Final exam for Advanced calculus, 410, on 12/16/2011

The test lasts 2 hours. No documents are allowed. The use of a calculator, cell phone or other equivalent electronic device is not allowed. There are 2 pages.

**Exercise 1** (20pts) Study the convergence of the following series

\[
\sum_{n \geq 1} \frac{\sin^2 n}{n^2}, \quad \sum_{n} \frac{2^n}{n!}.
\]

**Exercise 2** (20pts) Prove that the function \( f \) defined by \( f(x) = x^2 \sin(1/x^8) \) if \( x \neq 0 \) and \( f(0) = 0 \) is differentiable at \( 0 \) and that \( f'(0) = 0 \).

**Exercise 3** (10pts) Let \( (u_n) \) be a sequence in \( \mathbb{R} \) converging to \( l > 0 \). Use the definition of convergence to show that

\[ \exists N, \forall n \geq N, \quad u_n \geq 0. \]

**Exercise 4** (15pts) Using the Lagrange remainder theorem, study the following limits

\[
\lim_{x \to 0} \frac{\log(1 + x)}{x}, \quad \lim_{x \to 0} \frac{\sin x - x}{x^3}, \quad \lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}(\sin x)^2}{x^4}.
\]

**Exercise 5** (15pts) Let \( A \) be any subset of \([0, 1]\) s.t. \( A \) is dense in \([0, 1]\) and \([0, 1] \setminus A \) is also dense in \([0, 1]\). We define

\[ f(x) = \begin{cases} 
0 & \text{if } x \not\in A, \\
1 & \text{if } x \in A.
\end{cases} \]

1. (5pts) Show that for any interval \( I \subset [0, 1] \)

\[ \inf_I f = 0, \quad \sup_I f = 1. \]

2. (10pts) Conclude that for any partition \( P \) of \([0, 1]\)

\[ L(f, P) = 0, \quad U(f, P) = 1, \]
and hence that $f$ is not integrable in the Riemann sense.

**Exercise 6** (10pts+5 bonus pts)
1.(5pts) Explain why for $|x| < 1$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$ 

2.(5pts) Use the previous equality to compute the $k$-th derivative of $1/(1+x^2)$ at $x = 0$.

3. (5 bonus pts, difficult) Use again the first question to deduce that

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$ 

**Exercise 7** (10pts) Give an example of function $f : [0, 1] \to \mathbb{R}$ s.t. $f(x) \geq 0$ for all $x$, there exists $x$ s.t. $f(x) > 0$ and

$$\int_{0}^{1} f = 0.$$ 

**Exercise 8** (15 bonus pts, difficult) Assume that $f_n(x)$ is a family of continuous functions from $[0, 1]$ to $\mathbb{R}$. Assume that $\forall x \in [0, 1]$ fixed, the sequence $(f_n(x))$ is increasing in $n$ and converges to $f(x)$ which is assumed to be continuous on $[0, 1]$.

1.(5 bonus pts) For any $\varepsilon > 0$, we define

$$E_{n, \varepsilon} = \{ x \in [0, 1], \ f_n(x) \leq f(x) - \varepsilon \}.$$ 

Prove that $\bigcap_n E_n = \emptyset$.

2.(10 bonus pts) Using the uniform continuity of $f$, conclude that

$$\forall \varepsilon > 0, \ \exists N, \ \forall n \geq N, \ \forall x \in [0, 1], \ |f(x) - f_n(x)| < \varepsilon.$$ 

Hint: Use the previous question, the continuity of $f_n$ and $f$ and the fact that $f_n(x)$ is increasing in $n$. 

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