Compactness in Ginzburg-Landau energy by kinetic averaging

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Abstract

We consider a Ginzburg-Landau energy for two dimensional divergence free fields appearing in the gradient theory of phase transition for instance. We prove that, as the relaxation parameter vanishes, families of such fields with finite energy are compact in $L^p(\Omega)$. Our proof is based on a kinetic interpretation of the entropies which were introduced by Desimone, Kohn, Müller and Otto. The so-called *kinetic averaging lemmas* allow to generalize their compactness results. Also the method yields a kinetic equation for the limit where the right handside is an unknown *kinetic defect bounded measure* from which we deduce some Sobolev regularity. This measure also satisfies some cancellation properties depending on its local regularity, which seem to indicate several level of singularities in the limit.

1 Introduction

This paper is concerned with the compactness, as the parameter ε vanishes, for divergence free functions in $I\!R^2$ with a finite Ginzburg-Landau energy. Namely, we consider functions $u_\varepsilon : \Omega \subset I\!R^2 \to I\!R^2$ (Ω a smooth bounded domain of $I\!R^2$) such that

$$(1.11)\qquad \begin{cases} \text{div } u_\varepsilon = 0 \quad (\textit{i.e. } u_\varepsilon \textit{ is a curl}), \\ \varepsilon \int_\Omega |Du_\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_\Omega \left|1 - |u_\varepsilon|^{2\alpha}\right|^p dx \le E_0. \end{cases}$$

This problem, and variants, arises in many physical applications such as thin films or in the gradient theory of phase transition (see A. Desimone *et al* [6] and the references therein).
Throughout this paper, we use the notation

$$(1.12)\qquad \boldsymbol{\chi}(\xi, u) = \mathbb{1}_{\{\xi \cdot u > 0\}}$$

We prove the

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Theorem 1.1 *Assume that $2\alpha p > 1$ and $0 < p \le 2$. Then the sequence u_ε is relatively compact in $\mathrm{L}^q(\Omega)$ for $1 \le q < 2\alpha p$. After extraction of a subsequence, the limit u is a divergence free field and it satisfies $|u| = 1$ and*

$$\xi \cdot \nabla_x \boldsymbol{\chi}(\xi, u) = \sum \Big(\frac{1}{\beta} \partial_{\xi_i} (|\xi|^2 \partial_{\xi_j} (m^\beta_{ikj} \xi_k)) - \frac{3+\beta}{\beta} \xi_j \partial_{\xi_i} (m^\beta_{jki} \xi_k) \Big) \tag{1.13}$$

where the sum is taken for i, j, k equal to $1, 2$ and β is any real number with $0 < \beta \le 1$, $\beta < 2\alpha p - 1$, $\beta \le \alpha p$, and the $m^\beta_{ijk}(\xi, x)$ are measures such that

$$\int_\Omega \sup_{\xi \in I\!R^2} |m^\beta_{ijk}(\xi, x)| \, dx \le E_0/2, \qquad \sum_i m^\beta_{iij} = 0, \qquad \sum_k \xi_k \nabla_\xi m^\beta_{ijk} = 0, \tag{1.14}$$

$$\int_{\Omega \times B(R)} |\nabla_\xi m^\beta_{ijk}(\xi, x)| \, dx \, d\xi \le C(R) \; E_0, \quad \text{for all } R > 0. \tag{1.15}$$

Some regularity for the limiting function u as well as cancellation properties of the kinetic defect measures m^β_{ijk} are also available, especially the strange phenomenon of the parametrisation by β can be relaxed in some cases.

Proposition 1.2 *Additionally, we have for any open subset $\mathcal{O}$ of Ω and any $r > 1$*
1) $u \in W^{s,q}$ for all $0 \le s < \frac{1}{5}$, $q < \frac{5}{3}$.
2) If ∇u_ε is uniformly bounded in $L^r(\mathcal{O})$, then $m^\beta_{ijk} = 0$ on $\mathcal{O}$.
3) If $\nabla_x |u_\varepsilon|$ is uniformly bounded in $L^r(\mathcal{O})$, then $\sum_j m^\beta_{ijj} = 0$.
4) If $|u_\varepsilon| \to 1$ in $L^\infty(\mathcal{O})$, then $\frac{1}{\beta} m^\beta_{ijk} = m_{ijk}$ does not depend on β, $\sum_j m^\beta_{ijj} = 0$ and

$$\xi \cdot \nabla_x \boldsymbol{\chi}(\xi, u) = \sum_{i,j,k} |\xi|^2 \xi_k \partial_{\xi_i} \partial_{\xi_j} m_{ikj} \text{ on } \mathcal{O}. \tag{1.16}$$

This theorem is motivated by the compactness result proved in [5] by A. Desimone, R.W. Kohn, S. Müller and F. Otto for a similar functional. Our method of proof is also motivated by that of [5], and by the remarkable analogy between the Ginzburg-Landau model (1.11) and scalar conservation laws through a family of entropies as noted in that paper. Here, we introduce the additional idea of using the kinetic formulation which allows simpler and stronger compactness arguments through kinetic averaging lemmas just as it does it for nonlinear hyperbolic systems.

This method allows improvements in two directions; the exponents α and p are broader than those in [5] although the cases $p > 2$ are still open, and also we give a piece of information on the limiting problem $\varepsilon \to 0$. From the equation satisfied by $\boldsymbol{\chi}(\xi, u(x))$ we deduce some Sobolev regularity (first point of the proposition) by a direct application of kinetic averaging lemmas to the transport equation derived in the theorem 1.1. Notice that our proof gives regularity for the limit u but not uniform bounds on u_ε itself. Also as in the case of scalar conservation laws or isentropic gas dynamics with $\gamma = 3$, (see P.L. Lions, B. Perthame and E. Tadmor [9], [10]) we do not expect that the exponents obtained are sharp. This regularity question, especially BV regularity, appears fundamental in the example of L. Ambrosio, C. De Lellis and C. Mantegazza [1]. Notice however that in 2d, by Sobolev injections, BV functions also belong to the Sobolev space $W^{s,p}$ for the limiting values of s and p. Another information appears in the proposition on the limit u; it satisfies the condition $\mathrm{div} u = 0$, and $|u| = 1$. In two space dimensions this also writes

$$u = \nabla^T \psi(x), \quad |\nabla \psi| = 1.$$

But, except maybe for convex domains, the singularities of this Eikonal equation are not described by the socalled *viscosity solutions* as the counterexample by W. Jin and R. V. Kohn [8] proves it. The singularities might better be described by a defect energy studied in P. Aviles and Y. Giga [2]. On the other hand, from the results of the above proposition, it seems that the singularities could be due either to singularities of the gradient, either to the fact that $|u|$ might vanish at some points as in the more classical theory in F. Béthuel, H. Brézis and F. Hélein [3]. In particular the dependency upon β only comes from this points where u_ε does vanish in the limit. However, pushing forward the comparison with scalar conservation laws, more information, e.g. a sign condition on the measures appearing in the right hand side of (1.13), might rise the uniqueness of the limiting function u, as in B. Perthame [11] ; of course boudary conditions should also be specified, which is not necessary here.

Notice that a variant of the above results also holds true for variants of the energy in (1.11) such as the case when the divergence free constraint is relaxed but it is imposed that $|u_\varepsilon| = 1$ which is studied in T. Rivière and S. Serfaty [13]. Especially some of the assumptions for compactness can be removed. From their study it seems that this case yields the viscosity solution in the limit. We also would like to point out another method for compactness for the minimizers of (1.11), based on SBV spaces and which can be found in [1].

The energy (1.11) is a simplified form of energies involved in phase transitions of ferromagnetic materials as is detailed in [6].

The outline of this paper is as follows. In the second section, we give a general representation formula, then we consider (section 3) a particular setting adapted to the present situation. In the section 4, we prove the compactness result and the formula for the limit. The statements of the Proposition are proved in the last section.

2 A general representation formula

LEMMA 2.1 *For any smooth function u defined on Ω, we have, in the sense of distribution in ξ,*

$$|u|\xi\cdot\nabla_x\boldsymbol{\chi}(\xi,u)+|\xi|^2\ \nabla_x|u|\cdot\nabla_\xi\boldsymbol{\chi}=|\xi|^2\frac{u}{|u|}\cdot\nabla_\xi\boldsymbol{\chi}\ \mathrm{div}\ u. \tag{2.21}$$

PROOF OF LEMMA 2.1: We first approximate $\boldsymbol{\chi}(\xi,u)$ by regular functions $\boldsymbol{\chi}_n(\xi\cdot u)$ such that $\boldsymbol{\chi}_n'(r)\to\delta(r)$ in $w-M^1(I\!R)$ (weak topology of Radon measures). As a consequence

$$\begin{array}{l}\nabla_\xi\boldsymbol{\chi}_n(\xi\cdot u)\longrightarrow\nabla_\xi\boldsymbol{\chi}(\xi,u)=\delta(\xi\cdot u)u,\\ \nabla_x\boldsymbol{\chi}_n(\xi\cdot u)\longrightarrow\nabla_x\boldsymbol{\chi}(\xi,u).\end{array} \tag{2.22}$$

We compute, with the notation $\zeta\cdot\nabla u\cdot\zeta=\sum_{i,j}\zeta_i\partial_i u_j\zeta_j$

$$\begin{aligned}|u|\xi\cdot\nabla_x\boldsymbol{\chi}_n(\xi\cdot u)\ &=|u|\boldsymbol{\chi}_n'(\xi\cdot u)\ \xi\cdot\nabla_x u\cdot\xi\\ &=-|\xi|^2\nabla_x|u|\nabla_\xi\boldsymbol{\chi}_n(\xi\cdot u)\\ &\quad+|\xi|^2|u|\boldsymbol{\chi}_n'(\xi\cdot u)\left(\frac{\xi}{|\xi|}\cdot\nabla u\cdot\frac{\xi}{|\xi|}+\frac{u}{|u|}\cdot\nabla u\cdot\frac{u}{|u|}\right).\end{aligned} \tag{2.23}$$

Now, we fix a point x and argue on this expression as a measure in ξ. Either $u(x)=0$ and all the terms in (2.23) vanish in the limit $n\to\infty$. Or

$u(x) \neq 0$ and we choose a basis for $I\!R^2_\xi$ such that $u = (|u|, 0)$. We have, in $w - M^1$,

(2.24)
$$\delta(\xi_1)\left[\frac{\xi}{|\xi|} \cdot \nabla u \cdot \frac{\xi}{|\xi|} + \frac{u}{|u|} \cdot \nabla u \cdot \frac{u}{|u|}\right] = \delta(\xi_1)(\partial_2 u_2 + \partial_1 u_1) = \delta(\xi_1)\mathrm{div} u.$$

Therefore, taking the limit $n \to \infty$ in (2.23), we obtain in $\mathcal{D}'(I\!R^2)$ for all x

$$|u|\xi \cdot \nabla_x \boldsymbol{\chi} = -|\xi|^2 \nabla_x |u| \cdot \nabla_\xi \boldsymbol{\chi} + |\xi|^2 |u| \delta(\xi \cdot u)\mathrm{div} u, \tag{2.25}$$

which is precisely the lemma. ∎

3 Kinetic equation for the Ginzburg-Landau functional

A variant of the lemma 2.1 is, under the same assumptions and with a divergence free condition

LEMMA 3.1 *For any $\beta > 0$, $u \in H^1(\Omega)$ with* div $u = 0$*, we have a.e. in x and in $\mathcal{D}'(I\!R^2_\xi)$ (recalling that $\boldsymbol{\chi} u/|u| = 0$ when $u = 0$)*

(3.31)
$$\begin{aligned}\xi \cdot \nabla_x(|u|^{1+\beta}\boldsymbol{\chi}) = \; & \tfrac{1}{\beta}\mathrm{div}_x\Big\{-\nabla_\xi\big[|\xi|^2|u|\boldsymbol{\chi}(|u|^\beta - 1)\big] \\ & +(3+\beta)\big[\boldsymbol{\chi}|u|\xi(|u|^\beta - 1)\big]\Big\} \\ & +\frac{1}{\beta}\nabla_\xi \otimes \nabla_\xi : \big[|\xi|^2 \boldsymbol{\chi}(|u|^\beta - 1)\frac{u}{|u|} \otimes \nabla_x(u \cdot \xi)\big] \\ & -\frac{3+\beta}{\beta}\mathrm{div}_\xi\big[\boldsymbol{\chi}\frac{u}{|u|}(|u|^\beta - 1)\xi \cdot \nabla u \cdot \xi\big] \\ & -\frac{2}{\beta}\mathrm{div}_\xi\big[\boldsymbol{\chi}(|u|^\beta - 1)(\xi \cdot \frac{u}{|u|})\nabla_x(u \cdot \xi)\big] \\ & +\frac{3+\beta}{\beta}\boldsymbol{\chi}(|u|^\beta - 1)\frac{u}{|u|} \cdot \nabla_x(u \cdot \xi).\end{aligned}$$

Remark. This lemma is a generalized kinetic version of the analogous identities proved in [5] for non-linear functionals of u. The left hand side of the formula is just the term representing the stationary, free transport of $|u|^{1+\beta}\boldsymbol{\chi}$.

PROOF OF LEMMA 3.1: We first prove the result for u smooth. Then, a density argument in H^1 concludes because, on the set where u vanishes, all the quantities in this expression vanish almost everywhere. So once the lemma is proved for any regularization u_n of u, we let u_n converge to u in H^1. Then, both the left hand and right hand sides converge in $\mathcal{D}'(\mathbb{R}^2)$ toward the above expressions in the lemma, since u_n converges toward u in every L^p, $u_n/|u_n|$ toward $u/|u|$ where u does not vanish, in every L^p, and ∇u_n toward ∇u in L^2.

We want to obtain a stationary kinetic equation with source terms of the form

$$\xi \cdot \nabla_x(|u|^{1+\beta}\boldsymbol{\chi}) = div_x A(u) + B(u), \tag{3.32}$$

with A and B such that $A(u_\varepsilon)$ is compact and $B(u_\varepsilon)$ is uniformly bounded in $L^1_x(H^{-m}_\xi)$ for some integer m, if u_ε is a sequence satisfying (1.11).

Of course, thanks to the lemma 2.1

$$\begin{aligned}\xi \cdot \nabla_x(|u|^{1+\beta}\boldsymbol{\chi}) &= (1+\beta)\boldsymbol{\chi}|u|^\beta \xi\cdot\nabla|u| + |u|^{1+\beta}\xi\cdot\nabla_x\boldsymbol{\chi}\\ &= \frac{1+\beta}{\beta}|u|\boldsymbol{\chi}\xi\cdot\nabla(|u|^\beta) - |\xi|^2|u|^\beta\nabla_x|u|\cdot\nabla_\xi\boldsymbol{\chi}\\ &= \frac{1+\beta}{\beta}|u|\boldsymbol{\chi}\xi\cdot\nabla(|u|^\beta - 1) - \frac{1}{\beta}|\xi|^2|u|\nabla_\xi\boldsymbol{\chi}\cdot\nabla(|u|^\beta - 1).\end{aligned} \tag{3.33}$$

We then factorize all derivatives

$$\begin{aligned}&\xi\cdot\nabla_x(|u|^{1+\beta}\boldsymbol{\chi}) = -\frac{1}{\beta}\mathrm{div}_x\nabla_\xi\left[|\xi|^2|u|\boldsymbol{\chi}(|u|^\beta - 1)\right]\\ &+\frac{1+\beta}{\beta}\mathrm{div}_x\left[\boldsymbol{\chi}|u|\xi(|u|^\beta-1)\right] + \frac{1}{\beta}\mathrm{div}_\xi\left[|\xi|^2|u|(|u|^\beta-1)\nabla_x\boldsymbol{\chi}\right]\\ &+\frac{1}{\beta}\mathrm{div}_\xi\left[|\xi|^2\boldsymbol{\chi}(|u|^\beta-1)\nabla|u|\right] + \frac{2}{\beta}\mathrm{div}_x\left[\boldsymbol{\chi}|u|\xi(|u|^\beta-1)\right]\\ &-\frac{3+\beta}{\beta}|u|(|u|^\beta-1)\nabla_x\boldsymbol{\chi}\cdot\xi - \frac{3+\beta}{\beta}\boldsymbol{\chi}(|u|^\beta-1)\xi\cdot\nabla|u|.\end{aligned} \tag{3.34}$$

Now, we transform the x derivatives of $\boldsymbol{\chi}$ into ξ derivatives with the formula

$$\nabla_x\boldsymbol{\chi} = \frac{u}{|u|^2}\cdot\nabla_\xi\boldsymbol{\chi}\nabla_x(u\cdot\xi). \tag{3.35}$$

Notice that the formula (3.35) is proved by regularizing $\boldsymbol{\chi}$ like in the lemma 2.1 and by noticing that formally

$$\begin{aligned}\frac{u}{|u|^2}\cdot\nabla_\xi\boldsymbol{\chi}\nabla_x(u\cdot\xi) &= \frac{u}{|u|^2}\cdot u\delta(\xi\cdot u)\nabla_x(u\cdot\xi) = \delta(\xi\cdot u)\nabla_x(u\cdot\xi)\\ &= \nabla_x\boldsymbol{\chi}.\end{aligned} \tag{3.36}$$

Next, the combination of formulas (3.34) and (3.35) gives
(3.37)

$$
\begin{aligned}
&\xi\cdot\nabla_x(|u|^{1+\beta}\boldsymbol{\chi}) = -\frac{1}{\beta}\mathrm{div}_x\nabla_\xi[|\xi|^2|u|(|u|^\beta-1)]\\
&+\frac{3+\beta}{\beta}\mathrm{div}_x[\boldsymbol{\chi}|u|\xi(|u|^\beta-1)]\\
&+\frac{1}{\beta}\nabla_\xi\otimes\nabla_\xi : [|\xi|^2\boldsymbol{\chi}(|u|^\beta-1)\frac{u}{|u|}\otimes\nabla_x(u\cdot\xi)]\\
&-\frac{2}{\beta}\mathrm{div}_\xi[(\xi\cdot\frac{u}{|u|})\boldsymbol{\chi}(|u|^\beta-1)\nabla_x(u\cdot\xi)]-\frac{1}{\beta}\mathrm{div}_\xi[|\xi|^2\boldsymbol{\chi}(|u|^\beta-1)\nabla|u|]\\
&+\frac{1}{\beta}\mathrm{div}_\xi[|\xi|^2\boldsymbol{\chi}(|u|^\beta-1)\nabla|u|]-\frac{3+\beta}{\beta}\mathrm{div}_\xi[\boldsymbol{\chi}\frac{u}{|u|}(|u|^\beta-1)\xi\cdot\nabla u\cdot\xi]\\
&+\frac{3+\beta}{\beta}\boldsymbol{\chi}(|u|^\beta-1)\frac{u}{|u|}\cdot\nabla_x(u\cdot\xi)+\frac{3+\beta}{\beta}\boldsymbol{\chi}(|u|^\beta-1)\xi\cdot\nabla|u|\\
&-\frac{3+\beta}{\beta}\boldsymbol{\chi}(|u|^\beta-1)\xi\cdot\nabla|u|.
\end{aligned}
$$

After simplification, we obtain the equation (3.31) as stated in the Lemma 3.1. ∎

4 Compactness and proof of Theorem 1.1

We are now ready to prove the Theorem 1.1. It uses several steps. First, we prove a weaker statement (compactness for quantities vanishing with $|u_\varepsilon|$). Then, we prove the compactness of the sequence $|u_\varepsilon|$ itself. Finally we derive the cancellation properties in (1.14).

Following [5], the basic estimate we use is the following straightforward consequence of (1.11)

$$
\begin{aligned}
&\|\nabla u_\varepsilon\|_{L^2(\Omega)}\,\|(1-|u_\varepsilon|^{2\alpha})^{p/2}\|_{L^2(\Omega)}\le E_0/2,\\
&|u_\varepsilon|\longrightarrow 1 \ \mathit{in}\ L^q(\Omega),\ 1\le q\le 2\alpha p.
\end{aligned}
\tag{4.41}
$$

4.1 A first compactness result

We begin with a simple direct application of averaging lemmas which yields a partial compactness result.

Lemma 4.1 *With the bounds (4.41) and for α, p and β as in the Theorem 1.1, a family of solutions $|u_\varepsilon|^{1+\beta}\boldsymbol{\chi}(u_\varepsilon,\xi)$ to (3.31) has compact moments $<|u_\varepsilon|^{1+\beta}\boldsymbol{\chi}_\varepsilon>=\int\psi(\xi)|u_\varepsilon|^{1+\beta}\boldsymbol{\chi}(u_\varepsilon,\xi)d\xi$ in $L^r(I\!R^2)$ for all $1\le r<2\alpha p/(1+\beta)$, for any smooth and compactly supported function ψ.*

Proof of Lemma 4.1: Our aim is to apply kinetic averaging lemmas to the equation (3.31). Since the transport equation contains a full space derivative in the right hand side, the only version we can apply is the one of B. Perthame and P.E. Souganidis [12] (see also F. Bouchut [4], and the references therein for further references on this theory).

After choosing a value for β, we first prove that for some $r > 1$

$$|u_\varepsilon|(|u_\varepsilon|^\beta - 1) \longrightarrow 0 \textit{ in } \mathrm{L}^r, \tag{4.42}$$

$$\|(|u_\varepsilon|^\beta - 1)\nabla u_\varepsilon\|_{\mathrm{L}^1(I\!\!R^2)} \le \frac{1}{2} E_0. \tag{4.43}$$

As for the condition (4.42), it is an obvious consequence of the second assumption in (4.41) and of the relation $1 + \beta < 2\alpha p$.

Next, we prove that for $\beta q \le 2\alpha p$, $0 < \beta \le 2\alpha$ and $q \ge p$,

$$\int_\Omega |1 - |u_\varepsilon|^\beta|^q dx \le \int_\Omega |1 - |u_\varepsilon|^{2\alpha}|^p dx. \tag{4.44}$$

Indeed, with these conditions on β and q, we know that for any nonnegative real x

$$|1 - x^\beta|^q \le |1 - x^{2\alpha}|^p. \tag{4.45}$$

Then, the inequality (4.44) with $q = 2$ implies that

$$\|1 - |u_\varepsilon|^\beta\|_{L^2(\Omega)} \le \|(1 - |u_\varepsilon|^{2\alpha})^{p/2}\|_{L^2(\Omega)}. \tag{4.46}$$

Combining this inequality with the first condition of (4.41), we obtain

$$\|\nabla u_\varepsilon(|u_\varepsilon|^\beta - 1)\|_{L^1(\Omega)} \le \frac{1}{2} E_0. \tag{4.47}$$

The bounds (4.42), (4.43) are thus proved.

The first estimate (4.44) has for consequence that for any bounded domain $\mathcal{O}$ in ξ, the terms $\boldsymbol{\chi}|u_\varepsilon|\xi(|u_\varepsilon|^\beta - 1)$ and $\boldsymbol{\chi}|\xi|^2|u_\varepsilon|(|u_\varepsilon|^\beta - 1)$ vanish in $L^r(\Omega \times \mathcal{O})$ with $1 < r < \infty$. We also deduce from the estimate (4.43) that any term of the form $\boldsymbol{\chi}(|u|^\beta - 1)\frac{u_i}{|u|}\partial_j u_k$ multiplied by a polynomial function in ξ is bounded uniformly in ε in $L^1(\Omega \times \mathcal{O})$.

Now considering the equation (3.31), we notice that the terms with the full derivative in x are compact and that all the other terms are bounded. Applying averaging lemmas of [12], we conclude the compactness stated in the lemma 4.1. ∎

4.2 The full compactness proof

The compactness of the moments of $|u_\varepsilon|^{1+\beta}\boldsymbol{\chi}(\xi,u_\varepsilon)$ implies the relative compactness of the sequence u_ε as it is proved in the next lemma.

LEMMA 4.2 *With the bound (4.41) and for $2\alpha p > 1$, $p \le 2$, the sequence u_ε is relatively compact. Moreover after extracting a subsequence which converges to u, then $\boldsymbol{\chi}(\xi,u_\varepsilon)$ converges toward $\boldsymbol{\chi}(\xi,u)$ in $L^q(\Omega\times B(0,R))$ for any $1 \le q < \infty$ and any R.*

PROOF: We apply the lemma 4.1. For any $\phi(\xi)$ compactly supported, the moments $\int |u_\varepsilon|^\beta \boldsymbol{\chi}(\xi,u_\varepsilon)\phi(\xi)d\xi$ are relatively compact in some L^{q_0}. Choosing $\phi = \xi\psi(\xi)$ with ψ compactly supported in an annulus, we find that the sequence $u_\varepsilon|u_\varepsilon|^\beta$ is relatively compact in $L^{q_0}(\Omega)$.

We denote u the weak limit of u_ε in $L^{2\alpha p}$ after extraction. According to the estimate (4.44) in the lemma 4.1, $|u_\varepsilon|^\beta$ converges to 1 in $L^{\alpha p/\beta}$. Since Ω is bounded, and since $\beta < 2\alpha p - 1$, the product $u_\varepsilon|u_\varepsilon|^\beta$ converges weakly toward u. From the strong convergence of $u_\varepsilon|u_\varepsilon|^\beta$, we deduce that $|u| = 1 = \lim |u_\varepsilon||u_\varepsilon|^\beta$. The space $L^{2\alpha p}$ being uniformly convex because of the condition $2\alpha p > 1$, the extracted sequence of u_ε converges strongly toward u. The original sequence u_ε is thus relatively compact.

We now prove that $\boldsymbol{\chi}(\xi,u_\varepsilon) \to \boldsymbol{\chi}(\xi,u)$ strongly. The difficulty comes from the lack of continuity of $\boldsymbol{\chi}$ for $u_\varepsilon \approx 0$. For any extracted sequence u_ε converging toward some u, the same kind of argument shows that the sequence $u_\varepsilon/|u_\varepsilon|$ also converges strongly toward u in any L^q with $q < \infty$. For any $\eta > 0$ and any $\xi < R$, if $\xi \cdot \frac{u_\varepsilon}{|u_\varepsilon|} > \eta$ and $\xi \cdot u < -\eta$ or the contrary, since $|u| = 1$ we have

$$\left|\frac{u_\varepsilon}{|u_\varepsilon|} - u\right| > \frac{\eta}{R}. \tag{4.48}$$

We can then estimate the measure μ_ξ of the set of points x such that the condition $\xi \cdot \frac{u_\varepsilon}{|u_\varepsilon|} > \eta$ and $\xi \cdot u < -\eta$ or the contrary is fulfilled since

$$\int \left|\frac{u_\varepsilon}{|u_\varepsilon|} - u\right| dx \ge \frac{\eta}{R}\mu_\xi. \tag{4.49}$$

If we denote μ the measure of all couples (ξ, x) such that the previous condition is fulfilled, we have the following bound

$$\mu \le \frac{R^3}{\eta} \int \left|\frac{u_\varepsilon}{|u_\varepsilon|} - u\right| dx. \tag{4.410}$$

We also denote ν the mesure of the set of all couples (ξ, x) such that either $\left|\xi \cdot \frac{u_\varepsilon}{|u_\varepsilon|}\right| \leq \eta$ or $|\xi \cdot u| \leq \eta$. For any given x, the set of all ξ which satisfy this condition measures at most $4R\eta$, so

$$\nu \leq 4|\Omega|R\eta. \tag{4.411}$$

For any $1 \leq q < \infty$, since $\boldsymbol{\chi}(\xi, u)$ is equal to 1 for $\xi \cdot u > 0$ and zero otherwise, we know that

$$\int_{\Omega \times B(0,R)} |\boldsymbol{\chi}(\xi, u_\varepsilon) - \boldsymbol{\chi}(\xi, u)|^q dx d\xi \leq \mu + \nu. \tag{4.412}$$

Using the estimates (4.410) and (4.411), we thus obtain

$$\int_{\Omega \times B(0,R)} |\boldsymbol{\chi}(\xi, u_\varepsilon) - \boldsymbol{\chi}(\xi, u)|^q dx d\xi \leq \frac{R^3}{\eta} \int \left| \frac{u_\varepsilon}{|u_\varepsilon|} - u \right| dx + 4|\Omega|R\eta. \tag{4.413}$$

Since $u_\varepsilon/|u_\varepsilon|$ converges toward u in L^1, we deduce that $\boldsymbol{\chi}(\xi, u_\varepsilon)$ converges toward $\boldsymbol{\chi}(\xi, u)$ in $L^q(\Omega \times B(0,R))$ for any $1 \leq q < \infty$ and any R. ∎

4.3 Bounds and cancellation properties

We are now able to end the proof of the theorem 1.1. Since the term $\boldsymbol{\chi}(\xi, u_\varepsilon(x))\ (|u_\varepsilon|^\beta - 1)\ \frac{u_{\varepsilon k}}{|u_\varepsilon|}\ \partial_i u_{\varepsilon j}$ is bounded in L^1, we introduce the measures defined by the weak limit in $M^1(I\!R^2)$

$$m^\beta_{ijk} = \text{w}-\lim\ \boldsymbol{\chi}(\xi, u_\varepsilon(x))\ (|u_\varepsilon|^\beta - 1)\ \frac{u_{\varepsilon k}}{|u_\varepsilon|}\ \partial_i u_{\varepsilon j}. \tag{4.414}$$

The inequality

$$|\ \boldsymbol{\chi}(\xi, u_\varepsilon(x))\ (|u_\varepsilon|^\beta - 1)\ \frac{u_{\varepsilon k}}{|u_\varepsilon|}\ \partial_i u_{\varepsilon j}\ | \leq |(|u_\varepsilon|^\beta - 1)\ \partial_i u_{\varepsilon j}\ |$$

together with (4.43) gives the first bound (1.14) on the measure m^β_{ijk}.

As for the second bound (1.15), it follows from the fact that the ξ derivative of $\boldsymbol{\chi}$ is a one directional Dirac mass (see (2.22)).

We take the limit in all the terms of the equation (3.31) satisfied by the u_ε and we obtain

$$\begin{aligned}\xi\cdot\boldsymbol{\chi}(\xi,u)= &\sum_{i,j,k}\Big(\tfrac{1}{\beta}\partial_{\xi_i}\partial_{\xi_j}(|\xi|^2 m^\beta_{ikj}\xi_k)-\tfrac{3+\beta}{\beta}\partial_{\xi_i}(m^\beta_{jki}\xi_j\xi_k)\\ &-\tfrac{2}{\beta}\partial_{\xi_i}(m^\beta_{ijk}\xi_j\xi_k)\Big)+\sum_{ij}\tfrac{3+\beta}{\beta}m^\beta_{iji}\xi_j,\end{aligned} \tag{4.415}$$

which is, after straightforward simplifications, the equation written in the theorem 1.1. From the divergence free condition on u_ε and the definition of m^β_{ijk}, we deduce

$$\sum_i m^\beta_{iij}=0. \tag{4.416}$$

Since

(4.417)
$$\sum_k \xi_k\nabla_\xi(\boldsymbol{\chi}(|u_\varepsilon|^\beta-1)\partial_i u_{\varepsilon j}\frac{u_{\varepsilon k}}{|u_\varepsilon|})=u\delta(u_\varepsilon\cdot\xi)(|u_\varepsilon|^\beta-1)\partial_i u_{\varepsilon j}\frac{u_\varepsilon\cdot\xi}{|u_\varepsilon|}=0,$$

we also find at the limit

$$\sum_k \xi_k\nabla_\xi m^\beta_{ijk}=0. \tag{4.418}$$

The proof of the theorem 1.1 is thus completed.

5 Proof of proposition 1.2

This section is devoted to the proof of the results stated in the Proposition 1.2. It is divided in three subsections where we first prove the regularizing effect, second the statements regarding the set where m vanishes (regularity) and finally the weaker cancellation due to the points where $|u_\varepsilon|$ may vanish.

5.1 Proof of point 1

The first point of the Proposition 1.2 is obtained noticing that $\boldsymbol{\chi}(\xi,u(x))$ also satisfies the kinetic equation

$$\xi\cdot\nabla_x\boldsymbol{\chi}(\xi,u)=\mathrm{div}_\xi G(\xi,x),$$

with G a locally bounded measure. This is a consequence of the equation (1.13) and of the regularity for m^{β}_{ijk} in (1.15) once β is a fixed admissible positive value. Then, the regularity statement is an immediate consequence of the Besov regularity for kinetic averaging proved in R.J. DiPerna, P.L. Lions and Y. Meyer [7]. We use the Theorem 3, p283, with $p = 2$, $m = 1$, $\tau > 0$ and close to 0, and $q = (\frac{2}{\tau})'$ (close to 1). And, by the strict inequalities on the parameters, we allowed a little loss on the parameters to work in the classical Sobolev spaces rather than in the Lorentz based Besov spaces. Notice that this parameters are better than those of the regularizing effects for isentropic gas dynamics which shares the same kinetic structure (but with a double ξ derivative in the right hand side) in an evolution version (see [10]). For scalar conservation laws, better regularity is still possible combining the kinetic averaging lemmas with the contraction property in L^1 (see [9]), an argument which is not available here.

5.2 Proof of points 2 and 3

We choose β small enough such that $\beta r^* < \alpha p$ with r^* being the conjugate exponent to r. The estimate (4.44) then implies that $|u_\varepsilon|^\beta - 1$ converges to zero in $L^{r^*}(\mathcal{O})$.

Consequently, if ∇u_ε is uniformly bounded in L^r

$$(|u_\varepsilon|^\beta - 1)\nabla u_\varepsilon \longrightarrow 0 \ in \ L^1(\mathcal{O}). \tag{5.51}$$

And since $\boldsymbol{\chi}$ is bounded in L^∞

$$m^{\beta}_{ijk} = 0 \ in \ L^1(\mathcal{O}). \tag{5.52}$$

Notice also that

$$\sum_j m^{\beta}_{ijj} = \frac{1}{2}\mathrm{w-}\lim \boldsymbol{\chi}(\xi, u_\varepsilon)(|u_\varepsilon|^\beta - 1)\partial_i |u_\varepsilon|. \tag{5.53}$$

So by the mean of the same argument, if $\nabla |u_\varepsilon|$ is uniformly bounded in $L^r(\mathcal{O})$ then

$$\sum_j m^{\beta}_{ijj} = 0. \tag{5.54}$$

5.3 Proof of point 4

We are going to show that on $\mathcal{O}$, $\frac{1}{\beta}m^\beta_{ijk}$ does not depend on β. More precisely, we are going to prove that

$$\frac{1}{\beta}m^\beta_{ijk} = m_{ijk} = \text{w}-\lim \boldsymbol{\chi}(|u_\varepsilon|-1)\partial_i u_{\varepsilon j} u_{\varepsilon k}. \tag{5.55}$$

First of all, of course $u_\varepsilon - u_\varepsilon/|u_\varepsilon|$ converges to zero in $L^\infty(\mathcal{O})$ so that on $\mathcal{O}$

$$m^\beta_{ijk} = \text{w}-\lim \boldsymbol{\chi}(|u_\varepsilon|^\beta - 1)\partial_i u_{\varepsilon j} u_{\varepsilon k}. \tag{5.56}$$

Now, notice that for some constant C

$$\Big| |u_\varepsilon|^\beta - 1 - \beta(|u_\varepsilon|-1)\Big| \le C|u_\varepsilon|^{\beta-1}(|u_\varepsilon|-1)^2. \tag{5.57}$$

Using the uniform convergence of $|u_\varepsilon|$, we then find

(5.58)
$$\left|\frac{1}{\beta}m^\beta_{ijk} - \text{w}-\lim(\boldsymbol{\chi}(|u_\varepsilon|-1)\partial_i u_{\varepsilon j} u_{\varepsilon k})\right| \le C\text{w}-\lim(|u_\varepsilon|-1)^2|\nabla u_\varepsilon|.$$

To prove (5.55), it remains only to use the lemma

LEMMA 5.1 *If $|u_\varepsilon|$ converges uniformly to 1 on $\mathcal{O}$,*

$$(|u_\varepsilon|-1)^2|\nabla u_\varepsilon| \longrightarrow 0 \text{ in } L^1(\mathcal{O}). \tag{5.59}$$

PROOF OF LEMMA 5.1: We have

$$\|(|u_\varepsilon|-1)^2\nabla u_\varepsilon\|_{L^1} \le \|\nabla u_\varepsilon\|_{L^2}\|(|u_\varepsilon|-1)\|_{L^2}\|(|u_\varepsilon|-1)\|_{L^\infty}, \tag{5.510}$$

and

(5.511)
$$\|(|u_\varepsilon|-1)\|_{L^2} \le \|(|u_\varepsilon|^\beta-1)\|_{L^2}\left\|\frac{|u_\varepsilon|-1}{|u_\varepsilon|^\beta-1}\right\|_{L^\infty} \le \frac{1}{\beta}\|(|u_\varepsilon|^\beta-1)\|_{L^2}.$$

We then obtain thanks to the estimate (4.41) and the bound (4.44)

$$\|(|u_\varepsilon|-1)^2\nabla u_\varepsilon\|_{L^1} \le \frac{C}{\beta}\|(|u_\varepsilon|-1)\|_{L^\infty}, \tag{5.512}$$

which proves the lemma. ∎

We now turn to the property $\sum_j m^\beta_{ijj}=0$. Using the formula (5.55), we find

$$\begin{aligned}\textstyle\sum_j m^\beta_{ijj} &= \beta \textstyle\sum_j m_{ijj} = \tfrac12 \mathrm{w-}\lim \boldsymbol{\chi}(|u_\varepsilon|-1)\partial_i |u_\varepsilon|^2\\ &= \mathrm{w-}\lim \boldsymbol{\chi}(|u_\varepsilon|-1)|u_\varepsilon|\partial_i(|u_\varepsilon|-1)\\ &= \mathrm{w-}\lim \boldsymbol{\chi}(|u_\varepsilon|-1)\partial_i(|u_\varepsilon|-1),\end{aligned} \tag{5.513}$$

thanks to the uniform convergence of $|u_\varepsilon|$. But of course

$$\begin{aligned}\boldsymbol{\chi}(|u_\varepsilon|-1)\partial_i(|u_\varepsilon|-1) &= \tfrac12 \boldsymbol{\chi}\partial_i(|u_\varepsilon|-1)^2 = \tfrac12\partial_i\big(\boldsymbol{\chi}(|u_\varepsilon|-1)^2\big)\\ &\quad -\tfrac12(|u_\varepsilon|-1)^2\partial_i\boldsymbol{\chi}.\end{aligned} \tag{5.514}$$

Applying again the formula (3.35), we end up with

$$\begin{aligned}\boldsymbol{\chi}(|u_\varepsilon|-1)\partial_i(|u_\varepsilon|-1) &= \tfrac12\partial_i\big(\boldsymbol{\chi}(|u_\varepsilon|-1)^2\big) + \tfrac12\boldsymbol{\chi}(|u_\varepsilon|-1)^2\tfrac{u_\varepsilon}{|u_\varepsilon|^2}\cdot\partial_i u_\varepsilon\\ &\quad -\tfrac12\mathrm{div}_\xi\big(\boldsymbol{\chi}(|u_\varepsilon|-1)^2\tfrac{u_\varepsilon}{|u_\varepsilon|^2}\partial_i(u_\varepsilon\cdot\xi)\big).\end{aligned} \tag{5.515}$$

The lemma 5.1 and the uniform convergence of $|u_\varepsilon|$ prove that all the terms in the right hand side converge to zero in the distributional sense. Therefore, we have shown that

$$\sum_{j=1,2} m^\beta_{ijj} = 0 \ \textit{on}\ \mathcal{O}. \tag{5.516}$$

It remains to use (5.55) and (5.516) to simplify the kinetic equation obtained in the theorem 1.1. Since $\frac1\beta m^\beta_{ijk}$ is independent of β, we know that on $\mathcal{O}$

$$\sum_{i,j,k}\xi_j\xi_k\partial_{\xi_i}m_{jki} + \sum_{i,j} m_{jii}\xi_j = 0, \tag{5.517}$$

which, together with (5.516), implies that

$$\sum_{i,j,k} \xi_j \xi_k \partial_{\xi_i} m_{jki} = 0. \tag{5.518}$$

As a consequence, we have on $\mathcal{O}$

$$\begin{aligned} \xi \cdot \nabla_x \boldsymbol{\chi}(\xi, u) &= \partial_{\xi_i} \Big(\textstyle\sum_{i,j,k} |\xi|^2 \xi_k \partial_{\xi_j} m_{ikj} + \sum_{i,j} |\xi|^2 m_{ijj} \Big) \\ &= \textstyle\sum_{i,j,k} |\xi|^2 \xi_k \partial_{\xi_i} \partial_{\xi_j} m_{ikj} + 2 \sum_{i,j,k} \xi_i \xi_k \partial_{\xi_j} m_{ikj} \\ &\quad + \textstyle\sum_{i,j} |\xi|^2 \partial_{\xi_j} m_{iij} \\ &= \textstyle\sum_{i,j,k} |\xi|^2 \xi_k \partial_{\xi_i} \partial_{\xi_j} m_{ikj}. \end{aligned} \tag{5.519}$$

This is the equation stated in proposition 1.2.

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