Averaging lemmas

P-E Jabin (University of Nice)

Presentation of the course

Kinetic equations : Transport equation in phase space, *i.e.* on $f(x,v)$ of x and v

$$\partial_t f + v\cdot\nabla_x f = g, \qquad t\geq 0,\ x,v\ \in\ \mathbb{R}^d.$$

As for hyperbolic equation, the solution cannot be more regular than the initial data or the right hand-side. But averages in velocity like

$$\rho(t,x)=\int_{\mathbb{R}^d} f(t,x,v)\,\phi(v)\,dv, \quad \phi\in C_c^\infty(\mathbb{R}^d),$$

usually are, the question being of course how much ?

Plan of the course

1. Introduction
2. L^2 framework
3. General L^p framework
4. One limit case : Averaging lemma with a full derivative
5. An example of application : Scalar Conservation Laws

Plan of the first course

1. Well posedness of the basic equation
2. The $1d$ case
3. Local equilibrium
4. Application to the Vlasov-Maxwell system

Well posedness of the basic equation

During most of this course, we deal with the simplest

$$\partial_t f + \alpha(v)\cdot\nabla_x f = g(t,x,v), \qquad t\in\mathbb{R}_+,\ x\in\mathbb{R}^d,\ v\in\omega, \quad (1)$$

where $\omega = \mathbb{R}^d$ or a subdomain ; Or with the stationary

$$\alpha(v)\cdot\nabla_x f = g(x,v), \qquad t\in\mathbb{R}_+,\ x\in O,\ v\in\omega, \qquad (2)$$

where O is open, regular in $\mathbb{R}^d$ and ω is usually rather the sphere S^{d-1}.

Of course (1) is really a particular case of (2) with

$$d \longrightarrow d+1, \quad x\longrightarrow (t,x), \quad \alpha(v)\longrightarrow (1,\alpha(v)).$$

The fundamental relation for solutions to (1) is

$$f(t_2,x,v) = f(t_1,x-\alpha(v)(t_2-t_1),v) + \int_0^{t_2-t_1} g(t_2-s,x-\alpha(v)s,v)\,ds, \forall t_1,\ t_2$$

or for solutions to (2)

$$f(x,v) = f(x-\alpha(v)\,t,v) + \int_0^t g(x-\alpha(v)\,s,v)\,ds, \forall t.$$

Those two formulas may be used to solve the equation but these are not unique so an initial data must be given

$$f(t=0,x,v) = f^0(x,v), \tag{3}$$

and for (2) the incoming value of f on the boundary must be specified

$$f(x,v) = f^{in}(x,v), \quad x \in \partial O, \alpha(v)\cdot\nu(x) \le 0, \tag{4}$$

where $\nu(x)$ is the outward normal to O at x.

With that the equation is solvable as per

Theorem

Let $f^0 \in \mathcal{D}'(\mathbb{R}^d \times \omega)$ and $g \in L^1_{loc}(\mathbb{R}_+,\ \mathcal{D}'(\mathbb{R}^d \times \omega))$. Then there is a unique solution in $L^1_{loc}(\mathbb{R}_+,\ \mathcal{D}'(\mathbb{R}^d \times \omega))$ to (1) with (3) in the sense of distribution given by

$$f(t,x,v) = f^0(x - \alpha(v)\, t, v) + \int_0^t g(t-s, x - \alpha(v)\, s, v)\, ds. \quad (5)$$

Note that if f solves (1) then for $\phi \in C_c^\infty(\mathbb{R}^d \times \omega)$

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \omega} f(t,x,v)\, \phi(x,v) \in L^1_{loc}(\mathbb{R}_+),$$

so f has a trace at $t = 0$ in the weak sense and (3) perfectly makes sense.

On the other hand, the modified equation, which we will frequently use,

$$\alpha(v) \cdot \nabla_x f + f = g, \quad x \in \mathbb{R}^d, \ v \in \omega, \tag{6}$$

is well posed in the whole $\mathbb{R}^d$ without the need for any boundary condition

Theorem

Let $g \in \mathcal{S}'(\mathbb{R}^d \times \omega)$, there exists a unique f in $\mathcal{S}'(\mathbb{R}^d \times \omega)$ solution to (6). It is given by

$$f(x, v) = \int_0^\infty g(x - \alpha(v)\, t, v)\, e^{-t}\, dt. \tag{7}$$

Notice that taking the Fourier transform in x of (6)

$$(i\, \alpha(v) \cdot \xi + 1)\hat{f} = \hat{g},$$

and of course $1 + i\alpha(v) \cdot \xi$ never vanishes contrary to $i\alpha(v) \cdot \xi$.

The $1d$ case

Let us study the easiest case, namely

$$v\partial_x f = g.$$

Of course away from $v \neq 0$, if $g \in L^p(\mathbb{R}^2)$ then $\partial_x f \in L^p(\mathbb{R}^2)$. But what if f or g do not vanish around $v = 0$. For instance

$$f(x, v) = \rho(x)\delta(v),$$

then of course, whatever ρ, in the sense of distribution

$$v\partial_x f = 0.$$

So clearly concentrations have to be avoided. Let us be more precise. Take $\phi \in C_c^\infty(\mathbb{R})$, define

$$\rho(x) = \int_{\mathbb{R}} \phi(v)\, f(x, v)\, dv.$$

And compute for a bounded interval I

$$\|\rho\|_{W^{k,p}(I)} = \int_I \int_I \frac{|\rho(x) - \rho(y)|^p}{|x-y|^{1+kp}}\, dx\, dy.$$

Using the equation

$$\begin{aligned}|\rho(x) - \rho(y)| &\leq \int_{\mathbb{R}} |f(x,v) - f(y,v)|\phi(v)\, dv \\ &\leq \int_{|v|<R} |f(x,v) - f(y,v)|\phi(v)\, dv + \int_{|v|>R} \ldots\end{aligned}$$

and this last term is bounded by

$$\begin{aligned}&\int_0^1 \int_{|v|>R} |x-y|\, |\partial_x f(\theta\, x + (1-\theta)\, y, v)|\, \phi(v)\, dv\, d\theta \\ &\leq \int_0^1 \int_{|v|>R} \frac{|x-y|}{|v|}\, |g(\theta\, x + (1-\theta)\, y, v)|\, \phi(v)\, d\theta\, dv \\ &\leq C\, \frac{|x-y|}{|R|^{1/p}} \left(\int_0^1 \int_{\mathbb{R}} |g(\theta\, x + (1-\theta)\, y, v)|^p\, \phi(v)\, dv\, d\theta \right)^{1/p}.\end{aligned}$$

As for the first it is simply bounded by

$$C\,R^{1-1/p}\,\left(\int_{\mathbb{R}} |f(x,v)-f(y,v)|^p\,\phi(v)\,dv\right)^{1/p}.$$

Minimizing in R, one gets

$$|\rho(x)-\rho(y)| \leq C\,|x-y|^{1-1/p}\,\left(\int_{\mathbb{R}} |f(x,v)-f(y,v)|^p\,\phi(v)\,dv\right)^{1/p^2}$$
$$\times\left(\int_0^1\int_{\mathbb{R}} |g(\theta\,x+(1-\theta)\,y,v)|^p\,\phi(v)\,dv\,d\theta\right)^{(1-1/p)/p}.$$

So for k such that $p-1>kp$ or $k<1-1/p$

$$\|\rho\|_{W^{k,p}(I)} \leq C\,\|f\|_{L^p_{loc}}^{1/p}\,\|g\|_{L^p_{loc}}^{1-1/p}.$$

Consider (2) in the special case

$$f(x,v)=\rho(x)\,M(v).$$

This is a simplification but provides many examples of optimality later on. Some remarks :
We have

$$M(v)\,\alpha(v)\cdot\nabla_x\rho(x)=g.$$

Write $g=M(v)\,h(x,v)$.
If h is regular, this gives some regularity for ρ but not necessarily in term of Sobolev spaces.
Notice first that some assumption is needed on α. Indeed if $\exists\xi\in S^{d-1}$ s.t.

$$|O|=|\{v\in\mathbb{R}^d\mid\alpha(v)\parallel\xi\}|\neq 0,$$

and if $supp\, M \subset O$ then it is only possible to deduce that

$$\xi \cdot \nabla_x \rho \in L^\infty.$$

Nothing can be said about the derivatives in the other directions. Even if $\alpha(v)$ is not concentrated like $\alpha(v) = \xi$, some assumption is needed on M. If not, M it self may be concentrated along one direction ξ in which case the same phenomenon occurs.

The Vlasov-Maxwell system

It describes the evolution of charged particles

$$\partial_t f + v(p)\cdot\nabla_x f + (E(t,x) + v(p)\times B(t,x))\cdot\nabla_p f = 0, \quad t \geq 0,\ x, p \in \mathbb{R}^d.$$

E and B are the electric and magnetic fields

$$\partial_t E - \operatorname{curl} B = -j, \quad \operatorname{div} E = \rho,$$
$$\partial_t B + \operatorname{curl} E = 0, \quad \operatorname{div} B = 0,$$

where ρ and j are the density and current of charges

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,p)\,dp, \quad j(t,x) = \int_{\mathbb{R}^d} v(p)\,f(t,x,p)\,dp.$$

Initial data are required

$$f(t=0,x,p)=f^0(x,p), \quad E(t=0,x)=E^0(x), \quad B(t=0,x)=B^0(x).$$

$p =$ impulsion of the particles.

$$v(p)=p, \qquad \text{classical case.}$$

$$v(p)=\frac{p}{(1+|p|^2)^{1/2}}, \qquad \text{relativistic case.}$$

Goal : Weak Stability. Given f_n solution to the system, show that

$$f_n \longrightarrow f, \quad \text{solution to the system.}$$

A priori estimates

$$\|f_n(t,.,.)\|_{L^p(\mathbb{R}^{2d})} \leq \|f^0\|_{L^p(\mathbb{R}^{2d})}, \quad \forall t \geq 0, \quad \forall p \in [1,\ \infty].$$

Conservation of energy

$$\int_{\mathbb{R}^{2d}} E(p) f_n(t,x,p)\, dx\, dp + \int_{\mathbb{R}^d} (|E_n(t,x)|^2 + |B_n(t,x)|^2)\, dx \leq \int_{\mathbb{R}^{2d}} E(p) f^0(x,p)\, dx\, dp + \int_{\mathbb{R}^d} (|E^0(x)|^2 + |B^0(x)|^2)\, dx.$$

with

classical $\quad E(p) = |p|^2/2, \qquad$ relativistic $\quad E(p) = (1+|p|^2)^{1/2}.$

Weak convergence

After extraction, one has

$$f_n \longrightarrow f, \quad \text{in } w - * \, L^\infty(\mathbb{R}_+,\ L^p(\mathbb{R}^{2d})), \quad \forall 1 \leq p \leq \infty,$$

and

$$E_n \longrightarrow E, \quad B_n \longrightarrow B, \quad \text{in } w - * \, L^\infty(\mathbb{R}_+,\ L^2(\mathbb{R}^d)).$$

Through interpolation, it is also possible to prove

$$\rho_n \longrightarrow \rho, \quad j_n \longrightarrow j, \quad \text{in } w - * \, L^\infty(\mathbb{R}_+,\ L^p(\mathbb{R}^d)), \quad \forall 1 \leq p \leq p_0.$$

Problem : How to pass to the limit in

$$(E_n(t,x) + v(p) \times B_n(t,x))\, f_n \ ?$$

The solution found by DiPerna and Lions uses averaging lemmas. Take $\phi \in \mathcal{D}(\mathbb{R}^{2d})$

$$\int_{\mathbb{R}^{2d}} E_n(t,x)\, f_n(t,x,p)\, \phi(x,p)\, dx\, dp$$
$$= \int_{\mathbb{R}^d} E_n(t,x) \int_{\mathbb{R}^d} f_n(t,x,p)\, \phi(x,p)\, dp\, dx,$$

and what is only needed is the compactness of moments of f_n like

$$\int_{\mathbb{R}^d} f_n(t,x,p)\, \phi(x,p)\, dp.$$

Notice that

$$\partial_t f_n + v(p)\cdot\nabla_x f_n = -\nabla_p\cdot((E_n + v(p)\times B_n)\, f_n) \in L^2_{loc}(\mathbb{R}_+\times\mathbb{R}^d,\ H^{-1}_{loc}(\mathbb{R}^d)),$$

uniformly in n.

Averaging lemmas then implies that

$$\int_{\mathbb{R}^d} f_n(t,x,p)\,\phi(x,p)\,dp \in H^{1/4}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d),$$

uniformly in n.

Therefore compactness holds and we can pass to the limit in all the terms.

Second course : L^2 framework

1. The result
2. A serious computation
3. Maybe a second serious computation (if I did not talk too much)

Regularity in L^2

Assume that

$$\alpha(v) \cdot \nabla_x f = g, \quad x \in \mathbb{R}^d,\ v \in \omega,$$

with

$$\forall \zeta \in S^{d-1},\ \forall \varepsilon \in \mathbb{R}_+, \qquad |\{v \in \omega\,;\ |\alpha(v) \cdot \zeta| < \varepsilon\}| \leq \varepsilon^\theta.$$

Then

Theorem

Assume $|\omega| < \infty$, that f and g belong to $L^2(\mathbb{R}^d \times \omega)$ then ρ defined through

$$\rho(x) = \int_\omega f(x, v)\, dv$$

belongs to $H^{\theta/2}(\mathbb{R}^d)$.

Following Bouchut, simply write

$$\alpha(v)\cdot\nabla_x f + f = f + g,$$

and get

$$\rho(x) = T\,f + T\,g,$$

with

$$T\,f(x) = \int_\omega \int_0^\infty f(x-\alpha(v)t,\ v)\,e^{-t}\,dt\,dv.$$

The aim is to determine k s.t. T is continuous from $L^2(\mathbb{R}^d\times\omega)$ to $H^k(\mathbb{R}^d)$. For further use, define

$$T_s\,f(x) = \int_\omega \int_0^\infty f(x-\alpha(v)t,\ v)\,t^{-s}\,e^{-t}\,dt\,dv.$$

The L^2 estimate is the core result for averaging lemmas from which almost all other can be deduced.

A remark

The dual operator of T is simply

$$T^* h(x,v) = \int_0^\infty h(x+\alpha(v)\,t)\,e^{-t}\,dt$$

and is related to the X-ray transform $X:\ \mathbb{R}^d \longrightarrow \mathbb{R}^d \times S^{d-1}$

$$X\,h(x,v) = \int_{-\infty}^{\infty} h(x+vt)\,dt.$$

This operator was studied separately in harmonic analysis (see for instance Christ, Duoandikoetxea and Oruetxebarria, Wolff) but with emphasis on mixed type inequalities like the continuity from $L^p(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d,\ L^p(S^{d-1}))$ and not on the gain of differentiability which is our main goal here. These other inequalities are nevertheless very useful and can be seen as a kind of dispersion estimates.

The proof by Fourier transform

The Fourier transform in x is denoted $\mathcal{F}$ and $\hat{f} = \mathcal{F}f$

$$\mathcal{F}f = \hat{f} = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\, f(x)\, \frac{dx}{(2\pi)^{d/2}}.$$

We recall that $\mathcal{F}$ is an isometry on $L^2(\mathbb{R}^d)$ and that the Sobolev space is

$$H^k(\mathbb{R}^d) = \left\{\rho \in \mathcal{S}'(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d} (1+|\xi|)^{2k}\, |\mathcal{F}\rho(\xi)|^2\, d\xi < \infty\right\}.$$

The homogeneous Sobolev space is simply

$$\dot{H}^k(\mathbb{R}^d) = \left\{\rho \in \mathcal{S}'(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d} |\xi|^{2k}\, |\mathcal{F}\rho(\xi)|^2\, d\xi < \infty\right\}.$$

We mainly follow bouchut here. Applying Fourier transform

$$\mathcal{F}\, T_s\, f = \int_{\omega} \mathcal{F}\, f(\xi, v)\ \int_0^{\infty} e^{-i\,t\,\alpha(v)\cdot\xi}\ \frac{e^{-t}}{t^s}\, dt\, dv.$$

This is simply equal to

$$\int_{\omega} \frac{\mathcal{F}\, f(\xi, v)}{1 + i\alpha(v)\cdot\xi}\, dv,$$

if $s = 0$.

Denote

$$\chi(z) = \int_0^{\infty} e^{-i\,t\,z}\ \frac{e^{-t}}{t^s}\, dt.$$

Notice that of course

$$|\chi(z)| \leq \int_0^\infty \frac{e^{-t}}{t^s}\,dt \leq C < \infty,$$

provided that $s < 1$.
This already gives that

$$|\mathcal{F}\,T_s\,f| \leq \int_\omega |\mathcal{F}\,f(\xi, v)|\,dv,$$

and thanks to Cauchy-Schwarz that

$$\int_{\mathbb{R}^d} |T_s\,f(x)|^2\,dx \leq |\omega| \int_{\mathbb{R}^d\times\omega} |f(x,v)|^2\,dx\,dv.$$

On the other hand, if $|z| \geq 1$, we have in addition

$$
\begin{aligned}
|\chi(z)| &\leq \left| \int_0^K t^{-s}\, dt \right| + \left| \int_K^\infty e^{-i\,t\,z}\, \frac{e^{-t}}{t^s}\, dt \right| \\
&\leq C\, K^{1-s} + \left| \frac{1}{z} \int_K^\infty e^{-t}\, |t^{-s} - s\, t^{-s-1}|\, dt \right| \\
&\leq CK^{1-s} + \frac{C}{|z|}\, K^{-s} \leq \frac{C}{|z|^{1-s}},
\end{aligned}
$$

through minimization in K. The combination of both yields

$$
|\chi(z)| \leq \frac{C}{1 + |z|^{1-s}}.
$$

Now by Cauchy-Schwarz, we have that

$$|\mathcal{F}\, T_s\, f|^2 \leq \int_\omega |\mathcal{F}\, f(\xi, v)|^2\, dv \int_\omega |\chi(\xi \cdot \alpha(v))|^2\, dv$$
$$\leq \int_\omega |\mathcal{F}\, f(\xi, v)|^2\, dv \int_\omega \frac{C}{1 + |\alpha(v) \cdot \xi|^{2-2s}}\, dv.$$

We recall that for all $\phi \in C^1(\mathbb{R})$

$$\int_\omega \phi\left(|\alpha(v) \cdot \xi|\right)\, dv = -\int_0^\infty \phi'(y)\, |\{v \in \omega\, ;\ |\alpha(v) \cdot \xi| < y\}|\, dy.$$

Recall that

$$\forall \zeta \in S^{d-1},\ \forall \varepsilon \in \mathbb{R}_+, \qquad |\{v \in \omega\, ;\ |\alpha(v) \cdot \zeta| < \varepsilon\}| \leq \varepsilon^\theta.$$

We obtain that

$$\int_\omega \frac{C}{1 + |\alpha(v) \cdot \xi|^{2-2s}}\, dv \leq \int_0^\infty \frac{C}{1 + |y|^{3-2s}}\, \frac{y^\theta}{|\xi|^\theta}\, dy \leq \frac{C}{|\xi|^\theta},$$

provided that $\theta - 3 + 2s < -1$. If $|\omega| < \infty$, this gives

$$\int_{\mathbb{R}^d} (1 + |\xi|)^\theta\, |\mathcal{F}\, T_s\, f|^2\, d\xi \leq C \int_{\mathbb{R}^d \times \omega} |f(x, v)|^2\, dx\, dv.$$

We have proved

Theorem

Assume $|\omega| < \infty$, *that*

$$\forall \zeta \in S^{d-1},\ \forall \varepsilon \in \mathbb{R}_+, \qquad |\{v \in \omega\,;\ |\alpha(v)\cdot\zeta| < \varepsilon\}| \leq \varepsilon^{\theta}.$$

and that $\theta + 2s < 2$ *then* T_s *is continuous from* $L^2(\mathbb{R}^d \times \omega)$ *to* $H^{\theta/2}(\mathbb{R}^d)$.

Averaging lemmas rely on orthogonality properties of T so that a direct proof is difficult. The method presented here uses instead a $T\,T^*$ argument and is taken from Vega, J.
For simplicity, we restrict ourselves to

$$\alpha(v) = v, \quad \omega = S^{d-1},$$

The dual of operator T is

$$T^* h(x,v) = \int_0^\infty h(x+vt)\, e^{-t}\, dt.$$

Then $T \ : \ L^2(\mathbb{R}^d \times S^{d-1}) \longrightarrow H^{1/2}$ equivalent to $T^* \ : \ H^{-1/2} \longrightarrow L^2(\mathbb{R}^d \times S^{d-1})$ or

$$T^* \ : \ L^2(\mathbb{R}^d) \longrightarrow L^2(S^{d-1},\ H^{1/2}(\mathbb{R}^d))$$

Denote by Δ_x^θ the differentiation operator

$$\Delta_x^\theta h = \mathcal{F}^{-1}\,(|\xi|^{2\theta}\,\mathcal{F}\,h),$$

with obviously $\Delta_x^1 = -\Delta$ the laplacian.
Now compute

$$\int_{\mathbb{R}^{2d}} \Delta_x^{1/4}\, T^* h \cdot \Delta_x^{1/4}\, T^* h\, dx\, dv = \int_{\mathbb{R}^d} \Delta_x^{1/2}\, T\; T^* h \cdot h(x)\, dx.$$

We then observe that

$$\begin{aligned} T\;T^* h(x) &= \int_0^\infty \int_0^\infty \int_{S^{d-1}} h(x + (t-u)v) e^{-t-u}\, dv\, du\, dt \\ &= 2\int_0^\infty \int_0^t \int_{S^{d-1}} h(x + (t-u)v) e^{-t-u}\, dv\, du\, dt. \end{aligned}$$

With two changes of variables from $t-u$ to τ and from the polar coordinates τv to y

$$T\,T^* h(x) = 2\int_0^\infty \int_0^t \int_{S^{d-1}} h(x+(t-u)v)e^{-t-u}\,dv\,du\,dt.$$
$$= \int_0^\infty \int_0^t \int_{S^{d-1}} h(x+\tau v)e^{-2t+\tau}\,dv\,d\tau\,dt$$
$$= \int_0^\infty \int_{|y|\le t} h(x-y)\ \ e^{-2t+|y|}\ \frac{dy}{|y|^{d-1}}\,dt.$$

Hence when differentiating $T\,T^*$, we obtain exactly the structure of a Riesz transform. Therefore the operator $T\,T^*$ is continuous from $L^2(\mathbb{R}^d)$ to $\dot{H}^1(\mathbb{R}^d)$ or $\Delta_x^{1/2} T\,T^*$ is continuous inside $L^2(\mathbb{R}^d)$.

Third Course : L^p estimates

1. The result
2. Interpolation, Sobolev and Besov spaces
3. L^p estimate for the operator T
4. End of the proof
5. Counterexamples for optimality

The problem and the theorem

For simplicity take

$$v \cdot \nabla_x f = \Delta_x^a g, \quad x \in \mathbb{R}^d, \ v \in \mathbb{R}^d, \quad a < 1, \tag{8}$$

and for the average

$$\rho(x) = \int_{\mathbb{R}^d} f(x, v)\, \phi(v)\, dv.$$

Assume the following bounds on f and g

$$\begin{aligned} f &\in \dot{W}_v^{\beta,\, p_1}(\mathbb{R}^d,\ L_x^{p_2}(\mathbb{R}^d)), \qquad && \beta \geq 0, \\ g &\in \dot{W}_v^{\gamma,\, q_1}(\mathbb{R}^d,\ L_x^{q_2}(\mathbb{R}^d)), \qquad && -\infty < \gamma < 1, \end{aligned}$$

with $1 < p_2,\ q_2 < \infty$, $1 \leq p_1 \leq \min(p_2,\ p_2^*)$ and $1 \leq q_1 \leq \min(q_2,\ q_2^*)$, and $\gamma - 1/q_1 < 0$.

Then, see DiPerna-Lions-Meyer, Bézard, DeVore-Petrova, Bouchut, J.-Perthame, J.-Vega...

Theorem

With the previous assumptions

$$\|\rho\|_{\dot{B}^{s,r}_{\infty,\infty}} \leq C\|f\|^{1-\theta}_{W^{\beta,p_1}_v(L^{p_2}_x)} \times \|g\|^{\theta}_{W^{\gamma,q_1}_v(L^{q_2}_x)},$$

with

$$\frac{1}{r} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}, \quad s = (1-a)\theta,$$
$$\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}.$$

This result essentially uses the L^2 regularizing effect and a lot of interpolation.

See Bergh-Löfstrom for more details.

Definition

E and F be two Banach spaces. An interpolated space at order θ between E and F is a space $G \subset E + F$ s.t. $\forall T$ continuous in E and in F then T is continuous in G and

$$\|T\|_G \leq \|T\|_E^{1-\theta} \|T\|_F^{\theta}.$$

Note that there is no reason why the interpolate should be unique.

Proposition

Let T be a continuous operator from E_1 to E_2 and from F_1 to F_2. Let G_i be an interpolated space at order θ between E_i and F_i. Then T is continuous from G_1 to G_2 and

$$\|T\|_{G_1 \to G_2} \leq \|T\|_{E_1 \to E_2}^{1-\theta} \|T\|_{F_1 \to F_2}^{\theta}.$$

For example an interpolate at order θ between the spaces $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ is the space $L^r(\mathbb{R}^d)$ with

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

Recall the definition of Sobolev spaces

$$W^{1,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) \,|\, \nabla f \in L^p(\mathbb{R}^d)\},$$
$$W^{-1,p}(\mathbb{R}^d) = \{f = g + \nabla.h \,|\, g \in L^p(\mathbb{R}^d),\ h \in (L^p(\mathbb{R}^d))^d\},$$

and homogeneous Sobolev spaces

$$\dot W^{1,p}(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d) \,|\, \nabla f \in L^p(\mathbb{R}^d)\},$$
$$\dot W^{-1,p}(\mathbb{R}^d) = \{f = \nabla.h \,|\, h \in (L^p(\mathbb{R}^d))^d\},$$

with obvious extensions for $W^{k,p}$ where $k \in \mathbb{Z}$.

Then the spaces $W^{s,p}(\mathbb{R}^d)$ with $s \in \mathbb{R}$ can be obtained by interpolation : If $s \in [0,\ 1]$ then $\dot W^{s,p}(\mathbb{R}^d)$ is an interpolate at order s between $L^p(\mathbb{R}^d)$ and $\dot W^{1,p}(\mathbb{R}^d)$.

If $1 < p < \infty$ then an equivalent definition is that $f \in \dot W^{s,p}(\mathbb{R}^d)$ iff $\Delta^{s/2} f \in L^p(\mathbb{R}^d)$.

We use the so-called K-theory from Lions-Peetre.
For E and F two Banach spaces and $\rho \in E + F$ define

$$K_\rho(t) = \inf_{\rho=\rho_1+\rho_2} (\|\rho_1\|_E + t\|\rho_2\|_F).$$

Define $(E,\ F)_{\theta,k}$ as the space of functions ρ such that

$$\left(\int_0^\infty (K_\rho(t)\, t^{-\theta})^k \,\frac{dt}{t}\right)^{1/k} < \infty,$$

and if $k = \infty$

$$\sup_t K_\rho(t)\, t^{-\theta} < \infty.$$

All spaces $(E,\ F)_{\theta,k}$ are interpolated spaces at order θ.
This method generates all Besov spaces (and Lorentz spaces for the interpolation between L^p and L^q).
We will use it only for $k = \infty$.

The space $(W^{s_1,p}(\mathbb{R}^d),\ W^{s_2,p}(\mathbb{R}^d))_{\theta,\infty}$ is the Besov space $B^{s,p}_{\infty}(\mathbb{R}^d)$ with

$$s = (1-\theta)\, s_1 + \theta\, s_2.$$

This space is very close from the Sobolev space

$$W^{s,p}(\mathbb{R}^d) \subset B^{s,p}_{\infty}(\mathbb{R}^d) \subset W^{s',p}(\mathbb{R}^d) \quad \forall s' < s.$$

For the homogeneous spaces $(\dot{W}^{s_1,p}(\mathbb{R}^d),\ \dot{W}^{s_2,p}(\mathbb{R}^d))_{\theta,\infty}$, we obtain the homogeneous Besov space $\dot{B}^{s,p}_{\infty}(\mathbb{R}^d)$ with on a compact support Ω

$$\dot{W}^{s,p}(\Omega) \subset \dot{B}^{s,p}_{\infty}(\Omega) \subset \dot{W}^{s',p}(\Omega) \quad \forall s' < s.$$

Unfortunately the space $(W^{s_1,p}(\mathbb{R}^d),\ W^{s_2,q}(\mathbb{R}^d))_{\theta,\infty}$ is not a Besov space if $p \neq q$, we denote it $B^{s,r}_{\infty,\infty}$ but

$$W^{s,p}(\mathbb{R}^d) \subset B^{s,p}_{\infty,\infty}(\mathbb{R}^d) \subset W^{s',p}(\mathbb{R}^d) \quad \forall s' < s.$$

Estimate for the operator T

We perform the same trick and change into

$$(\lambda + v \cdot \nabla_x) f(x, v) = \Delta_x^{\alpha/2} g(x, v) + \lambda f(x, v).$$

We denote by T_λ the operator

$$T_\lambda f(x) = \int_0^\infty \int_{\mathbb{R}^d} f(x - vt,\ v)\, e^{-\lambda t}\, \phi(v)\, dv\, dt.$$

Consequently

$$\rho(x) = \int_{\mathbb{R}^d} f(x, v) \phi(v)\, dv = \lambda T_\lambda f + \Delta_x^{a/2} T_\lambda g.$$

We first study this operator T_λ.

We prove

Proposition

For any $1 \leq p_1 \leq \min(p_2,\, p_2^*)$ *with* $1 < p_2 < \infty$, *for any* s *with* $s \leq 1/p_1$, *we have for* $s \geq 0$

$$T_\lambda : \ \dot{W}^{s,p_1}_{loc,v}(\mathbb{R}^d,\ L^{p_2}_x(\mathbb{R}^d)) \longrightarrow \dot{W}^{1+s-1/p_1,\,p_2}(\mathbb{R}^d),$$

$$\textit{with norm } C\lambda^{s-1/p_1}.$$

Notice first that with a simple change of variable

$$T_\lambda f(x) = \frac{1}{\lambda}\int_0^\infty \int_{\mathbb{R}^d} f(x - vt/\lambda,\ v)\, e^{-t}\, \phi(v)\, dv\, dt = \frac{1}{\lambda}\, Tf_\lambda(\lambda x),$$

with $f_\lambda(x) = f(x/\lambda, v)$. Therefore it is enough to do the proof for $\lambda = 1$, *i.e.* for the operator T.

Lemma

L^1 case : $\forall\, 0 \le s < 1,\ T:\ \dot{W}^{s,1}_{loc,v}(\mathbb{R}^d,\ L^p_x(\mathbb{R}^d)) \longrightarrow \dot{W}^{s,p}(\mathbb{R}^d)$, *for every* $1 \le p \le \infty$.

Proof. It is a direct computation, noticing

$$\partial_{x_i} f(x - vt, v) = -\frac{1}{t}\partial_{v_i}(f(x - vt,\ v)) + \frac{1}{t}(\partial_{v_i} f)(x - vt,\ v).$$

First of all, simply by commuting

$$\left\| \int_{\mathbb{R}^d} f(x - vt,\ v)\,\phi(v)\,dv \right\|_{L^p} \le C\|f\|_{L^1_v L^p_x},$$

where C does not depend on t. Then

$$\begin{aligned}\left\|\partial_{x_i}\int_{\mathbb{R}^d} f(x - vt,\ v)\,dv\right\|_{L^p} &\le \left\|\frac{1}{t}\int_{\mathbb{R}^d}\partial_{v_i}(f(x - vt,\ v))\,\phi(v)\,dv\right\|_{L^p} \\ &\quad + \left\|\frac{1}{t}\int_{\mathbb{R}^d}(\partial_{v_i} f)(x - vt,\ v)\phi(v)\,dv\right\|_{L^p} \\ &\le \frac{C}{t}\|f\|_{W^{1,1}_v L^p_x}.\end{aligned}$$

By interpolation, we conclude that for any $s < 1$

$$\left\| \int_{\mathbb{R}^d} f(x - vt,\, v)\phi(v)\, dv \right\|_{\dot{W}^{s,p}} \leq \frac{C}{t^s} \|f\|_{W_v^{s,1} L_x^p},$$

and by integrating in t against e^{-t} we get the desired result.

With exactly the same idea, one obtains for negative derivatives,

Lemma

$\forall\, s \leq 0,\ \ T:\ \ \dot{W}^{s,1}_{loc,v}(\mathbb{R}^d,\ L^p_x(\mathbb{R}^d)) \longrightarrow \dot{W}^{s,p}(\mathbb{R}^d).$

It remains to combine this with the L^2 case. In fact for any $s \in \mathbb{R}$

$$\Delta_x^s h(x+vt) = \Delta_v^s h(x+vt)\ t^{-s},$$

which implies for the dual operator T^* with $s<1$

$$\Delta_x^{s/2} T^* h = \phi(v)\,\Delta_v^{s/2} \int_0^\infty h(x+vt)\frac{e^{-t}}{t^s}\,dt = \phi(v)\Delta_v^{s/2}(\phi^{-1} T_s^* h),$$

according to the definition of T_s.
From the L^2 estimate on T_s

Lemma
(L^2 setting) $\forall s<1/2$, $T:\ \dot{H}^s_{loc,v}(L^2_x) \longrightarrow \dot{H}^{s+1/2}$.

To obtain the behaviour of T on any space of the form $\dot{W}_v^{s,p_1}(L_x^{p_2})$, we cannot simply interpolate between the two lemmas because we would be restricted to $s<1/2$. Instead we have to interpolate before integrating in t. A slight problem arises because the operator $\Delta_x^{s/2}$ does not operate nicely on L^1.
This would require the use of Hardy space, which we skip here...

The end of the proof

We first make the additional assumption that $\beta < 1/p_1$. Indeed with that we may apply the proposition to both f and g.
We have

$$\rho = \rho^1 + \rho^2 = \lambda T_\lambda f + \Delta_x^{a/2}\, T_\lambda g,$$

with by the proposition

$$\|\rho^1\|_{\dot W^{1+\beta-1/p_1,\, p_2}} \leq C\,\lambda \times \lambda^{\beta-1/p_1} \times \|f\|_{\dot W_v^{\beta,p_1} L_x^{p_2}},$$
$$\|\rho^2\|_{\dot W^{1+\gamma-1/q_1-a,\, q_2}} \leq C\lambda^{\gamma-1/q_1} \times \|g\|_{\dot W_v^{\gamma,q_1} L_x^{q_2}}.$$

We interpolate between $\dot W^{1+\beta-1/p_1,\, p_2}$ and $\dot W^{1+\gamma-1/q_1-a,\, q_2}$ using the K-method

$$K(t) = \inf_{\rho=\rho_1+\rho_2} (\|\rho_1\|_{\dot W^{1+\beta-1/p_1,\, p_2}} + t\|\rho_2\|_{\dot W^{1+\gamma-1/q_1-a,\, q_2}}).$$

Take

$$\lambda = t^{1/(1+\beta-1/p_1-\gamma+1/q_1)},$$

and indeed find

$$K(t) \le t^{\theta} \times \|f\|^{1-\theta}_{\dot{W}^{\beta,p_1}_v L^{p_2}_x} \times \|g\|^{\theta}_{\dot{W}^{\gamma,q_1}_v L^{q_2}_x},$$

with

$$\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1},$$

as given by the theorem.

Consequently ρ belongs to the space $\dot{B}^{s,r}_{\infty,\infty}$ as the interpolation of order (θ,∞) of the two spaces $\dot{W}^{1+\beta-1/p_1,\,p_2}$ and $\dot{W}^{1+\gamma-1/q_1-a,q_2}$.

The case $\beta \geq 1/p_1$

The problem is that the proposition is not true anymore. If one tries to prove any of the lemmas for $\beta \geq 1/p_1$, there is not enough integrability in t.
More precisely, we have to integrate a term in t^{-k} with $k \geq 1$ which is not possible. However

$$
\begin{aligned}
T_\lambda f &= \int_0^\infty \int_{\mathbb{R}^d} \partial_t(t) f(x - vt,\, v) e^{-\lambda t}\, \phi(v)\, dv\, dt \\
&= \int_0^\infty \int_{\mathbb{R}^d} f(x - vt,\, v)\, \lambda t e^{-\lambda t}\, \phi(v) \\
&\qquad + \int_0^\infty \int_{\mathbb{R}^d} v \cdot \nabla_x f(x - vt,\, v) t e^{-\lambda t}\, \phi(v)
\end{aligned}
$$

So eventually

$$T_\lambda f = \int_0^\infty \int_{\mathbb{R}^d} f(x - vt,\, v)\, \lambda t e^{-\lambda t}\, \phi(v) \\ + \frac{1}{\lambda} \int_0^\infty \int_{\mathbb{R}^d} \Delta_x^{a/2} g(x - vt,\, v) \lambda t e^{-\lambda t}\, \phi(v).$$

The first term has the same homogeneity as $T_\lambda f$ but with more integrability around the origin in t. The second term, once it is multiplied by λ behaves exactly like the usual $T_\lambda g$.
Therefore, repeating this simple trick as many times as necessary, we avoid any problem of integrability in t for $T_\lambda f$ and we may consider β as large as we want.

This is a slight generalization of two notes of Lions. The examples are given in dimension two for simplicity.

Consider two C_c^∞ functions a and b and take

$$f_N(x,v) = N^{\delta(1/p_1-\beta)} \times a(N\,x_1, x_2/N)\; b(N^\delta v_1),$$
$$g_N(x,v) = N^{1-\delta+\delta/p_1-\delta\beta} \times \partial_1 a(N\,x_1, x_2/N)\; N^\delta v_1\, b(N^\delta v_1).$$

Then simply choose δ such that g_N belongs to the space $W_v^{\gamma,q_1}(L_x^{q_2})$ uniformly in N for every q_2, so

$$\delta = \frac{1}{1-1/p_1+\beta+1/q_1-\gamma}.$$

Notice that if $\gamma < 0$, we also have to require that $wb(w)$ be the γ derivative of some function.

Now

$$v \cdot \nabla_x f_N = g_N + h_N,$$

with for any r

$$\|h_N\|_{L^1_v(W^{1,r}_x)} \leq C N^{-2\delta}.$$

Therefore the contribution from h_N to the regularity of the average is one full derivative and it can be neglected.
To finish, notice that for any $1 \leq r \leq \infty$

$$\|\rho_N\|_{\dot{W}^{s,r}} \geq N^{s-\delta(1-1/p_1+\beta)}.$$

Hence for this norm to be bounded uniformly in N, we need that

$$s \leq \delta(1-1/p_1+\beta) = \frac{1-1/p_1+\beta}{1-1/p_1+\beta+1/q_1-\gamma},$$

which is precisely the value given by the theorem.

Optimality of the r exponent

Consider

$$f_N(x,v) = N^{1/p_2+\delta(1/p_1-\beta)} \times a(N\,x_1,x_2)\; b(N^\delta v_1),$$
$$g_N(x,v) = N^{1+1/p_2-\delta+\delta/p_1-\delta\beta} \times \partial_1 a(N\,x_1,x_2)\; N^\delta v_1\, b(N^\delta v_1).$$

To bound uniformly g_N in the correct space

$$\delta = \frac{1+1/p_2-1/q_2}{1-1/p_1+\beta+1/q_1-\gamma}$$

We again have

$$v\cdot\nabla_x f_N = g_N + h_N,$$

with h_N more regular than g_N and so negligible for our purpose. Finally

$$\|\rho_N\|_{W^{s,r}} \geq N^{s+1/p_2-1/r-\delta(1-1/p_1+\beta)}.$$

We plug the correct value of s (seen before) and find

$$\frac{1}{r} = \frac{1}{p_2} - \frac{s}{p_2} + \frac{s}{q_2},$$

which is again the predicted value.

Plan of the course

1. The case with a full derivative

1.1 The result

1.2 Proof

2. The L^1 case

2.1 Known results

2.2 The theorem to be proved

2.3 The proof

The case with a full derivative

The main result here was obtained by Perthame-Souganidis. We deal with

$$v \cdot \nabla_x f = \mathrm{div}_x g, \quad x \in \mathbb{R}^d,\ v \in S^{d-1}.$$

Very little can be expected in this case : All f satisfy the equation with a right hand side just as regular as themselves. Nevertheless it is enough to ensure some compactness for the average

$$\rho(x) = \int_{S^{d-1}} f(x, v)\, dv.$$

Assume that

$$f \ \in\ \dot{W}_v^{\beta,\, p_1}(S^{d-1},\ L_x^{p_2}(\mathbb{R}^d)), \qquad \beta \geq 0,$$
$$g \ \in\ \dot{W}_v^{\gamma,\, q_1}(S^{d-1},\ L^{q_2}(\mathbb{R}^d)), \qquad -\infty < \gamma < 1,$$

with $1 < p_2,\ q_2 < \infty$, $1 \leq p_1 \leq \min(p_2,\ p_2^*)$ and $1 \leq q_1 \leq \min(q_2,\ q_2^*)$ and assume moreover that $\gamma - 1/q_1 < 0$.

Then

Theorem

One has

$$\|\rho\|_{B^{0,r}_{\infty,\infty}} \leq C\|f\|^{1-\theta}_{W^{\beta,p_1}_v(L^{p_2}_x)} \times \|g\|^{\theta}_{W^{\gamma,q_1}_v(L^{q_2}_x)},$$

with

$$\frac{1}{r} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2},$$
$$\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}.$$

The space $B^{0,r}_{\infty,\infty}$ is again obtained by interpolation but here as ρ trivially belongs to $L^{p_2}(\mathbb{R}^d)$ we have that ρ belongs to all $L^{r'}$ with $r' \in [p_2,\ r[$ or $]r,\ p_2]$.

It is possible to deduce

Corollary

Consider two sequences f_n and g_n of solutions. Assume moreover that f_n is uniformly bounded in $\dot W_v^{\beta,p_1}(S^{d-1},\ L^{p_2}(\mathbb{R}^d))$ with

$$\beta \geq 0,\ 1 < p_2 < \infty,\ 1 \leq p_1 \leq \min(p_2,\ p_2^*),$$

and that g_n is uniformly bounded and compact in $\dot W_v^{\beta,q_1}(S^{d-1},\ L^{q_2}(\mathbb{R}^d))$ with

$$-\infty < \gamma < 1,\ 1 < q_2 < \infty,\ 1 \leq q_1 \leq \min(q_2,\ q_2^*).$$

Then the sequence ρ_n is compact in any $L^{r'}$ with $r' \in]p_2,\ r[$ or $]r,\ p_2[$ and r given by the previous theorem.

This may replace compensated compactness in some situations (convergence of the vanishing viscosity approximation to scalar conservation laws for instance).

Proof of the corollary

As f_n is uniformly bounded, $f_n \longrightarrow f, \; w-*$ (at least after extraction). On the other hand, still after extraction, $g_n \longrightarrow g$. Thus

$$v \cdot \nabla_x f = \mathrm{div}_x \, g,$$

or

$$v \cdot \nabla_x (f_n - f) = \mathrm{div}_x \, (g_n - g).$$

Applying now the theorem to $f_n - f$ and $g_n - g$, we find that

$$\|\rho - \rho_n\|_{B^{0,r}_{\infty,\infty}} \le C \|f - f_n\|^{1-\theta}_{W^{\beta,p_1}_v (L^{p_2}_x)} \times \|g - g_n\|^{\theta}_{W^{\gamma,q_1}_v (L^{q_2}_x)}.$$

As $g_n - g$ strongly converges toward 0 and f_n is uniformly bounded, we deduce that

$$\rho_n - \rho \longrightarrow 0, \text{ in } B^{0,r}_{\infty,\infty}.$$

Therefore it is the same in all $L^{r'}$ with $r' \in]p_2, \; r[$ or $]r, \; p_2[$ since $\rho - \rho_n$ is uniformly bounded in L^{p_2}.

Proof of the Theorem

We follow the steps described in the third course and decompose

$$\rho = \rho_1 + \rho_2 = \lambda T_\lambda f + \mathrm{div}_x\, T_\lambda g.$$

From the main proposition

$$\|\rho^1\|_{\dot W^{1+\beta-1/p_1,\, p_2}} \leq C\,\lambda \times \lambda^{\beta-1/p_1} \times \|f\|_{\dot W_v^{\beta,p_1} L_x^{p_2}},$$

$$\|\rho^2\|_{\dot W^{\gamma-1/q_1,\, q_2}} \leq C\lambda^{\gamma-1/q_1} \times \|g\|_{\dot W_v^{\gamma,q_1} L_x^{q_2}}.$$

So again minimizing in λ in the functional $K(t)$, we take

$$\lambda = t^{1/(1+\beta-1/p_1-\gamma+1/q_1)},$$

and we indeed find

$$K(t) \leq t^\theta \times \|f\|^{1-\theta}_{W_v^{\beta,p_1} L_x^{p_2}} \times \|g\|^{\theta}_{W_v^{\gamma,q_1} L_x^{q_2}},$$

with

$$\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}.$$

Therefore ρ belongs to $B^{s,r}_{\infty,\infty}$ and it only remains to notice that

$$s = (1-\theta)\,(1+\beta-1/p_1) + \theta\,(\gamma-1/q_1) = 0,$$

which finishes the proof.

The L^1 case

A situation of interest is

$$v \cdot \nabla_x f = g,$$

where f is only in $L^1(\mathbb{R}^d \times S^{d-1})$.
It is crucial for collisional models : See DiPerna-Lions for the existence of renormalized solutions to Boltzmann equation, and Golse, Saint-Raymond for the derivation of hydrodynamic limits.
Here ρ is not in any Sobolev spaces. But some compactness property still holds

Theorem

Let f_n and g_n be two sequences of uniformly bounded solutions in the space $L^1(\mathbb{R}^d \times S^{d-1})$. Assume moreover that the sequence f_n is uniformly equi-integrable in v. Then the sequence of averages ρ_n is compact in $L^1_{loc}(\mathbb{R}^d)$.

The proof relies first on the fact that if f_n is equi-integrable in velocity then it is in both variables :

Proposition

Let f_n and g_n be two sequences of uniformly bounded solutions in $L^1(\mathbb{R}^d \times S^{d-1})$. If the sequence f_n is uniformly equi-integrable in $v \in S^{d-1}$ then it is uniformly equi-integrable in $(x, v) \in \mathbb{R}^d \times S^{d-1}$.

It is then possible to get

Theorem

Let f_n and g_n be two sequences of uniformly bounded solutions in $L^1(\mathbb{R}^d \times S^{d-1})$. Assume moreover that the sequence f_n is uniformly equi-integrable in $(x, v) \in \mathbb{R}^d \times S^{d-1}$. Then the sequence of averages ρ_n is compact in $L^1_{loc}(\mathbb{R}^d)$.

With the additional assumption that g_n is equi-integrable, this last result was already noticed in Golse-Lions-Perthame-Sentis.
We only give here the proof of the last theorem with a slight variant of the method used by Golse and Saint-Raymond.

The result to be proved

Take f and g a couple of solutions, and assume $\exists \Phi \in C(\mathbb{R}_+)$ with $\phi(\xi)/\xi$ increasing and $\Phi(\xi)/\xi \longrightarrow \infty$ as $\xi \to \infty$ and s.t.

$$I(f) = \int_{\mathbb{R}^d \times S^{d-1}} \Phi(|f(x,v)|)\, dx\, dv < \infty,$$

Then $\exists \varepsilon(h)$ depending only on Φ with $\lim \varepsilon(h) = 0$ as $h \to 0$ and such that for any $\phi \in C_c^1(\mathbb{R}^d,\ \mathbb{R}_+)$

$$\int_{\mathbb{R}^d} |\rho(x+h) - \rho(x)|\, \phi(x)\, dx \leq C_\phi\, \varepsilon(h)\, (\|f\|_{L^1} + \|g\|_{L^1} + I(f)).$$

Of course this property gives the compactness of any sequence and thus the theorem.

The proof

Notice that

$$v \cdot \nabla_x(\phi f) = g\,\phi + f\, v \cdot \nabla_x \phi.$$

Now decompose

$$(\lambda + v \cdot \nabla_x)\,(\phi\, f) = \bar{g} + \lambda f_1^M + \lambda f_2^M,$$

with

$$f_1^M = \phi\, f\, \mathbb{I}_{|f| \le M}, \quad f_2^M = \phi\, f\, \mathbb{I}_{|f| > M}, \quad \bar{g} = g\,\phi + f\, v \cdot \nabla_x \phi.$$

Then

$$\phi\,\rho = T_\lambda\, \bar{g} + \lambda\, T_\lambda f_1^M + \lambda\, T_\lambda\, f_2^M.$$

Obviously

$$\int_{\mathbb{R}^d} |\rho(x+h) - \rho(x)|\, \phi(x)\, dx \le \int_{\mathbb{R}^d} |\phi(x+h)\rho(x+h) - \phi(x)\rho(x)| + h \|\nabla \phi\|_{L^\infty}\, \|\rho\|_{L^1}$$

On the other hand

$$\int_{\mathbb{R}^d} |\phi(x+h)\rho(x+h) - \phi(x)\rho(x)| \leq \int_{\mathbb{R}^d} |T_\lambda \bar{g}(x+h) - T_\lambda \bar{g}|\, dx$$
$$+ \int_{\mathbb{R}^d} |\lambda T_\lambda f_1^M(x+h) - \lambda T_\lambda f_1^M|\, dx$$
$$+ \int_{\mathbb{R}^d} |\lambda T_\lambda f_2^M(x+h) - \lambda T_\lambda f_2^M|\, dx$$

So that finally

$$\int_{\mathbb{R}^d} |\rho(x+h) - \rho(x)|\, \phi(x)\, dx \leq 2\|T_\lambda \bar{g}\|_{L^1} + 2\lambda \|T_\lambda f_2^M\|_{L^1}$$
$$+ \int_{\mathbb{R}^d} |\lambda T_\lambda f_1^M(x+h) - \lambda T_\lambda f_1^M|\, dx + C_\phi h\|f\|_{L^1}.$$

From the main proposition in the third course we have

$$\|T_\lambda \bar{g}\|_{L^1} \leq \frac{C}{\lambda}\|\bar{g}\|_{L^1} \leq \frac{C}{\lambda}(\|g\|_{L^1} + C_\phi\|f\|_{L^1}),$$

and

$$\|T_\lambda f_2^M\|_{L^1} \leq \frac{C}{\lambda}\|f_2^M\|_{L^1} \leq \frac{C}{\lambda}\frac{M}{\Phi(M)} I(f),$$

as (remember that $\phi(\xi)/\xi$ is increasing)

$$\begin{aligned}\int_{\mathbb{R}^d\times S^{d-1}} |f(x,v)|\, \mathbb{I}_{|f|>M}\, dx\, dv \\ &= \int_{\mathbb{R}^d\times S^{d-1}} \Phi(|f(x,v)|)\, \mathbb{I}_{|f|>M}\, \frac{|f|}{\Phi|f|}\, dx\, dv \\ &\leq \sup_{\xi>M}\frac{\xi}{\Phi(\xi)} \int_{\mathbb{R}^d\times S^{d-1}} \Phi(|f(x,v)|)\, dx\, dv.\end{aligned}$$

For the last term $T_\lambda f_1^M$, notice that is is compactly supported in the support of ϕ so

$$\| T_\lambda f_1^M \|_{W^{1/2,1}(\mathbb{R}^d)} \leq C_\phi \| T_\lambda f_1^M \|_{H^{1/2}(\mathbb{R}^d)}.$$

Furthermore as f_1^M belongs to $L^2(\mathbb{R}^d \times S^{d-1})$ then

$$\| T_\lambda f_1^M \|_{H^{1/2}(\mathbb{R}^d)} \leq C\, \lambda^{-1/2} \, \| f_1^M \|_{L^2(\mathbb{R}^d \times S^{d-1})} \leq C\, \lambda^{-1/2}\, M^{1/2} \, \| f_1^M \|_{L^1}^{1/2}.$$

Consequently

$$\int_{\mathbb{R}^d} |\lambda T_\lambda f_1^M (x+h) - \lambda T_\lambda f_1^M| \, dx \leq h^{1/2} \, \| T_\lambda f_1^M \|_{W^{1/2,1}(\mathbb{R}^d)}$$
$$\leq C_\phi \, h^{1/2} \, \lambda^{1/2} \, M^{1/2} \, \| f_1^M \|_{L^1}^{1/2}.$$

Combining all estimates, one obtains

$$\int_{\mathbb{R}^d} |\rho(x+h)-\rho(x)|\,\phi(x)\,dx \leq \frac{C}{\lambda}\,(\|g\|_{L^1}+C_\phi\|f\|_{L^1}) + C\,\frac{M}{\Phi(M)}\,I(f)$$
$$+ C_\phi\,\lambda^{1/2}\,h^{1/2}\,M^{1/2}\,\|f_1^M\|_{L^1}^{1/2} + C_\phi\,h\,\|f\|_{L^1}.$$

For any h, it only remains to minimize in λ and M to conclude. Notice finally that in most applications, $\Phi(\xi)=\xi\,\log\xi$ (from entropy bounds). In that case, the function $\varepsilon(h)$ is

$$\varepsilon(h)=\frac{1}{\log 1/h}.$$

Fifth course : Application to scalar conservation law

Plan of the course :

1. Introduction of entropy solution
2. Propagation of L^p bounds
3. Existence I : The transport-collapse method
4. Existence II : Passing to the limit in the method
5. Existence III : Compactness thanks to averaging lemma
6. Uniqueness and Propagation of BV bounds.
7. Regularity by averaging lemmas.
8. Other regularity results.

For most of this part of the course, the convenient reference is Perthame

Scalar Conservation Law

Scalar conservation laws are hyperbolic equations on a scalar $u(t,x) \in \mathbb{R}$

$$\begin{aligned} &\partial_t u + \nabla_x \cdot (A(u(t,x))) = 0, \quad t \geq 0,\ x \in \mathbb{R}^d, \\ &u(t=0,x) = u^0(x), \end{aligned} \tag{9}$$

where the flux A is regular, namely $A \in C^2(\mathbb{R},\ \mathbb{R}^d)$.
The characteristics for Eq. (9) are lines. More precisely if u is a regular (C^1) solution then

$$u(t, x + t\, a(u^0(x))) = u^0(x),$$

where $a(\xi) = A'(\xi)$.
Of course this also shows that regular solutions cannot exist in general for all times : if $x = x_1 + t\, a(u^0(x_1)) = x_2 + t\, a(u^0(x_2))$, then $u(t,x)$ would have to be equal to both $u^0(x_1)$ and $u^0(x_2)$.

$\Longrightarrow$ Necessity of weak solutions and entropy for uniqueness

Entropy solution by kinetic formulation

Assume that u is a classical solution to (9). Define then

$$f(t,x,v) = \begin{cases} 1 & \text{if } 0 \le v < u(t,x), \\ -1 & \text{if } u(t,x) < v \le 0, \\ 0 & \text{in the other cases.} \end{cases} \tag{10}$$

Compute (in the sense of distribution)

$$\partial_t f = \partial_t u\, \delta(u(t,x)-v) = -a(u(t,x)) \cdot \nabla_x u(t,x)\, \delta(u(t,x)-v)$$
$$= -a(v) \cdot \nabla_x u(t,x)\, \delta(u(t,x)-v) = -a(v) \cdot \nabla_x f.$$

When u is no more C^1 this computation cannot be done. Instead

Definition : A function $u \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ is an entropy solution to (9) if and only if there exists $m \ge 0$ in $M^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^{2d})$, s.t. f defined through (10) satisfies

$$\partial_t f + a(v) \cdot \nabla_x f = \partial_v m. \tag{11}$$

u can be recovered through

$$u(t,x) = \int_{\mathbb{R}} f(t,x,v)\, dv$$

Note that if f is a solution then f is of bounded variation in time, in $BV_{loc}(\mathbb{R}_+,\ W^{-1-0,1}(\mathbb{R}^{d+1}))$. Therefore the trace of f at $t=0$ ($t=0+$ more precisely) is well defined.
So the trace of u is also well defined and we can impose

$$u(t=0,x) = u^0(x).$$

Assume

$$\exists C,\ \forall \xi \in \mathbb{R}^d,\ \forall \tau,\ \forall \varepsilon \in \mathbb{R}_+, \qquad |\{v \in \mathbb{R};\ |a(v)\cdot\xi - \tau| \le \varepsilon\}| \le C\,\varepsilon.$$

Theorem

For any $u^0 \in L^1(\mathbb{R}^d)$, $\exists! u \in L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^d))$, entropy solution to (9) with $u(t=0)=u^0$. Moreover if $u^0 \in L^\infty$ the solution satisfies (i) $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ and $u \in W^{s,3/2}_{loc}(\mathbb{R}_+^* \times \mathbb{R}^d)$ for any $s < 1/3$.

The easiest property of entropy solution is

Proposition

Take any $\phi \in C^2(\mathbb{R})$, convex and assume that

$$\int_{\mathbb{R}^d} \phi(u^0(x))\,dx < \infty,$$

then $\forall t > 0$, if u is an entropy solution with initial data u^0

$$\int_{\mathbb{R}^d} \phi(u(t,x))\,dx \le \int_{\mathbb{R}^d} \phi(u^0(x))\,dx.$$

In particular if $u^0 \in L^p$ then $u \in L^\infty(\mathbb{R}_+,\ L^p(\mathbb{R}^d))$.

Proof. Define $\phi_n \longrightarrow \phi$ with $\phi_n'' \in C_c(\mathbb{R})$. Because of the definition of f

$$\int_{\mathbb{R}^d} \phi_n(u(t,x))\,dx = \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\,f(t,x,v)\,dx\,dv.$$

Now multiplying the equation by $\phi_n'(v)$ and integrating

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\,f(t,x,v)\,dx\,dv = \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\,\partial_v m\,dx\,dv$$
$$= -\int_{\mathbb{R}^d\times\mathbb{R}} \phi_n''(v)\,m\,dx\,dv \le 0,$$

because $\phi_n'' \ge 0$ and $m \ge 0$. Consequently

$$\int_{\mathbb{R}^d} \phi_n(u(t,x))\,dx = \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\,f(t,x,v)\,dx\,dv$$
$$\le \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\,f(0,x,v)\,dx\,dv = \int_{\mathbb{R}^d} \phi_n(u^0(x))\,dx,$$

and passing to the limit in n, one obtains the proposition.

It was introduced by Brenier. For any n we define f_n recursively on the $]i/n,\ (i+1)/n]$. u_n is then always given by

$$u_n(t,x) = \int_{\mathbb{R}} f_n(t,x,v)\,dv.$$

Step 0 : Initialization

$$f_n(0,x,v) = \begin{cases} 1 & \text{if } 0 \le v < u^0(t,x), \\ -1 & \text{if } u^0(t,x) < v \le 0, \\ 0 & \text{in the other cases.} \end{cases}$$

Step 1 : Transport. Given $f_n(i/n,x,v)$, f_n on $]i/n,\ (i+1)/n[$ is the solution to

$$\partial_t f_n + a(v)\cdot\nabla_x f_n = 0, \quad t \in [i/n,\ (i+1)/n[,$$

with the corresponding initial data at $t = i/n$.

This explicitly gives

$$f_n(t,x,v) = f_n(i/n, x - a(v)\,(t - i/n), v).$$

But it is not true that f_n is an indicatrix.
Step 2 : Collapse. Define

$$Lf(v) = \begin{cases} 1 & \text{if } 0 \le v < \int_{\mathbb{R}} f(v)\,dv, \\ -1 & \text{if } \int_{\mathbb{R}} f(v)\,dv < v \le 0, \\ 0 & \text{in the other cases.} \end{cases}$$

Then pose

$$f_n((i+1)/n, x, v) = L\,(f_n(i/n, x - a(v)/n, v)) = L\,f_n((i+1)/n-, x, v),$$

where $f_n((i+1)/n-, x, v)$ is the limit of $f_n(t,x,v)$ for $t \to (i+1)/n$ with $t < (i+1)/n$.

Therefore one recovers for all i

$$f_n(i/n, x, v) = \begin{cases} 1 & \text{if } 0 \le v < u_n(i/n, x), \\ -1 & \text{if } u_n(i/n, x) < v \le 0, \\ 0 & \text{in the other cases.} \end{cases}$$

Finally the main property of the collapse operator : $\forall f$ with $\sup |f| \le 1$ and $\forall \phi(v) \in C^1$ with $\phi'(v) \ge 0$

$$\int_{\mathbb{R}} \phi(v) L f(v)\, dv \le \int_{\mathbb{R}} \phi(v) f(v)\, dv.$$

Convergence

In the sense of distribution f_n satisfies

$$\partial_t f_n + a(v) \cdot \nabla_x f_n = g_n,$$

with

$$g_n = \sum_{i=1}^{\infty} \delta(t - i/n)\,(f_n(i/n, x, v) - f_n(i/n-, x, v)).$$

Moreover

$$\sup |f_n(0, x, v)| = 1, \quad \int_{\mathbb{R}^{d+1}} |f_n(0, x, v)\,dx\,dv = \int_{\mathbb{R}^d} u^0(x)\,dx < \infty,$$

and by induction on the intervals $[i/n,\ (i+1)/n]$, for any $t > 0$

$$\|f_n(t, ., .)\|_{L^1(\mathbb{R}^{d+1})} = \|u_n(t, .)\|_{L^1(\mathbb{R}^d)} = \|u^0\|_{L^1}, \qquad \sup_{x,v} |f_n(t, x, v)| = 1.$$

Hence we may extract a converging subsequence, still denoted f_n,

$$f_n \longrightarrow f, \quad w - *L^{\infty}.$$

In addition use the property of the collapse operator : $\forall \Phi(x, v)$ with $\partial_v \Phi \geq 0$

$$\int_{\mathbb{R}^{d+1}} \Phi(x, v) \left(f_n(i/n, x, v) - f_n(i/n-, x, v)\right) dx\, dv \leq 0.$$

Hence there exists a measure $M_{i,n}(x, v) \geq 0$ s.t.

$$\left(f_n(i/n, x, v) - f_n(i/n-, x, v)\right) = \partial_v M_{i,n}(x, v).$$

Obviously this implies that

$$g_n = \partial_v m_n, \quad m_n \geq 0,$$

with

$$m_n(t, x, v) = \sum_{i=1}^{n} \delta(t - i/n)\, M_{i,n}(x, v).$$

Now define $\Phi_M = v\mathbb{I}_{|v|\le M} + M\mathbb{I}_{v>M} - M\mathbb{I}_{v<-M}$.
Multiplying the kinetic equation by Φ_M and integrating,

$$\int_{\mathbb{R}^{d+1}} \Phi_M \left(f_n(T,x,v) - f_n(0,x,v)\right) dx\, dv$$
$$= -\int_0^T \int_{\mathbb{R}^{d+1}} \partial_v \Phi_M \, dm_n(t,x,v).$$

So from the L^1 estimate on f_n

$$\int_0^T \int_{-M}^M \int_{\mathbb{R}^d} dm_n(t,x,v) \le 2M \|f_n(t,.,.)\|_{L^1} \le 2M\|u^0\|_{L^1(\mathbb{R}^d)}.$$

Therefore still extracting a subsequence, we obtain

$$m_n \longrightarrow m, \quad w-*M^1_{loc}$$

with $m \ge 0$ in $M^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1})$. The limit f then satisfies

$$\partial_t f + a(v)\cdot\nabla_x f = \partial_v m.$$

It remains to show that the constraint on f holds at the limit.
Assuming that u_n is compact in L^1 then this follows from the fact that it is satisfied at every $t = i/n$.

Compactness of u_n

Take a function $\Phi \in C^\infty(\mathbb{R})$ satisfying

$$\Phi(v) = 1 \quad \text{if } |v| \leq 1, \quad \Phi(v) = 0 \quad \text{if } |v| \geq 2, \quad 0 \leq \Phi(v) \leq 1 \quad \forall v.$$

Then define

$$u_n^R = \int_{\mathbb{R}} f_n(t,x,v)\, \Phi(v/R)\, dv.$$

This u_n^R is an average of f_n for which we can apply averaging lemmas.

Remember that

$$\partial_t f_n + a(v) \cdot \nabla_x f_n = \partial_v m_n.$$

The measure m_n is in any $W^{-r,p}([0,\ T] \times \mathbb{R}^d \times [-R,\ R]$ for $r > 0$ and $p < (1 - r/d)^{-1}$ as

$$\|m_n\|_{W^{-r,1}([0,\ T]\times\mathbb{R}^d\times[-R,\ R]} \le C_r \int_{W^{-r,1}([0,\ T]\times\mathbb{R}^d\times[-R,\ R]} dm_n \le C_r\, R\, \|u^0\|_{L^1}.$$

Next $\|f_n\|_{L^\infty} \le 1$ so $f_n \in L^p_{loc}$ for any p and in particular

$$\|f_n\|_{L^2([0,\ T]\times B(0,K)\times[-R,\ R])} \le C\,\sqrt{TKR}.$$

Using averaging lemmas, u_n^R belongs to $W^{s,5/3}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ for any $s < 1/5$ with

$$\|u_n^R\|_{W^{s,5/3}([0,\ T]\times B(0,K))} \le C(s,T,K,R),$$

and therefore u_n^R is locally compact so that

$$u_n^R \longrightarrow u^R = \int_{\mathbb{R}} f(t,x,v)\,\Phi(v/R)\,dv. \qquad \text{in } L^{5/3}_{loc}.$$

Now as $u^0 \in L^1$ there exists an even convex function $\chi \in C^2(\mathbb{R})$ with $\chi(\xi)/|\xi| \longrightarrow +\infty$ as $|\xi| \to +\infty$ and s.t.

$$\int_{\mathbb{R}^d} \chi(u^0(x))\, dx < \infty.$$

From the definition of f_n this implies that

$$\int_{\mathbb{R}^d \times \mathbb{R}} \chi'(v)\, f_n(t=0,x,v)\, dv\, dx = \int_{\mathbb{R}^d} \chi(|u^0(x)|)\, dx < \infty.$$

Multiplying the kinetic equation by χ' and integrating, one gets

$$\frac{d}{dt}\int_{\mathbb{R}^d \times \mathbb{R}} |\chi'(v)|\, |f_n(t,x,v)|\, dv\, dx = \frac{d}{dt}\int_{\mathbb{R}^d \times \mathbb{R}} \chi'(v)\, f_n(t,x,v)\, dv\, dx$$
$$= \int_{\mathbb{R}^d \times \mathbb{R}} g_n\, \chi'\, dx\, dv = -\int_{\mathbb{R}^d \times \mathbb{R}} m_n\, \chi''(v)\, dv\, dx \le 0.$$

This shows that

$$\int_{\mathbb{R}^d} |u_n - u_n^R| \, dx \leq \int_{\mathbb{R}^d} \int_{|v| \geq R} |f_n(t,x,v)| \, dv$$
$$\leq \frac{1}{|\chi'(R)|} \int_{\mathbb{R}^d \times \mathbb{R}} \chi' \, f_n \, dx \, dv \leq \frac{1}{|\chi'(R)|} \int_{\mathbb{R}^d} \chi(u^0(x)) \, dx,$$

and so $u_n - u_n^R \longrightarrow 0$ in L^1 as R tends to infinity, uniformly in n. From the compactness of u_n^R, we deduce the compactness of u_n in L^1_{loc} and we are done.

Uniqueness was first obtained by Kruzkov. The formal argument here corresponds to the proof by Perthame.
Consider two entropy solutions u_1 and u_2, then

Proposition

L^1 contractivity : We have for any $t > 0$

$$\|u_1(t,.) - u_2(t,.)\|_{L^1(\mathbb{R}^d)} \leq \|u_1^0 - u_2^0\|_{L^1(\mathbb{R}^d)}.$$

This of course implies the uniqueness of the solution but it does even more than that (see next).
Denote f_1 and f_2 the two functions defined from u_1 and u_2 and m_1, m_2 the measures in the kinetic equations. For simplicity assume that $u_1 \geq 0$ and $u_2 \geq 0$ and hence $f_1 \geq 0$ and $f_2 \geq 0$.

First note that as $f_i \geq 0$, $f_i^2 = f_i$. f_i^2 solves the same equation but multiplying the equation by $2f_i$ we also get

$$\partial_t f_i^2 + a(v) \cdot \nabla_x f_i^2 = 2f_i \, \partial_v m_i.$$

Thus

$$2f_i \, \partial_v m_i = \partial_v m_i,$$

and

$$\int_{\mathbb{R}} f_i \, \partial_v m_i \, dv = 0. \tag{12}$$

Of course this is only formal. The rigourous argument requires the use of convolution.

Now use the kinetic equation for f_1 and f_2 and compute

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}} |f_1 - f_2|^2 \, dx \, dv = \int_{\mathbb{R}^d \times \mathbb{R}} (f_1 - f_2)(\partial_v m_1 - \partial_v m_2)$$
$$= - \int_{\mathbb{R}^d \times \mathbb{R}} (f_1 \, \partial_v m_2 + f_2 \, \partial_v m_1),$$

by our crucial relation.

As f_i is non increasing

$$\int_{\mathbb{R}^d\times\mathbb{R}} f_1\,\partial_v m_2\,dx\,dv = -\int_{\mathbb{R}^d\times\mathbb{R}} \partial_v f_1\, m_2\,dx\,dv \geq 0,$$

and the same is true for the other term. Finally

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} |f_1 - f_2|^2\,dx\,dv \leq 0.$$

To conclude note that $|f_1 - f_2|$ is equal to 0 if $0 \leq v \leq u_1$ and $0 \leq v \leq u_2$ or if $v > u_1$ and $v > u_2$; It is equal to 1 if $u_1 < v < u_2$ or $u_2 < v < u_1$. Therefore

$$\int_{\mathbb{R}^d\times\mathbb{R}} |f_1 - f_2|^2\,dx\,dv = \int_{\mathbb{R}^d} |u_1 - u_2|\,dx,$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u_1 - u_2|\,dx \leq 0.$$

Propagation of BV norm

Take $h \in \mathbb{R}^d$ and apply the contractivity for $u(t,x)$ and $u(t,x+h$ (corresponding to $u^0(x+h)$), it shows that

$$\int_{\mathbb{R}^d} |u(t,x+h) - u(t,x)|\, dx \leq \int_{\mathbb{R}^d} |u^0(x+h) - u^0(x)|\, dx,$$

and so

$$\int_{\mathbb{R}^d} \frac{|u(t,x+h) - u(t,x)|}{|h|}\, dx \leq \int_{\mathbb{R}^d} |\nabla_x u^0(x)|\, dx.$$

Hence

Corollary

Let u be an entropy solution and assume that $u^0 \in BV(\mathbb{R}^d)$ then $u(t,.) \in BV(\mathbb{R}^d)$ and

$$\|u(t,.)\|_{BV} \leq \|u^0\|_{BV}.$$

There are many ways to prove this result.
For example take the sequence f_n obtained before

$$\|f_n(t)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))} = \|f_n(i/n+)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))}, \quad \forall t \in [\frac{i}{n},\ \frac{i+1}{n}[.$$

The collapse operator contracts the BV norm so

$$\|f_n(i/n+,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))} \leq \|f_n(i/n-,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))}.$$

One then gets that $\|f_n(t)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))} = \|f_n(0)\|_{BV} = \|u^0\|_{BV}$.
Going back to the estimate on f the uniqueness proof gives

$$\int_{\mathbb{R}^d\times\mathbb{R}} \frac{|f(t,x+h,v) - f(t,x,v)|^2}{|h|}\,dx\,dv,$$

which is not BV but in fact like a $H^{1/2}$ norm. Of course

$$\|u(t,.)\|_{BV} = \|f(t,.,.)\|_{BV_x(M^1_v)},$$

and this in turn dominates any $H^s_x(L^2_v)$ norm of f with $s < 1/2$.

However it is only the very specific form of f which gives the bound the other way around. In fact the uniqueness argument be used to directly bound

$$\|f\|^2_{H^s_x(L^2_v)} = \int_{\mathbb{R}^{2d}\times\mathbb{R}} \frac{|f(t,x,v) - f(t,y,v)|^2}{|x-y|^{2s+d}}\, dx\, dy\, dv.$$

Define as before for a regular Φ

$$u^R = \int_{\mathbb{R}} f(t,x,v)\,\Phi(v/R)\,dv.$$

Note that from the definition of f

$$\int_{\mathbb{R}} |\partial_v f(t,x,v)| = 1.$$

so that

$$\|f\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d, BV_{loc}(\mathbb{R}))} \leq C.$$

As $\|f\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^{d+1})} = 1$, by interpolation

$$\|f\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d,\ H^s(\mathbb{R}))} \leq C, \quad s < 1/2.$$

Because $\|f\|_{L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^{d+1}))} = \|u\|_{L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^d))}$, with a last interpolation

$$\|f\|_{L^2([0,\ T]\times\mathbb{R}^d,\ H^s(\mathbb{R}))} \leq C\left(\|u\|_{L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^d))}\right), \quad s < 1/2.$$

The measure m belongs to $W^{s,1}_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1})$. So we may apply averaging lemmas and get

$$u^R \in W^{s,3/2}_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1}), \quad \forall\, s < 1/3.$$

Now if $u \in L^\infty$ then for $R > \|u\|_{L^\infty}$, $u^R = u$ and

$$u \in W^{s,3/2}_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1}), \quad \forall\, s < 1/3.$$

This is the promised regularity

If u is only in L^p, then the argument would be more complicated.

Oleĭnik BV regularization

It is possible to show that the solution immediately becomes BV in the particular case of a strictly convex flux in dimension 1 : $\inf a'(v) > 0$.
The original argument was given for the vanishing viscosity approximation, with first proving a semi-Lipschitz bound on u.
Here we instead use the transport collapse scheme.
To simplify assume that

$$a(v) = v, \quad u^0 \geq 0, \quad u^0 \in L^\infty(\mathbb{R}).$$

The following holds for f_n, u_n defined by Transport-Collapse

Proposition

For any $t > 0$, any $R > 0$

$$\|t\,\partial_x u_n(t,.) - 1\|_{M^1([-R,\ R])} \leq 2R\|u^0\|_{L^\infty} + 2t\,\|u^0\|_{L^\infty}^2.$$

Proof. We argue by induction on every interval $]i/n,\ (i+1)/n]$. Start with $]0,\ 1/n]$, f_n is simply the solution to the free transport

$$f_n(t,x,v) = f(0,x-vt,v).$$

So

$$\begin{aligned}
\partial_x u_n(t,x) &= \int_{\mathbb{R}} \partial_x f_n(0,x-vt,v)\,dv \\
&= \int_{\mathbb{R}} \Big(-\frac{1}{t}\partial_v(f_n(0,x-vt,v)) + \frac{1}{t}(\partial_v f_n)(0,x-vt,v)\Big)\,dv \\
&= \frac{1}{t}\int_{\mathbb{R}} (\partial_v f_n)(0,x-vt,v)\,dv.
\end{aligned}$$

As such for $0 < t < 1/n$, by the definition of $f(0)$

$$\begin{aligned}
t\,\partial_x u_n(t,x) - 1 &= \int_{\mathbb{R}} (\delta(v) - \delta(v-u^0(x-vt)))\,dv - 1 \\
&= -\int_{\mathbb{R}} \delta(v-u^0(x-vt))\,dv.
\end{aligned}$$

Therefore

$$\int_{-R}^{R} |\partial_x u_n(t,x) - 1|\, dx = \int_{\mathbb{R}} \int_{-R+vt}^{R+vt} \delta(v - u^0(x))\, dx\, dv$$

$$\leq \int_{-R-\|u^0\|_{L^\infty}\, t}^{R+\|u^0\|_{L^\infty}\, t} \int_{\mathbb{R}} \delta(v - u^0(x))\, dx\, dv \leq 2\, R\, \|u^0\|_{L^\infty}.$$

u_n is continuous at $t = i/n$ so the same is true at $t = 1/n$.

Next, assume that the estimate is true at time $t = i/n$. Define

$$g_n(i, x, v) = f_n(i/n+, x + v\, i/n, v),$$

and notice that

$$\partial_v g_n = (\partial_v f_n)(i/n+, x + v\, i/n, v) + \frac{i}{n} \partial_x f_n(i/n+, x + v\, i/n, v).$$

On the other hand for $t \in]i/n,\ (i+1)/n]$

$$u_n(t,x) = \int_{\mathbb{R}} f_n(t,x,v)\, dv = \int_{\mathbb{R}} g_n(i, x - vt, v)\, dv.$$

So with the same argument as before

$$\begin{aligned}\partial_x u_n =& \frac{1}{t}\int_{\mathbb{R}} (\partial_v g_n)(i, x - vt.v)\, dv \\ &= \frac{1}{t}\int_{\mathbb{R}} (\partial_v f_n)(i/n+, x + v\,(i/n - t), v) \\ &+ \frac{1}{t}\,\frac{i}{n}\int_{\mathbb{R}} \partial_x f_n(i/n+, x + v\,(i/n - t), v)\, dv.\end{aligned}$$

By the definition of $f_n(i/n+)$, one gets the induction relation

$$\begin{aligned}&t\,\partial_x u_n - 1 = \int_{\mathbb{R}} (\delta(v) - \delta(v - u_n(i/n, x + v\,(i/n - t))))\, dv - 1 \\ &+ \frac{i}{n}\int_{\mathbb{R}} \partial_x u_n(i/n, x + v\,(i/n - t), v)\,\delta(v - u_n(i/n, x + v\,(i/n - t)))\, dv \\ &= \int_{\mathbb{R}} (\frac{i}{n}\partial_x u_n(i/n, x + v\,(i/n - t)) - 1) \\ &\qquad\qquad \times\,\delta(v - u_n(i/n, x + v\,(i/n - t)))\, dv.\end{aligned}$$

Consequently for $i/n < t < (i+1)/n$

$$\int_{-R}^{R} |t\,\partial_x u_n - 1|\,dx \le \int_{-R-(t-i/n)\,\|u^0\|_{L^\infty}}^{R+(t-i/n)\,\|u^0\|_{L^\infty}} \int_{\mathbb{R}} |i/n\partial_x u_n(i/n,x) - 1|\,\delta(v - u_n(i/n,x))\,dv\,dx$$

$$\le \int_{-R-(t-i/n)\,\|u^0\|_{L^\infty}}^{R+(t-i/n)\,\|u^0\|_{L^\infty}} |i/n\partial_x u_n(i/n,x) - 1|\,dx$$

$$\le 2(R + (t - i/n)\,\|u^0\|_{L^\infty})\,\|u^0\|_{L^\infty} + \frac{2\,i}{n}\,\|u^0\|_{L^\infty}^2$$

$$\le 2R\|u^0\|_{L^\infty} + 2\,t\|u^0\|_{L^\infty},$$

because we have assumed that $u(i/n,x)$ satisfies the estimate.

Conclusion

In $1d$ there is a wide gap between the previous BV regularity and the $1/3$ derivative provided by averaging lemmas.
So can we improve averaging lemmas in higher dimensions and maybe get BV ?
There is an example by DeLellis, Otto, Westdickenberg showing that for solutions with bounded entropy production, it is not possible. For entropy solutions it is open.
Regularity in Sobolev spaces is not the only interesting property of solutions. for example, strong traces are proved to exist for the solution by Vasseur. More recently it was shown that the solutions enjoy a "BV like" structure (see Crippa, Otto, Westdickenberg).

And finally kinetic formulations and the corresponding averaging results are not limited to scalar conservation laws...