(\mathbb{R}^{2\mathrm{d}})\right) \ \forall \, T > 0 \ . \tag{9}$$

Condition (9) is rather restrictive, thus we will use also another one

$$f, F[f] \ \in \ C\left([0, \ T], L^1(\mathbb{R}^{2\mathrm{d}})\right) \ \forall \, T > 0 \ . \tag{10}$$

Condition (10) allows us to speak directly of the limit of f but is not enough alone. We will need another asumption

$$(11) \quad \begin{cases} \textit{There are sequences } f_m^0,\ f_m,\ F_m[f_m] \textit{ solution to } (1), \textit{ which} \\ \textit{satisfy } (3)-(5),\ (7)-(9), \textit{ such that } \rho_m \rightharpoonup \rho \ w - M^1(\mathrm{I\!R}^{\mathrm{d}}) \\ \textit{for a.e } t \textit{ and } \sup_m e_m(t) \to 0 \textit{ as } t \to \infty, \end{cases}$$

where we have defined

$$e_m(t) \ = \ \int_t^\infty \sqrt{E_m(s)} ds \ .$$

Also notice an elementary estimate which we will use later on

$$(12) \qquad \int_{\mathrm{I\!R}^{2\mathrm{d}}} f(t,x,v) dv dx \ \leq \ \int_{\mathrm{I\!R}^{2\mathrm{d}}} f^0(x,v) dv dx \ .$$

It is immediately deduced from the conservative form of the equation.

Solutions satisfying (10)–(12) will be called *weak solutions*.

The finite energy assumption gives us estimates about the mass density $\rho(t,x) = \int_{\mathrm{I\!R}^{\mathrm{d}}} f dv$ and about the quantities $j^\alpha(t,x) = \int_{\mathrm{I\!R}^{\mathrm{d}}} |v|^\alpha f dv$ for $0 < \alpha < 2$ namely the usual

$$(13) \qquad \|\rho(t,.)\|_{L^1} \ \leq \ \|f(t,.,.)\|_{L^1} \ ,$$

$$(14) \qquad \|\rho(t,.)\|_{L^{\frac{d+2}{d}}} \ \leq \ C \|f(t,.,.)\|_{L^\infty}^{\frac{2}{d+2}} E(t)^{\frac{d}{d+2}} \ ,$$

$$(15) \qquad \|j^\alpha(t,.)\|_{L^1} \ \leq \ C \|f(t,.,.)\|_{L^1}^{1-\frac{\alpha}{2}} \ E(t)^{\frac{\alpha}{2}} \ ,$$

$$(16) \qquad \|j^\alpha(t,.)\|_{L^{\frac{d+2}{d+\alpha}}} \ \leq \ C \|f(t,.,.)\|_{L^\infty}^{\frac{2-\alpha}{d+2}} E(t)^{\frac{d+\alpha}{d+2}} \ .$$

With our assumptions the only bounds (13) and (15) are uniform in time. However if we have

$$(17) \qquad \|f(t,.,.)\|_{L^\infty}^{\frac{2}{d+2}} \times E(t)^{\frac{d}{d+2}} \ \in \ L^\infty([0,+\infty]) \ ,$$

Then the bound (14) is also uniform in time and we can expect some weak convergence of ρ. The decay of the energy of f suggests that the solution concentrates in large times near the small velocities and indeed we are able to prove the following theorem :

Theorem 1 **:** *Under the assumptions* (3) – (5), (7), (8), (10), *a solution to the equation* (1) *obtained through* (11) *satisfies, as* t *tends to infinity*

$f(t,x,v) \rightharpoonup \bar{f}(x,v) = \bar{\rho}(x)\delta(v)$ *in* $w - M^1(\mathrm{I\!R}^{2d})$ *(weak topology on Radon measures) and*

$\rho(t,x) \rightharpoonup \bar{\rho}(x)$ *in* $w - M^1(\mathrm{I\!R}^{d})$, *and in* $w - L^p(\mathrm{I\!R}^{d})$ *for* $1 \leq p \leq \frac{d+2}{d}$ *if* (17) *holds.*

Theorem 2 **:** *Suppose* (3), *suppose also that* $f \in L^1 \cap L^\infty(\mathrm{I\!R}^{2d})$ *is a nonnegative solution to* (1), *in a distributional sense, which satisfies the bound* (12) *and that its mass density* ρ *satisfies the continuity equation*

$$\partial_t \rho + \mathrm{div} j = 0 , \tag{18}$$

and the two following conditions which generalize (7) *and* (8)

$$\lim_{t\to\infty} \int_{\mathrm{I\!R}^{2d}} |v|^2 \phi(x) f(t,x,v) dx dv = 0 , \forall \phi \in C_0^\infty(\mathrm{I\!R}^d) \text{ nonnegative,} \tag{7'}$$

$$\int_{\mathrm{I\!R}^{2d}} |v|^2 \phi(x) f(t,x,v) dx dv \in L^{\frac{1}{2}}([0, \infty]) , \forall \phi \in C_\infty(\mathrm{I\!R}^d) . \tag{8'}$$

Then there exists a Radon measure $\bar{\rho}$ *such that, when* t *tends to infinity,* $f(t,.,.)$ converges to $\bar{\rho}(x)\delta_0(v)$ in $w - M^1(\mathrm{I\!R}^{2d})$

$$\lim_{t\to\infty} \int_{\mathrm{I\!R}^{2d}} \psi(x,v) f(t,x,v) dv dx = \int_{\mathrm{I\!R}^{d}} \psi(x,0) \bar{\rho}(x) dx, \quad \forall \psi \in C_0(\mathrm{I\!R}^{2d}) .$$

Remarks.

1. The limit $\bar{\rho}$ depends heavily on the complete initial datum (not only on ρ^0 for example) and of course on the operator F. In fact this can be seen on a very simple example, let us suppose that $F = -\lambda v f$. Such an operator satisfies obviously all the conditions we have set and we know explicitely the solution :

$$f(t,x,v) = e^{d\lambda t} f^0(x - \frac{e^{\lambda t} - 1}{\lambda} v, \ e^{\lambda t} v) .$$

In this case the limit is

$$\bar{\rho}(x) = \int_{\mathrm{I\!R}^d} f^0(x - \frac{v}{\lambda}, \ v) \, dv .$$

2. Theorem 2 demands very few regularity on the solution compared with theorem 1. Notice that although f is not continuous in time here, $\int_{\mathbb{R}^{2d}} \psi(x,v) f(t,x,v) dv dx$ is continuous provided ψ is at least C^1 and compactly supported. If we want still less regularity on the solution we can replace f by $\int \Phi(t+s) f(s,.,.) ds$.

3. The important point in theorems 1 and 2 is that the full limit exists : the whole family $f(t,.,.)$ converges and not only sub-sequences. This relies on assumptions (8) or (8′).

4. We show in section 3 that the equation (2) enters in the case of theorem 1 for strong solutions and theorem 2 for weak solutions.

2. Proof of theorems 1 and 2.

First of all, we recall that ρ is bounded for all time and that all j^α converges to zero as $t \to \infty$ thanks to (7) and (8) and (15)

$$\|\rho(t,.)\|_{L^1} \ \le \ C \ ,$$
$$\|j^\alpha(t,.)\|_{L^1} \ \in \ L^1([0, \ +\infty]) \ .$$

As a consequence $\{f(t,.,.) \ |t \ge 0\}$ is weakly compact in $M^1(\mathbb{R}^{2d})$ and $\{\rho(t,.) \ |t \ge 0\}$ is weakly compact in $M^1(\mathbb{R}^{d})$.

We divide the proof of theorem 1 in two parts. First we will show the special form of the possible limits. Second we demonstrate the uniqueness of the limit. We then explain quickly how to prove theorem 2.

First Step. The form of the limit

Lemma 1 : *Consider a sequence of functions $f_n(.,.)$ uniformly bounded in $L^1(\mathbb{R}^{2d})$ which converges weakly to a measure $\bar f$ in $M^1(\mathbb{R}^{2d})$. Suppose the kinetic energy E_n of each f_n exists and converges to zero and set $\rho_n(x) = \int_{\mathbb{R}^d} f_n(x,v) dv$, then ρ_n converges weakly in $M^1(\mathbb{R}^d)$ to a $\bar\rho$ and $\bar f(x,v) = \bar\rho(x)\delta(v)$.*

Proof of lemma 1.

We show that, for all ϕ_1 and ϕ_2 in $C_0(\mathbb{R}^d)$, $\int_{\mathbb{R}^{2d}} \phi_1(x)\phi_2(v) f_n dx dv$ and $\phi_2(0)\int_{\mathbb{R}^d} \phi_1(x)\rho_n(x) dx$ can be made as close as we wish provided n is large enough. Indeed : for all $\epsilon > 0$, there exists η so that whatever $|v| < \eta$ we have

$$|\phi_2(v) - \phi_2(0)| \ < \ \frac{\epsilon}{2\|\phi_1\|_{L^\infty} . \sup_n \|\rho_n\|_{L^1}} \ .$$

Recalling now that E_n converges to 0, we choose N so that for all $n \ge N$, we have

$$2\eta^{-2}.\|\phi_1\|_{L^\infty}.\|\phi_2\|_{L^\infty}.E_n \;\leq\; \frac{\epsilon}{2}\;,$$

therefore for all $n \geq N$

$$\begin{aligned}\Delta^1(n) \;&=\; \left|\int_{\mathrm{I\!R}^{2\mathrm{d}}} \phi_1\phi_2 f_n dxdv - \phi_2(0)\int_{\mathrm{I\!R}^{\mathrm{d}}} \phi_1\rho_n dx\right| \\ &\leq\; \int_{\mathrm{I\!R}^{\mathrm{d}}} |\phi_1| \left(\int_{|v|<\eta} |\phi_2(v)-\phi_2(0)| f_n dv\right) dx \\ &\quad + \int_{\mathrm{I\!R}^{\mathrm{d}}} |\phi_1| \left(\int_{|v|\geq\eta} |\phi_2(v)-\phi_2(0)| f_n dv\right) dx\;.\end{aligned}$$

As a consequence we get

$$\begin{aligned}\Delta^1(n) \;&\leq\; \|\phi_1\|_{L^\infty}.\|\rho_n\|_{L^1}.\sup_{|v|<\eta}|\phi_2(v)-\phi_2(0)| \\ &\quad + \int_{\mathrm{I\!R}^{\mathrm{d}}} |\phi_1| \int_{|v|\geq\eta} |\phi_2(v)-\phi_2(0)|\frac{|v|^2}{\eta^2} f_n dvdx \\ &\leq\; \frac{\epsilon}{2} + \frac{2}{\eta^2}.\|\phi_1\|_{L^\infty}.\|\phi_2\|_{L^\infty}.E_n \;<\; \epsilon\;.\end{aligned}$$

Since $\int_{\mathrm{I\!R}^{2\mathrm{d}}} \phi_1(x)\phi_2(v)f_n(x,v)dxdv$ has a limit we deduce that the other integral $\phi_2(0)\int_{\mathrm{I\!R}^{\mathrm{d}}}\phi_1(x)\rho_n(x)dx$ also has one, which shows that ρ_n converges weakly towards some $\bar\rho$ in $M^1(\mathrm{I\!R}^{\mathrm{d}})$. The identity of the two limits demonstrates then that

$$\bar f(x,v) \;=\; \bar\rho(x).\delta(v)\;,$$

which ends the proof of the lemma. □

With the assumption (7) in theorem 1, this lemma allows to say that any weakly converging subsequence $f(t_n,.,.)$ converges to a $\bar\rho(x)\delta_0(v)$ and that we also have the weak convergence of $\rho(t_n,.)$ towards $\bar\rho$.

Finally, if we assume (17) we notice that $\rho(t_n,.)$ is weakly compact in all $L^p(\mathrm{I\!R}^{\mathrm{d}})$ with $1 < p \leq \frac{d+2}{d}$. Therefore every extracted sequence of $\rho(t_n,.)$ that converges weakly in a L^p converges towards the same limit which can only be $\bar\rho$. Hence $\bar\rho$ belongs to all L^p, $1 < p \leq \frac{d+2}{d}$, and $\rho(t_n,.)$ converges to this function weakly in these spaces without extraction. Of course $\bar\rho$ also belongs to L^1 with a norm less or equal to the norm of ρ which is constant.

Second step. Uniqueness of the limit

We now prove that the full sequence converges. By contradiction suppose that for two sequences, we have

$$\rho(t_n^1, x) \rightharpoonup \bar\rho_1(x) \ ,$$
$$\rho(t_n^2, x) \rightharpoonup \bar\rho_2(x) \ .$$

We prove that for a large class of functions ψ

$$\int_{\mathbb{R}^d} \bar\rho_1(x)\psi(x)dx = \int_{\mathbb{R}^d} \bar\rho_2(x)\psi(x)dx \ . \tag{19}$$

We use here condition (11). We thus get a sequence of solutions f_m which are strong solutions. For any ψ in $C_0^1(\mathbb{R}^d)$, we prove (18) as follows ; we first claim that

$$\begin{aligned} \Delta^2(n,m) &= \left| \int_{\mathbb{R}^d} \rho_m(t_n^1, x)\psi dx - \int_{\mathbb{R}^d} \rho_m(t_n^2, x)\psi dx \right| \\ &\le (\sup_{\mathbb{R}^d} |\nabla\psi|).\|f_m^0\|_{L^1}^{\frac{1}{2}}.e_m\big(min(t_n^1, t_n^2)\big) \ , \end{aligned} \tag{20}$$

first notice that since ρ_m and ψ are non-negative the above integrals are non-negative and we can take their square root.

For these solutions inequality (20) comes from the following simple calculation

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{R}^d} \rho_m(t,x)\psi(x)dx \right| &= \left| \int_{\mathbb{R}^{2d}} v\cdot\nabla\psi(x) f_m(t,x,v)dvdx \right| \\ &\le \left(\int_{\mathbb{R}^{2d}} |v|^2 f_m(t,x,v)dvdx \right)^{\frac{1}{2}} \times \\ &\quad \left(\int_{\mathbb{R}^d} |\nabla\psi|^2(x)\rho_m(t,x)dx \right)^{\frac{1}{2}} \\ &\le (\sup_{\mathbb{R}^d} |\nabla\psi|).\sqrt{E_m(t)}.\|f_m^0\|_{L^1}^{\frac{1}{2}} \ . \end{aligned}$$

We now use the fact that ρ_m converges weakly in $M^1(\mathbb{R}^d)$ at any fixed time to get that the left handside of (20) for ρ_m converges towards $\left|\int \rho(t_n^1, .)\psi dx - \int \rho(t_n^2, .)\psi dx\right|$. Condition (11) (the uniform convergence of e_m in fact) yields then that this is as small as we want provided t_n^1 and t_n^2 are large enough.

Thus (19) is true for any function ψ in $C_0^1(\mathbb{R}^d)$, a dense subspace of $C_0(\mathbb{R}^d)$, which proves that $\bar\rho_1 = \bar\rho_2$.

As a consequence, recalling the weak compactness of $\rho(t,.)$, we deduce that the full function $\rho(t,.)$ converges weakly towards some $\bar{\rho}(x)$ in $w-M^1(\mathbb{R}^d)$, and in all L^p with $1<p\leq\frac{d+2}{d}$ with assumption (17), as t tends to infinity.

And at last, assuming (17) and that we have strong solutions (if not we use again condition (11)), since ρ^0 belongs to L^1 there exists some function $\beta(|x|)$ such that $\beta\rho^0$ also belongs to L^1, and which satisfies that $\beta(|x|)$ converges to infinity when $|x|$ goes to infinity and $\nabla\beta$ is bounded.
Thanks to the equation (18) of mass conservation, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\beta(|x|)\rho dx &= \int_{\mathbb{R}^{2d}} f v\cdot\nabla_x\beta dvdx \\ &\leq \|\nabla\beta\|_{L^\infty}\sqrt{E(t)}.\|f(t,.,.)\|_{L^1}^{\frac{1}{2}}\end{aligned}$$

This shows that $\beta\bar{\rho}$ belongs to L^1 thanks to (8) and consequently that $\|\rho(t,.)\|_{L^1}$ has a limit which is the L^1 norm of $\bar{\rho}$. Thus $\rho(t,.)$ converges weakly also in L^1 which concludes the proof of theorem 1.

Indications on the proof of theorem 2.

We do not prove here theorem 2 since its proof follows exactly that of theorem 1. Consider first $\int\Phi(t+s)f(s,.,.)ds$ for any non negative function Φ in C_0^∞ with $\int\Phi(t)dt=1$ and replace f by this integral.

The first step of the proof of theorem 1 is done the same way with $\int\Phi(t_n+s)f(s,.,.)ds$ instead of $f(t_n,.,.)$ and test functions instead of continuous and compactly supported functions thanks to assumption ($7'$). We thus obtain that any converging subsequence of $\int\Phi(t+s)f(s,.,.)ds$ converges to some $\bar{\rho}(x)\delta_0(v)$.

Also, the second step is not more difficult for $\int\Phi(t+s)f(s,.,.)ds$ than it was previously for $f(t,.,.)$ the only assumptions (18) and ($8'$) are enough. This step proves that for any Φ non negative in $C_0^\infty(R)$, $\int\Phi(t+s)f(s,.,.)ds$ converges in distributional sense towards a $\bar{\rho}(x)\delta_0(v)$ and that $\bar{\rho}$ does not depend on Φ.

Now, letting first Φ converge to a Dirac mass in time and then the test functions ϕ to any continuous and compactly supported function, we get theorem 2.

3. An example of application

We have sofar presented some general results about the limit in large time of solutions of kinetic equations. In this section we detail the precise example (2), and we explain how we can apply theorem 1 or 2 to this particular case.

3.1 Presentation of the equation and main result

Let us consider the specific example (2) of equation of the form (1) which corresponds to the problem of the suspension of particles in a viscous fluid (see Jabin and Perthame [9] and the references therein for a derivation of these models).

Here A is a given matrix, $A:\ \mathbb{R}^{\mathrm{d}} \to \mathrm{M_d}(\mathbb{R})$. We now let $F[f] = (A \star_x j - \lambda v) f$ in (1). We will moreover assume, for instance, that for some real numbers α and C we have

$$(21) \qquad |A(x)| \ \le \frac{C}{|x|^\alpha}, \quad \forall\, x \in \mathbb{R}^{\mathrm{d}}\ .$$

The force term in equation (2) is formed of $A \star_x j$ which represents the interaction between particles and of $-\lambda v$ which represents a friction effect and which is responsible for the decay of the kinetic energy. Indeed this kinetic energy satisfies for strong solutions of (2)

$$(22) \quad E(t) \ = \ 2\int_0^t \int_{\mathbb{R}^{\mathrm{d}}} j(s,x)\cdot (A \star_x j)(s,x) dx ds - 2\lambda \int_0^t E(s) ds + E^0\ .$$

In general we do not know the sign of $\int j \cdot (A \star_x j) dx$ and thus we do not get any decay of the kinetic energy. However it is natural to consider the following case

$$(23) \qquad \begin{cases} A(x) \ = \ A^1(x) + A^2(x) \ with \\ \hat{A}^1 \ \le \ 0\ , \\ \|A^2\|_{L^\infty} \ \le \ \mu < \lambda\ , \end{cases}$$

where $\hat{g}$ represents the Fourier transform of g. Let now ν be defined as

$$(24) \qquad \nu \ = \ 2(\lambda - \mu)\ ,$$

with the condition (23), we have the following inequality on the energy

$$(25) \qquad E(t) \ \le \ E^0 - \nu \int_0^t E(s) ds,\ E(t) \ \le \ E^0 e^{-\nu t}\ .$$

Since (25) is a stronger inequality than (7) or (8), we will assume only condition (22) and we will prove in the next paragraph that (25) and hence (7) and (8) stands true.

We prove in subsections 3.2 and 3.3 the following theorem

Theorem 3 :

Assume (23) *and* (21) *with* $0 \leq \alpha \leq 4d/(d+2)$ *and that the initial data* f^0 *satisfies assumptions* (3) *and* (4)*, then the equation* (2) *has a solution* $f \in L^\infty([0,\ T], L^1 \cap L^\infty(\mathbb{R}^{\mathrm{d}}))$ *in distributional sense (and the non-linear term* $A \star_x j$ *belongs to some* $L^\infty([0,\ T], L^q(\mathbb{R}^{\mathrm{d}}))$ *). This solution satisfies* (5)*,* (7)*,* (8) *and the condition of continuity in time* (10)*. Since it is obtained through* (11) *theorem 1 applies and* f *converges to a distribution* $\bar{\rho}(x)\delta_0(v)$ *in* $w - M^1(\mathbb{R}^{2\mathrm{d}})$*.*

3.2 Existence of solutions

In order to state rigorous results on the equation (2), we give in this section an existence result of weak solutions which satisfy the properties required in section 1. The results are obtained with classical methods close to those used for Vlasov-Poisson (see results by A.A. Arsenev in [1], E. Horst in [8] or by R.J. DiPerna and P.L. Lions in [6]). Consequently we only sketch the proofs.

Let us begin with

Proposition 1 :

Suppose A *belongs to* $L^\infty(\mathbb{R}^{\mathrm{d}})$*, is locally lipschitz and the initial data* f^0 *is non-negative, compactly supported and belongs to* $L^1 \cap L^\infty(\mathbb{R}^{2\mathrm{d}})$ *then equation* (2) *has a unique, global, non-negative solution* $f \in C([\,0,\ \infty\,[\,; L^1)$ *in the distributional sense and we have the following estimates :*

(26) $\|f\|_{L^\infty([0,\ +\infty], L^1(\mathbb{R}^{\mathrm{d}}))} \ = \ \|f^0\|_{L^1(\mathbb{R}^{\mathrm{d}})}$,

(27) $\|f(t,.,.)\|_{L^\infty(\mathbb{R}^{2\mathrm{d}})} \ \leq \ e^{d\lambda t}\|f^0\|_{L^\infty(\mathbb{R}^{\mathrm{d}})}$, *for a.e.* $t > 0$.

Additionally if the initial data has finite energy E^0 *then for a.e.* $t > 0$ *the equality* (22) *holds true.*

A possible proof consists in approximating the initial data with some $\frac{\alpha}{N}\sum_{i=1}^N \delta(x - X_i)\delta(v - V_i)$. For this new initial datum our equation has a unique solution since the equation is equivalent to a dynamic system which can be solved by the Cauchy-Lipschitz theorem. The three estimates are then standard results for transport equations with a field regular enough, which is the case here. Notice also that the solution is continuous in time, that we can put equality in those estimates and more generally that almost any kind of regularity for the initial data propagates, assuming a corresponding regularity for A : if f^0 and A are C^n for example then for any time $f(t,.,.)$ is also C^n.

The proposition 2 is a first step towards the construction of weak solutions under weaker assumptions, especially on A.

We now have the following

Proposition 2 :

We suppose (23)*,* (21) *with* $0 \leq \alpha < d$ *and the initial data* f^0 *satisfies* (3)*,* (4)*, then the system* (2) *has a non-negative solution in distributional sense* $f \in L^\infty([0,\ T], L^1 \cap L^\infty(\mathbb{R}^d))$ *with* $j \in L^\infty([0,\ T], L^p(\mathbb{R}^d))$*,* $A \star_x j \in L^\infty([0,\ T], L^q(\mathbb{R}^d))$ *for* $1 \leq p \leq (d+2)/(d+1)$ *and* $1/q = 1/p + \alpha/d - 1$*. Also* f *satisfies the* L^1 *bound* (12)*, the* L^∞ *bound* (27) *and the energy decay* (25)*.*

Notice here that the bounds on the energy (25) and (27) and the L^∞ norm of f imply that $j(t,.)$ belongs to $L^1 \cap L^{\frac{d+2}{d+1}}$ for almost every t. So, if these bounds are true, assumption (21) and standards results in harmonic analysis imply that $A \star_x j$ belongs to some $L^p(\mathbb{R}^d)$. The non-linear term $(A \star_x j)f$ and consequently the system (2) have thus a precise signification for $(A \star_x j)f$ belongs at least to $L^1_{loc}(\mathbb{R}^{2d})$.

We can obtain this theorem by taking regular and converging approximations of A and of f^0. Then we apply proposition 1 and extract a sub-sequence f_n of the solutions that converges weakly towards a limit f which belongs to $L^\infty([0,\ T], L^1 \cap L^\infty(\mathbb{R}^d))$. Then (12), (27) and (25) are true for each solution f_n of the approximated system and since these estimates do not depend on n and are lower semi-continuous so they are also true for the limit f.

We only have to show that f is a solution of (2), any f_n is already a solution of (2) for approximated f_n^0 and A_n. Since f_n converges weakly towards f, all the linear terms of (2) approximated converge towards the corresponding linear terms of (2). To get the conclusion we only have to prove that $(A_n \star_x j_n)f_n$ converges weakly towards $(A \star_x j)f$ in some $L^r([0,\ T], L^s_{loc}(\mathbb{R}^{2d}))$ and indeed $A_n \star_x j_n$ converges strongly in some $L^p(\mathbb{R}^d)$ towards $A \star_x j$ because A_n converges strongly towards A and j_n converges weakly towards j, extracting one more sub-sequence if necessary. Thus the only difficulty is to pass from a weak convergence at each fixed time to a weak convergence in time. This is done the same way as for Vlasov-Poisson system, which ends the proof.

Remark :

The proof of proposition 2 is somehow important in itself for it proves that we have a sequel of strong solutions of an approximated equation which converges (weakly in all L^p for $1 < p < \infty$) towards a weak solution of the exact equation. As a consequence we do have condition (11).

We now present some conditions on A that imply the continuity of f and thus end the proof of theorem 3.. R.J. DiPerna and P.L. Lions proved a similar result for Vlasov-Poisson system (see [6]).

Proposition 3 :

Assume (23)*,* (21) *with* $0 \leq \alpha \leq 4d/(d+2)$*, that the initial data is non-negative and belongs to* $L^1 \cap L^\infty(\mathbb{R}^{2d})$ *and that its kinetic energy is finite (conditions* (3) *and* (4) *). Then the system* (2) *has a distributional solution* $f \in L^\infty([0,\ T], L^1 \cap L^\infty(\mathbb{R}^{2d}))$*, which satisfies conditions* (5)*,* (7)*,* (8) *(its kinetic energy is finite at all time and decays fast enough) and* (10) *and* (11) *(it is continuous in time and is the limit of regular solutions).*

Proof of proposition 3.

Proposition 3 only adds to proposition 2 the time continuity of f. Thus we only have to prove this continuity, which is condition (10). We first use a result due to R.J. DiPerna and P.L. Lions (see [5])

Lemma 2 :

Let $b \in L^1([0,\ T],\ (W^{1,p}_{loc}(\mathbb{R}^n)))$ *with* c*,div* $b \in L^1([0,\ T],\ L^\infty(\mathbb{R}^n))$*, and consider* $u \in L^1([0,\ T],\ L^q_{loc}(\mathbb{R}^n))$*, with* $p \geq q$*, a solution of*

$$\frac{\partial}{\partial t}u + b \cdot \nabla u \ = \ c \ ,$$

then u *belongs to* $C([0,\ T],\ L^1_{loc}(\mathbb{R}^n))$*.*

Letting $n = 2d$, $b = (v,\ A \star_x j - \lambda v)$ and $u = f$, this lemma shows that f is continuous from $[0,\ T]$ to $L^1_{loc}(\mathbb{R}^{2d})$.

Now we only have to get some control on the integral of f outside a bounded domain in x and v. Indeed if ϕ_R and ψ_R are any functions of $C^\infty_0(R^d)$

$$\begin{aligned}\partial_t \int_{\mathbb{R}^{2d}} f\phi_R(x)\psi_R(v)dvdx \ &= \ -\int_{\mathbb{R}^{2d}} \nabla_x\phi_R(x)\cdot v\psi_R(v)fdvdx \\ &\quad -\int_{\mathbb{R}^{2d}} \phi_R(A\star_x j - \lambda v)f\cdot\nabla_v\psi_R dvdx \ .\end{aligned}$$

We have also

$$\int_{\mathbb{R}^{2d}} \nabla_x\phi_R(x)\cdot v\psi_R(v)fdvdx \ \leq \ \|j\|_{L^1}\|\nabla\phi_R\|_{L^\infty}\|\psi_R\|_{L^\infty} \ ,$$

and

$$\int_{\mathbb{R}^{2d}} \phi_R A \star_x j f \cdot \nabla_v \psi_R dv dx \ \leq \ \|\phi_R\|_{L^\infty} \|A \star_x j\|_{L^r} \| \int_{\mathbb{R}^d} f |\nabla_v \psi_R| dv \|_{L^{r'}} \ ,$$

with, choosing $\nabla\psi_R$ supported in $B(0,2R) - B(0,R)$ and less than C/R with a fixed constant C,

$$\begin{aligned} \int_{\mathbb{R}^d} f |\nabla_v \psi| dv \ &\leq \ \|f\|_{L^\infty}^{\frac{3}{d+2}} \left(\int_{\mathbb{R}^d} |v|^2 f dv \right)^{\frac{d-1}{d+2}} \times \\ & \qquad \left(\int_{\mathbb{R}^d} \frac{1}{R^{2(d-1)/3}} |\nabla_v \psi_R|^{\frac{d+2}{d}} dv \right)^{\frac{3}{d+2}} \\ &\leq \ K \left(\int_{\mathbb{R}^d} |v|^2 f dv \right)^{\frac{d-1}{d+2}} \ . \end{aligned}$$

Thus we are able to take any r' less than $(d+2)/(d-1)$, so any r less than $(d+2)/3$ and $A \star_x j$ belongs to such an L^r for any α less or equal than $4d/(d+2)$.

Consequently if $0 \leq \alpha \leq 4d/(d+2)$, for any $\phi_R \in C_0^\infty(\mathbb{R}^d)$ and any $\psi_R \in C_0^\infty(\mathbb{R}^d)$ such that $\nabla\psi_R$ is supported in $B(0,2R) - B(0,R)$ and less than C/R, we have

$$\|\partial_t \int_{\mathbb{R}^{2d}} f^p \phi_R(x) \psi_R(v) dv dx\|_{L^\infty([0,\ T]} \leq K(\|\phi\|_{W^{1,\infty}}, \|\psi\|_{W^{1,\infty}}) \ . \tag{29}$$

From this we can deduce that the integrals of $f(t,.,.)$ for $|x| > R$ and $|v| > R$ are uniformously small in time. With lemma 3, this is enough to prove that f belongs to $C([0,\ T],\ L^1(R^{2d}))$ which ends the proof of proposition 3.

□

ACKNOWLEDGEMENTS

I am grateful to F. Bouchut, a fruitful talk with him is at the origine of most of part 2. I also want to thank B. Perthame whose constant advices have much helped me.

REFERENCES

[1] A.A. Arsenev. Global existence of a weak solution of Vlasov's system of equations. U.S.S.R. Comp. Math. and Math. Phys. 15, (1975), 131–141.

[2] N. Bellomo, Palcezwski, and G. Toscani. Mathematical topics in kinetic theories. World Scientific, Singapore (1988).

[3] D. Benedetto, E. Caglioti and M. Pulvirenti. A kinetic equation for granular media. Modl. Math. Anal. Numr. 31, (1997) 615–641.

[4] F. Bouchut and J. Dolbeault. On long time asymptotics of the Vlasov-Fokker-Planck equation and of the Vlasov-Poisson-Fokker-Planck system with coulombic and newtonian potentials. Diff. Eq. 8, (1995) 487–514.

[5] R.J. DiPerna and P.L. Lions. Ordinary differential eqautions, transport theory and Sobolev spaces. Invent. Math. 98, (1989), 511–547.

[6] R.J. DiPerna and P.L. Lions. Solutions globales d'quations du type Vlasov-Poisson. C.R. Acad. Sci. Paris Sr. I, 307, (1988), 655–658.

[7] H. Herrero, B. Lucquin-Desreux and B. Perthame. On the motion of dispersed bubbles in a potential flow. Publications du Laboratoire d'Analyse Numrique, Universit Pierre et Marie Curie, R 97021.

[8] E. Horst. On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation. Math. Meth. in the Appl. Sci. 3, (1981) 229–248.

[9] P-E Jabin and B. Perthame. Work in preparation.

[10] B. Perthame. Time decay, propagation of low moments and dispersive effects for kinetic equations. Comm. P.D.E. 21, (1996) 659–686.

[11] G. Russo and P. Smereka. Kinetic theory for bubbly flow I and II. SIAM J. Appl. Math. 56, (1996) 327–371.