A coupled Boltzmann & Navier–Stokes fragmentation model induced by a fluid-particle-spring interaction*

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Abstract

This paper is concerned with the modelling and analysis of the interaction between particles and fluids with particular regarding to fragmentation processes. We simplify the model by assuming that the particles are constituted by spheres jointed by springs. Then the aim is to deduce the terms appearing in the Navier–Stokes-type equations for the fluid and the counterpart influence in the Boltznmann system for the particles. The resulting coupled system is analysed by means of a refined averaging lemma.

1 Introduction and main results

sec:intro

Modeling complex multiphase fluids (two-phase fluids to fix the ideas) is an interesting problem which finds important applications in biotechnology, medicine, ecology, astrophysics, combustion theory or meteorology, such as the production of aerosols, sprays, polymers or diesel motors, for example, see []. The dynamics of the fluids is affected by their mutual interaction and may produce fragmentation or coagulation between the particles constituting the fluids, which modifies the density or the velocity of them. There are different ways to model this situation, depending on the nature of the fluids, their densities and all relevant physical parameters. The so-called fully

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Eulerian or Eulerian–Eulerian description provides a formalism under which the phases are given by physical quantities depending on position and time, such us velocities, densities or energies associated with each phase, see for example []. Another approach consists in a fluid-kinetic (Eulerian–Lagrangian) description in which the particles (or droplets) are inmersed into the surrounding fluid. The dynamics of the particles is described in this case by a probability density function (depending on time, position, velocity, mass or other variables such as the internal energy of the particles) which solves a kinetic equation in the phase space, see for example the pioneering work [26] or the more recent [17] for references. This last approach is more adapted if the particles are very diluted and therefore far from thermodynamical equilibrium.

This paper is concerned with the understanding and analysis of the evolution of a two-phase fluids system described by a fluid-kinetic approach that includes the possibility of particles or droplets fragmentation. The model consisting in a coupled Boltzmann & Navier–Stokes system is deduced from first principles. From the modeling point of view, this issue is rather complex. For instance notice that fragmentation creates kinetic energy in the sense that the sum of the kinetic energies of the daughter particles is always larger than the kinetic energy of the mother particle, provided that conservation of mass and momentum holds. Therefore the model should explain where this energy comes from, typically directly from the fluid or from the "internal" energy of the mother particle. In both cases, it is necessary to describe how the fluid influences the deformation of the particle. As we do not see how to handle the general case, we make the hypothesis that the particles moving by the action of a kinetic equation of Vlasov/Boltzmann–type can be represented by two spherical balls joined together by means of a spring. These particle structures are moving in a surrounding fluid governed by the Navier–Stokes system. Under the hypothesis on the particle structure representation, the number of spherical balls connected by springs and the distribution of the mass among them are not relevant for our modeling arguments.

We now briefly comment the different approaches to this problem studied in the literature.

In the coupling between fluid and kinetic (macro and micro) models different problems can be studied: sedimentation, collisions, fragmentation or coagulation and also the exchanges of mass between a particle and the environment (vaporization or chemical reactions, for example).

The sedimentation and dynamics of spherical particles sinking in a viscous fluid have been recently investigated when the inertia of the particles and the fluid are neglected, being the fluid flow quasi–stationary and described by the incompressible Stokes system, see [15]. Mathematically, in [6] it has been shown that the dynamics has a solution as long as particles do not get too close. The problem of finding a macroscopic system for the dynamics of rigid particles in a sedimentation process has been studied by different authors for particles with or without inertia in compressible or incompressible fluids, see [5, 9, 11, 12, 13, 16, 22, 23]. This approach is particularly suited to polydispersed flows, i.e. flows in which the size of the droplets can vary in a wide range, but each particle has a constant mass.

Fragmentation and coagulation have been studied from different points of view. The T.A.B. model is a description of the fragmentation founded on the hypothesis that fragmentation is due to the increase of the amplitude of the oscillations on the surface of the particles induced by the turbulent character of the surrounding fluid, see [24, 4]. Another interesting approach to determine fragmentation–coagulation kernels is founded on statistical models based on energy principles. This approach describes the transient evolution of the (particle) bubble-size probability density functions resulting from the break-up of the bubble moving in a turbulent fluid (see [18]). Another approach to this problem is given by the study of the time evolution of the average concentration of particles of a given size by means of Smoluchowski-type equations, see [2, 25] for a stochastic point of view. Deterministic studies for the Smoluchowski diffusive models with coagulation-fragmentation kernels have been performed in [20] while the connection between the deterministic discrete and the continuous coagulation-fragmentation models has been investigated in [21].

Let us introduce our main results as well as comment the techniques used in this paper.

Assuming that the particles are constituted by balls connected by a elastic spring and that the probability of fragmentation depends only on the dynamics of the length of the spring, we deduce in the limit a Boltzmann-type equation for the particle distribution function f. The particles are immersed in a fluid and analyzing the exterior Stokes problem first and Navier Stokes then, we deduce the interaction forces acting between the particles and the fluid that induce velocity and deformation for the particles and vorticity in the surrounding fluid. Thus, the particle distribution function depends on the variables t, x, v, p, q, and r, i.e. time, position, velocity, deformation vector, velocity of deformation and radius, and we obtain the following expression for its evolution

$$
\begin{aligned}
\frac{\partial f}{\partial t} \;&+\; v\cdot\nabla_x f + \nabla_v \cdot \left[\left(\frac{a}{r^2}(u-v) + b\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(u-v)\right) f\right] + q\cdot\nabla_p f \\
&+\; \nabla_q \cdot \Bigg[\Bigg(\frac{c}{r^2}(\omega\wedge p - q) - d\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(\omega\wedge p - q) \\
&\qquad\qquad -\mu p - \varphi(p,r)\Bigg) f\Bigg] = Q(f), \quad \boxed{\texttt{Vlasov-i}}
\end{aligned}
$$

where $Q(f)$ is the fragmentation kernel and u is the velocity of the fluid which is governed by the Navier–Stokes equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} \;&+\; u\cdot\nabla u - \nabla\Pi = \nu\Delta u - 2a\int_0^\infty\int_{\mathbb{R}^9}(u-v)rf\,dv\,dp\,dq\,dr \\
&-\; 2b\int_0^\infty\int_{\mathbb{R}^9}\frac{r^4}{|p|}\frac{p\otimes p}{|p|^2}(u-v)f\,dv\,dp\,dq\,dr \\
&\boxed{\texttt{NS-i}}
\end{aligned}
$$

The main difficulty in order to analyze the previous coupled system is that no control on the moments in q or p is available. To overcome this difficulty, we reintroduce in the fluid equation the correction up to the second order in the typical size parameter coming from an additional term in the energy which is the dissipation energy

$$
\int_0^t\int_0^\infty\int_{\mathbb{R}^{12}} r\,|\omega\wedge p - q|^2 f.
$$

This gives this new term in the right hand side of the previous Navier–Stokes equation

$$
2c\eta\,\mathrm{curl}\,\left[\int_0^\infty\int_{\mathbb{R}^9} r(p\wedge(\omega\wedge p - q))f\,dv\,dp\,dq\,dr\right].
$$

Then, by extending our previous analysis concerning the fragmentation processes kinetic equation in [17] together with the use of classical results about weak existence for the Navier–Stokes system [7] combined with a refined averaging lemma of type of those proved in [10], allow to prove a stability result under the hypotheses that the energy, entropy and moments are initially bounded.

The paper is structured as follows. Section 2 is devoted to model our problem making the main assumptions and starting our analysis from the exterior Stokes problem until the complete model including fragmentation. In Section 3 we deal with the formal analysis of the model that includes the study of the different conservation laws. Finally, section 4 deals with the existence and stability properties of weak solutions.

2 Modeling

sec:mod

2.1 General Assumptions

We consider particles moving freely within a fluid. They are assumed to be dispersed enough such that their effect on the fluid is additive: The interaction between the fluid and the particles is just the sum of the interactions the fluid would have with every particle taken separately.

Without any particles, the velocity and pressure of the fluid would be regular solutions to the incompressible Navier-Stokes equations.

In order to determine the interaction between a particle and a fluid, each particle is represented by two balls connected through a elastic spring. The probability of break-up is assumed to depend only on the length of the spring. Note that the possibility of break-up implies that this representation is only a way of making computations possible, indeed as each daughter particle would be composed of two balls, that should make at least four for the mother particle. This is nevertheless very useful for the computations and very much in agreement with the idea behind the T.A.B. model for instance.

Our last assumption is that the length of the spring is much larger than the diameter of the balls composing the particle. This is more a way of simplifying the computations than an absolute requirement and it is of not much consequence with respect to the previous one. In agreement with this assumption we neglect the rotation of each ball.

2.2 Case of a single particle: reduction to Stokes equation

Consider two spherical particles, $B(X^i, R)$, both of radius R centered in the points X^1 and X^2, respectively, such that the distance between them is $|X^1 - X^2| = l$. The particles are moving with velocities V^1 and V^2, respectively. We also assume that these particles are connected by a spring. Let ε be the mean scaled dimensionless path associated with the distance between the particles and let us assume that ε is very small. We denote X the center of the spring, i.e. $(X^1 + X^2)/2 = X$. The spring is moving in a fluid governed by the Navier–Stokes equations that, in velocity–pressure formulation, can be written as follows

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nabla\Pi \;=\; \nu\Delta u + \mathcal{I}\mathcal{P}, \tag{2.1}$$ n-s

with boundary condition on each sphere

$$u(t,x) \;=\; V^i(t,x), \quad \text{on } \partial B(X^i,R). \tag{2.2}$$

n-s-bc

In these relations u is the velocity, Π the pressure and ν the (dynamic) viscosity of the fluid, and $\partial B(X^i,R)$ denotes the boundary of the spherical ball $B(X^i,R)$. The term $\mathcal{IP}$ denotes the influence of the particles in the fluid evolution and needs to be modelled according to the hypothesis on the oscillating particles.

To allow the Navier–Stokes system to observe the spring, we decompose the velocity field (and therefore also the pressure) into a slowly varying part $U(t,x)$ and a second part u_ε which may change over the length of the spring. The idea is that U represents the natural evolution of the fluid (without the influence of any spring) and u_ε the local modification due to the spring. Hence

$$\begin{aligned} u(t,x) &= U(t,x) + u_\varepsilon\left(t, \frac{x-X}{\varepsilon}\right), \\ \Pi(t,x) &= P(t,x) + \frac{1}{\varepsilon}\,\Pi_\varepsilon\left(t, \frac{x-X}{\varepsilon}\right). \end{aligned} \tag{2.3}$$

escales

These lead to the rescaled Navier–Stokes system for u_ε and Π_ε

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} &+ \frac{1}{\varepsilon}(u_\varepsilon \cdot \nabla) u_\varepsilon - \frac{1}{\varepsilon^2}\nabla \Pi_\varepsilon - \nabla P(t,x) \\ &= \nu \frac{1}{\varepsilon^2}\Delta u_\varepsilon - \frac{1}{\varepsilon} U(t,x)\cdot \nabla u_\varepsilon - u_\varepsilon \cdot \nabla U(t,x) + O(\varepsilon), \end{aligned} \tag{2.4}$$

escaled-n-s

which is complemented with the condition at infinity

$$u_\epsilon(t,x) \longrightarrow 0, \quad \text{as } |x| \to \infty, \tag{2.5}$$

stokes-inf

where $U(t,x) = U(t,X) + O(\varepsilon)$.

At the first order in ϵ, the rescaled Navier–Stokes system (escaled-n-s 2.4) becomes the Stokes system, *i.e.*

$$-\nu \Delta u_\varepsilon = \nabla \Pi_\varepsilon + O(\varepsilon), \tag{2.6}$$

stokes

with boundary condition (stokes-inf 2.5) and

$$u_\varepsilon(t,x) \;=\; \tilde{V}^i(t,x) = V_i - \tilde{U}(t, X^i_\varepsilon), \quad \text{on } \partial B(X^i_\varepsilon, R_\varepsilon), \tag{2.7}$$

n-s-bcn

where $X^i_\varepsilon = (X - X^i)/\varepsilon$ is the rescaled center of the corresponding ball and we define the rotational part of the fluid

$$\tilde{U}(t,x) = U(t,X) + \epsilon\omega(t,X) \wedge x,$$

with $\omega = \operatorname{curl} U$ the vorticity of the fluid and "$\wedge$" the crossed product.

2.3 The effect of the spring: An approximation of u_ε

Let $l_\varepsilon = |X_\varepsilon^1 - X_\varepsilon^2|$ and $R_\varepsilon = R/\varepsilon$. We denote by G the fundamental solution of Stokes' equations in dimension three with zero condition at infinity, i.e.

$$G(x) = C\ \frac{\mathrm{Id}}{|x|} - \frac{x \times x}{|x|^3},$$

where Id is the identity matrix.

Proposition 2.1. *The velocity u_ϵ can be approximated by $u_\epsilon = u_\epsilon^{1,1} + u_\epsilon^{1,2} + u_\epsilon^{2,1} + u_\epsilon^{2,2}$, where*

$$u_\epsilon^{i,1}(t,x) = CR_\epsilon G(x - X_\varepsilon^i)\tilde{V}^1 + O\left(\frac{R_\epsilon^2}{|x - X_\varepsilon^1|^2}\right), \qquad (2.8)$$ u11

and with ($j = 2$ if $i = 1$ and $j = 1$ if $i = 2$)

$$u_\epsilon^{i,2}(t,x) = CR_\epsilon G(x - X_\varepsilon^j) u_\epsilon^{i,1}(t, X_\varepsilon^j) + O\left(\frac{R_\epsilon}{l_\epsilon}\left(\frac{R_\varepsilon}{l_\varepsilon} + \frac{R_\varepsilon^2}{|x - X_\varepsilon^2|^2}\right)\right). \qquad (2.9)$$ u12

compueps

Proof. We have to solve Stokes' equation in the domain out of two balls. Depending on the respective orientation of $\tilde{V}^1$ and $\tilde{V}^2$, some explicit formulas are known (using bispherical coordinates), see [], but here as $R_\varepsilon << l_\varepsilon$ we may obtain the desired result very simply.

Let us start by decomposing u_ε into $u_\varepsilon^1 + u_\varepsilon^2$, with u_ε^i the solution to Stokes' equations out of the two balls, vanishing at infinity and with boundary condition $\tilde{V}^i$ on $\partial B(X_\varepsilon^i, R_\varepsilon)$ and 0 on $\partial B(X_\varepsilon^j, R_\varepsilon)$ with $j \neq i$.

Then we use the method of reflection to compute each u_ε^i. We refer to JO [15] for a full description of the method with many particles and the exact conditions under which it works. Let us focus on u_ε^1, u_ε^2 being dealt with in the same manner. We pose

$$u_\varepsilon^1 = \sum_{k=1}^{\infty} u_\varepsilon^{1,k},$$

with $u_\varepsilon^{1,1}$ the solution to

$$\Delta u_\varepsilon^{1,1} = \nabla p_\varepsilon^{1,1}, \quad \operatorname{div} u_\varepsilon^{1,1} = 0, \quad \text{in } \mathbb{R}^3 \setminus B(X_\varepsilon^1, R_\varepsilon),$$
$$u_\varepsilon^{1,1}(\infty) = 0, \quad u_\varepsilon^{1,1} = \tilde{V}^1 \text{ on } \partial B(X_\varepsilon^1, R_\varepsilon),$$

and for $k \geq 1$

$$\Delta u_\varepsilon^{1,2k} = \nabla p_\varepsilon^{1,2k}, \quad \text{div}\, u_\varepsilon^{1,2k} = 0, \quad \text{in } \mathbb{R}^3 \setminus B(X_\varepsilon^2, R_\varepsilon),$$
$$u_\varepsilon^{1,2k}(\infty) = 0, \quad u_\varepsilon^{1,2k} = -u^{1,2k-1} \text{ on } \partial B(X_\varepsilon^2, R_\varepsilon),$$

and finally

$$\Delta u_\varepsilon^{1,2k+1} = \nabla p_\varepsilon^{1,2k+1}, \quad \text{div}\, u_\varepsilon^{1,2k+1} = 0, \quad \text{in } \mathbb{R}^3 \setminus B(X_\varepsilon^1, R_\varepsilon),$$
$$u_\varepsilon^{1,2k+1}(\infty) = 0, \quad u_\varepsilon^{1,2k+1} = -u^{1,2k} \text{ on } \partial B(X_\varepsilon^1, R_\varepsilon).$$

It is easy, at least formally, to check that u_ε^1 satisfies the right equations. Moreover, as $R_\varepsilon << l_\varepsilon$ the convergence of the series defining u_ε^1 poses no difficulty.

Now, note first that $u_\varepsilon^{1,1}$ may be computed explicitly very easily and

$$u_\epsilon^{1,1}(t,x) = R_\epsilon \left(\frac{\tilde{V}^1(t, X_\epsilon^1)}{|x - X_\epsilon^1|} + \frac{R_\varepsilon^2 - |x - X_\varepsilon^1|^2}{4} \nabla \left(\frac{\tilde{V}^1 \cdot (x - X_\varepsilon^1)}{|x - X_\varepsilon^1|^3} \right) \right),$$

which after neglecting the term in $R_\varepsilon^2/|x - X_\varepsilon^1|^2$ gives the desired result. Therefore, in a neighbourhood of $B(X_\varepsilon^2, R_\varepsilon)$, $u_\varepsilon^{1,1}$ is equal to $u_\varepsilon^{1,1}(X^2)_\varepsilon$ up to a correction of order $R_\varepsilon^2/l_\varepsilon^2$. This leads to the corresponding formula for $u_\varepsilon^{1,2}$.

Finally we remark that $u_\varepsilon^{1,k}$ is automatically of order $R_\varepsilon^{k-1}/l_\varepsilon^{k-1}$, justifying our approximation at order 2. □

In itself u_ε is of no interest. It is however required to compute the forces acting on each ball. Still neglecting the rotation we define with the usual formula, see [],

$$F^i = \int_{\partial B(X^i, R)} \sigma \cdot n dS, \quad \sigma = -\Pi\, Id + \nu(\nabla u + \nabla u^T),$$

where the ^{T} superscript denotes the transposed matrix. This means that we have to compute

$$F_\varepsilon^i = \varepsilon \int_{\partial B(X_\varepsilon^i, R_\varepsilon)} \sigma_\varepsilon \cdot n dS, \quad \sigma_\varepsilon = -\Pi_\varepsilon\, Id + \nu(\nabla u_\varepsilon + \nabla u_\varepsilon^T).$$

The previous approximation of u_ε leads to the following

Corollary 2.1. *The local force acting on the particle is*

$$F_\varepsilon^1 = -6\pi\nu R \left(\tilde{V}^1 + C \frac{R_\varepsilon}{l_\varepsilon} \left(\tilde{V}^2 + \frac{\tilde{V}^2 \cdot (X_\varepsilon^1 - X_\varepsilon^2)}{l_\varepsilon^2} (X_\varepsilon^1 - X_\varepsilon^2) \right) \right) + O\left(\frac{R_\varepsilon^3}{l_\varepsilon^2} \right).$$

Proof. We use the decomposition introduced in Proposition compueps 2.1. Note first that Stokes' equations imply that σ_ε is divergence free in the fluid domain. Therefore $u_\varepsilon^{1,1}$ induces no force on the second ball $B(X_\varepsilon^2, R_\varepsilon)$ and $u_\varepsilon^{1,2}$ does not contibrute to the force term on $B(X_\varepsilon^1, R_\varepsilon)$.

Moreover the force acting on a particle alone, moving with velocity W, may be computed explicitly very easily and is just $-6\pi\nu W$, see [] . Consequently, the contribution from $u_\varepsilon^{1,1}$ on $B(X_\varepsilon^1, R_\varepsilon)$ is exactly

$$-6\pi\nu R_\varepsilon \tilde{V}^1.$$

The contribution from $u_\varepsilon^{2,1}$ on $B(X_\varepsilon^1, R_\varepsilon)$ is then

$$-6\pi\nu R_\varepsilon\, C\frac{R_\varepsilon}{l_\varepsilon}\left(\tilde{V}^2 + \frac{\tilde{V}^2\cdot(X_\varepsilon^1 - X_\varepsilon^2)}{l_\varepsilon^2}\,(X_\varepsilon^1 - X_\varepsilon^2)\right) + O\left(\frac{R_\varepsilon^3}{l_\varepsilon^2}\right).$$

□

Note that the exact value of the constants is not very important, as this computation relies on the assumption that the particle is composed of two identical balls and this constant is affected if we apply our model to a chain of $N > 2$ particles jointed by springs.

2.4 The model without fragmentation

Let us denote by m the mass of the particle under consideration, V the velocity of its center of mass $V = V^1/2 + V^2/2$, P its rescaled deformation vector $P = (X_\varepsilon^1 - X_\varepsilon^2)/\varepsilon$ and $Q = \dot{P} = (V^1 - V^2)/\varepsilon$.

Given the computations of the forces that we have performed, we may take (up to the second order in R/l) the following equations for these quantities

$$\begin{aligned}
&\dot{X} = V,\ \dot{V} = \alpha\frac{R\,\nu}{m}\left(U(t,X) - V + \beta\frac{R}{\varepsilon}\,\frac{(U(t,X)-V)\cdot P}{|P|^3}\,P\right),\\
&\dot{P} = Q,\\
&\dot{Q} = \gamma\frac{R\,\nu}{m}\left(\omega(t,X)\wedge P - Q - \delta\frac{R}{\varepsilon}\,\frac{(\omega(t,X)\wedge P - Q)\cdot W}{|P|^3}\,P\right) - \mu P - \varphi_\varepsilon(P),
\end{aligned} \tag{2.10}$$

where $-\mu P$ is the term due to the spring and $\alpha, \beta, \gamma, \delta$ are numerical constants, which could be computed but whose exact value is most certainly irrelevant. In fact, the model can be modified by adding any kind of non-linear spring $g(P)$ (instead of the linear one $-\mu P$) such that $g(P)P > 0$. The function φ_ε represents the repulsive force preventing the two spheres composing the particle from overlapping.

Considering now a large number of such particles, we introduce $\bar{R}$ the average radius, with ρ the density of each particle (assumed to be uniform). Let us define the following constants

$$a = \alpha \frac{\nu}{\rho \bar{R}^2}, \quad b = a\,\beta \frac{\bar{R}}{\varepsilon}, \quad c = \gamma \frac{\nu}{\rho \bar{R}^2}, \quad d = c\,\delta \frac{\bar{R}}{\varepsilon}. \tag{2.11}$$

The regime in which we are interested $R << \varepsilon$ corresponds to the case $a = O(1)$ (in which case $c = O(1)$ as well). Note that b and d are typically small, first order corrections, which we keep as they are reasonably simple.

As the number of particles is taken to be too large to write down a set of equations for each, we consider the particle distribution function f of the variables t, x, v, p, q, and r, i.e. time, position, velocity, deformation vector, velocity of deformation and radius. This function satisfies the kinetic equation

$$\begin{aligned} \frac{\partial f}{\partial t} &+ v \cdot \nabla_x f + \nabla_v \cdot \left[\left(\frac{a}{r^2}(u-v) + b \frac{r}{|p|} \frac{p \otimes p}{|p|^2} \cdot (u-v) \right) f \right] + q \cdot \nabla_p f \\ &+ \nabla_q \cdot \left[\left(\frac{c}{r^2}(\omega \wedge p - q) - d \frac{r}{|p|} \frac{p \otimes p}{|p|^2} \cdot (\omega \wedge p - q) \right.\right. \\ &\qquad \left.\left. -\mu p - \varphi_\varepsilon(p,r) \right) f \right] = 0, \end{aligned} \tag{2.12}$$

Vlasov

where φ can be defined by $\varphi_\varepsilon(p) = \nabla_p \left[(|p| - 2\frac{R}{\varepsilon} r)_+ \right]^{-1}$ or alternatively by, for instance, $\varphi(p) = \nabla_p (|p| - 2\frac{R}{\varepsilon} r)^{-2}$. Notice that there is naturally a R/ε factor in this function which is due to the rescaling of the length of the string by ε whereas the size of each ball was R.

This has to be coupled with an equation for the evolution of the surrounding fluid. The simplest way of obtaining it is through the balance of forces. That means that the fluid should satisfy a Navier–Stokes equation with a force term locally equal to the opposite of the sum of the forces acting on the particle at the same point. This gives

$$\begin{aligned}\frac{\partial u}{\partial t} &+ \quad u\cdot\nabla u-\nabla\Pi=\nu\Delta u-2a\int_0^\infty\int_{\mathbb{R}^9}(u-v)rf\,dv\,dp\,dq\,dr\\ &- \quad 2b\int_0^\infty\int_{\mathbb{R}^9}\frac{r^4}{|p|}\frac{p\otimes p}{|p|^2}(u-v)f\,dv\,dp\,dq\,dr\end{aligned} \tag{2.13}$$

NS

It is also possible to obtain (NS 2.13) directly from our modeling, which has the advantage of making explicit the scaling between the number of particles and their size. Let us first take a finite but large number of particles, numbered from 1 to N. Denote by u^i_ε the correction to the fluid velocity due to the i-th particle, which is computed in the previous sections. Then the velocity u satisfies in the whole $\mathbb{R}^3$

$$\begin{aligned}\frac{\partial u}{\partial t} &+ \quad u\cdot\nabla u-\nabla\Pi=\nu\Delta u\\ &+ \quad \sum_{i=1}^N\frac{1}{\varepsilon^2}(\nu\Delta u^i_\varepsilon+\Pi^i_\varepsilon)-\frac{1}{\varepsilon}((U+u^i_\varepsilon)\cdot\nabla u^i_\varepsilon)-\partial_t u^i_\varepsilon-u^i_\varepsilon\cdot\nabla U,\end{aligned}$$

where the terms in u^i_ε and Π^i_ε depend on t and $(x-X)/\varepsilon$. Moreover if we denote by $F^{1,i}_\varepsilon$ and $F^{2,i}_\varepsilon$ the forces acting on each sphere of the i-th particle, we have that, with X^i the center of the corresponding particle,

$$\frac{1}{\varepsilon^2}\left(\nu\Delta u^i_\varepsilon+\Pi^i_\varepsilon\right)((x-X)/\varepsilon)\sim-(F^{1,i}_\varepsilon+F^{2,i}_\varepsilon)\,\delta_{X^i}(x).$$

Each of this force term behaves like $\nu\bar{R}$. The number of particles at a given space point is given by

$$N\int_0^\infty\int_{\mathbb{R}^6}f\,dv\,dp\,dq\,dr$$

when N is large, which induces the scaling

$$\lambda=N\,\nu\,R\sim 1.$$

In the sense of distributions, one may then easily prove that

$$\sum_{i=1}^N\frac{1}{\varepsilon^2}(\nu\Delta u^i_\varepsilon+\Pi^i_\varepsilon)\longrightarrow\lambda\left(-\rho u+j-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{r}{|p|}\frac{p\otimes p}{|p|^2}(u-v)f\,dv\,dp\right).$$

The other terms converge toward zero (as they are at least one order less in ε) and we obtain (2.13).

Let us remark that there is no term in (2.13) corresponding to the deformation of the spring in (2.12) (the term with the divergence in q). This is again a matter of scaling, as the length of the string (and therefore the force which is applied to it) is small.

It is however possible to derive the corresponding additional terms by doing an expansion in ε in the previous computation. Indeed

$$\frac{1}{\varepsilon^2}\left(\nu\Delta u^i_\varepsilon + \Pi^i_\varepsilon\right)((x-X)/\varepsilon) = -\,(F^{1,i}_\varepsilon + F^{2,i}_\varepsilon)\delta_{X^i} - \varepsilon(F^{1,i}_\varepsilon + F^{2,i}_\varepsilon)\; P^i\cdot\nabla\delta_{X^i}$$
$$-\,\frac{\varepsilon^2}{2}(F^{1,i}_\varepsilon + F^{2,i}_\varepsilon)\; P^i\cdot D^2\delta_{X_i}\cdot P^i + O(\varepsilon^2),$$

where P^i is the rescaled deformation of the i-th particle. Note that, in this expansion, both terms after the first are of order ε^2, the difference $F^{1,i}_\varepsilon - F^{2,i}_\varepsilon$ being itself of order ε (and contrary to the sum which is of order 1).

Passing to the limit in the number of particles and taking $b = d = 0$ so as to simplify the expressions, this would give in the fluid a correction like

$$2c\varepsilon^2\,\mathrm{curl}\,\left[\int_0^\infty\int_{\mathbb{R}^9} r(p\wedge(\omega\wedge p - q))f\,dv\,dp\,dq\,dr\right]$$
$$-\,2a\varepsilon^2\nabla\otimes\nabla : \left[\int_0^\infty\int_{\mathbb{R}^9} rp\otimes p\otimes(u-v)\,f\,dv\,dp\,dq\,dr\right].$$

Finally note that in this case other corrections should be added from the low order terms like $\frac{1}{\varepsilon}((U+u^i_\varepsilon)\cdot\nabla u^i_\varepsilon)$ as this term for instance should contribute at order ε.

2.5 The complete model including fragmentation

Consider the scaling

$$R << \varepsilon, \quad \frac{\nu}{\rho\,\bar R^2}\sim 1, \quad N\,\nu\,\bar R\sim 1,$$

where ν is the viscosity, N the number of particles, ε the average size and $\bar R$ the average radius of each sphere composing a particle.

Then we obtain the equations

$$
\begin{aligned}
\frac{\partial f}{\partial t} &+ v\cdot\nabla_x f + \nabla_v\cdot\left[\left(\frac{a}{r^2}(u-v) + b\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(u-v)\right)f\right] + q\cdot\nabla_p f \\
&+ \nabla_q\cdot\Bigg[\Bigg(\frac{c}{r^2}(\omega\wedge p - q) - d\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(\omega\wedge p - q) \\
&\qquad\qquad -\mu p - \varphi(p,r)\Bigg)f\Bigg] = Q(f),
\end{aligned}
\tag{2.14}
$$

coupled with (2.13). The fragmentation kernel $Q(f)$ reads

$$
\begin{aligned}
Q(f) &= -\frac{1}{2} f(t,x,v,p,q,r)\, B_1(p,q,r) \\
&+ \int_{\mathbb{R}^6} f(t,x,v-q',p',q',2^{1/3}r)\, B_2(p',q',2^{1/3}r,p,q,)\, dp'\, dq'
\end{aligned}
\tag{2.15}
$$

with

$$
\int_{\mathbb{R}^6} B_2(p',q',r',p,q)\, dq\, dp \;=\; 2^{1/3}\, B_1(p',q',r'). \tag{2.16}
$$

This corresponds to the fact that one particle with parameters x, v', p', q', r' may break-up into two identical particles with parameters x, v, p, q, r and x, v^*, p^*, q^*, r^*. Those two particles correspond to the two spheres of which the first was composed. Therefore, their size $r = r^*$ is exactly such that $2\,r^3 = r'^3$ and their velocities are the same $v = v' = v^*$. In fact, remembering the physical scalings, one would have $v = v' + \varepsilon q'$ and $v^* = v' - \varepsilon q'$, which gives $v = v' = v^*$ at the first order in ε. Finally the process is assumed to be invariant under galilean transformations, which means that the probability that it occurs does not depend on the position or velocity of the mother particle.

We refer to [17] and the references therein for a generic study of fragmentation kernels.

Note that the model we propose does not induce itself any extra effect on the fluid, but it is via the distribution function f how the interaction with the fluid is produced.

3 Formal analysis of the model

Let us first check the consistence of our model (2.13), (2.14) and (2.15) by analyzing the balance of conservation laws associated to it, such as mass, moments, energy, ... The precise study must be done as usual in the distributional formulation of (2.13), (2.14) and (2.15) by choosing especial test functions, truncations and approximations of the unity moments, energy, ... We omit here this standard method and the calculations are kept in a formal ambience. We begin this analysis with the mass preservation law for the kinetic equation

Lemma 3.1. *The system (2.14) and (2.15) preserves mass, i.e.*

$$\frac{d}{dt}\int_0^\infty \int_{\mathbb{R}^{12}} 2r^3 f \, d(x,v,q,p,r) \;=\; 0. \tag{3.1}$$ cl1

mass-cons

Proof. Since the other terms have divergence form, to prove mass conservation it is enough to check that

$$\int_0^\infty \int_{\mathbb{R}^{12}} r^3 Q(f)\, d(x,v,q,p,r) \;=\; 0.$$

Using (2.16) and making the change of variables $2^{\frac{1}{3}} r \to r$, we have

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^{12}} r^3 Q(f)\, d(x,v,q,p,r) \\ = -&\frac{2^{-\frac{1}{3}}}{2} \int_0^\infty \int_{\mathbb{R}^{12}} r^3 f(t,x,v,p,q,r)\, B_2(p',q',r',p,q)\, d(x,v,q,p,r) \\ +&\frac{2^{-\frac{1}{3}}}{2} \int_0^\infty \int_{\mathbb{R}^{18}} r^3 f(t,x,v,p',q',r)\, B_2(p',q',r,p,q,)\, d(x,v,q,p,p',q',r) \end{aligned}$$

mass-cons2

Obviously the right hand side of the above equality is zero. □

Let us now analyze the balance of the first momentum with respect to velocity.

Lemma 3.2. *The balance of momentum for the whole system (2.13), (2.14) and (2.15) is preserved and defined by the equation*

$$\frac{d}{dt}\int_0^\infty \int_{\mathbb{R}^{12}} 2r^3 v\, f \, d(x,v,q,p,r) \;=\; -\frac{d}{dt}\int_{\mathbb{R}^3} u\, dx.$$ mom-cons0

mom-cons

Proof. We first deal with calculus for the momentum of the fragmentation kernel

$$\int_0^\infty \int_{\mathbb{R}^{12}} r^3 v\, Q(f)\, d((x,v,q,p,r)$$
$$= -\frac{2^{-\frac{1}{3}}}{2} \int_0^\infty \int_{\mathbb{R}^{12}} r^3 v\, f(t,x,v,p,q,r)\, B_2(p',q',r',p,q)\, d(x,v,q,p,r)$$
$$+\frac{2^{-\frac{1}{3}}}{2} \int_0^\infty \int_{\mathbb{R}^{18}} r^3 v\, f(t,x,v,p',q',r)\, B_2(p',q',r,p,q,)\, d(x,v,q,p,p',q',r)$$

mom-cons2

which is zero as in the previous lemma. Then, the balance of momentum for the Vlasov equation (Vlasov2 2.14) reads

$$\frac{d}{dt}\int_0^\infty \int_{\mathbb{R}^{12}} r^3 v\, f\, d(x,v,q,p,r)$$
$$= \int_0^\infty \int_{\mathbb{R}^{12}} r^3 \left[\left(\frac{a}{r^2}(u-v) + b\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(u-v)\right) f \right] d(x,v,q,p,r).$$ mom-cons3

Taking into account the fluid equation (NS 2.13) we can identify the term in the right hand side as

$$\int_0^\infty \int_{\mathbb{R}^{12}} 2r^3 \left[\left(\frac{a}{r^2}(u-v) + b\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(u-v)\right) f \right] d(x,v,q,p,r)$$
$$= -\frac{d}{dt}\int_{\mathbb{R}^3} u\, dx,$$ mom-cons4

form which we deduce the announced result. □

In the next step we deal with the study of the energy balance. Define the energy associated with (Vlasov2 2.14)-(NS 2.13)

$$e(t) = \int_0^\infty \int_{\mathbb{R}^{12}} r^3 |v|^2 f\, d(x,v,q,p,r) + \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx.$$

Note that this energy does not include the deformation of the particles. The full energy would be

$$\int_0^\infty \int_{\mathbb{R}^{12}} r^3 (|v|^2 + \varepsilon^2 |q|^2 + \mu\varepsilon^2 |p|^2 + \varepsilon^2 \Phi(p,r))\, f\, d(x,v,q,p,r)$$
$$+\frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx, \qquad (3.2)$$ energytotal

with Φ a primitive of φ in p. Taking ε to 0, the formula for $e(t)$ is recovered.

Lemma 3.3. *The energy is a decreasing function and verifies the following balance law*

$$\frac{d}{dt}e(t) \leq -2\int_0^\infty \int_{\mathbb{R}^{12}} \Big[a\, r(u-v)^2 + b\frac{r^4}{|p|^3}\Big(p\cdot(u-v)\Big)^2\Big] f\, d(x,v,q,p,r)$$
$$- \int_{\mathbb{R}^3} \nu |\nabla u|^2 dx.$$

As a consequence, the moments defining the energy are bounded as well as the following quantity

$$\int_0^t \Big\{ \int_0^\infty \int_{\mathbb{R}^{12}} \Big[r(u-v)^2 + \frac{r^4}{|p|^3}\Big(p\cdot(u-v)\Big)^2\Big] f + \int_{\mathbb{R}^3} |\nabla u|^2 \Big\}.$$ ener-cons1/2

ener-cons

Remark. If instead of $e(t)$, one uses (3.2) as the energy, it is also necessary to include in the equation for the fluid all corrections up to the order ε^2. There is then an additional term in the energy dissipation which reads

energytotal

$$\int_0^t \int_0^\infty \int_{\mathbb{R}^{12}} r(\omega \wedge p - q)^2 f.$$

Proof. From now on in this lemma we avoid to mention the differential under integrals for simplicity. As in the previous analysis for the mass and momentum conservation laws, we have

$$\int_0^\infty \int_{\mathbb{R}^{12}} 2r^3 |v|^2 Q(f) = 0.$$ ener-cons1

Indeed, as before

$$\int_0^\infty \int_{\mathbb{R}^{12}} r^3|v|^2 Q(f)$$
$$= \frac{2^{-\frac{1}{3}}}{2}\Big\{ -\int_0^\infty \int_{\mathbb{R}^{12}} r^3|v|^2 f(t,x,v,p,q,r) B_2(p',q',r',p,q)$$
$$+ \int_0^\infty \int_{\mathbb{R}^{18}} r^3|v|^2 f(t,x,v,p',q',r) B_2(p',q',r,p,q)$$

ener-cons2

which is zero.

Let us turn now on the energy moment for the other terms of the system. We first compute it for the kinetic part of the model obtaining

$$-2\int_0^\infty \int_{\mathbb{R}^{12}} r^3 \Big[v\cdot\Big(\frac{a}{r^2}(u-v) + b\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(u-v)\Big). \qquad (3.3)$$ ener-cons3

We now proceed with the contribution to the energy of the fluid coupled equation:

$$-\int_{\mathbb{R}^3} \nu |\nabla u|^2$$
$$-2\int_0^\infty \int_{\mathbb{R}^{12}} r^3 u \cdot \left[\frac{a}{r^2}(u-v)r + b\frac{r}{|p|}\frac{p\otimes p}{|p|^2}(u-v)\right] f. \tag{3.4}$$
ener-cons4

Combining (3.3) and (3.4) we deduce

$$\frac{d}{dt}e(t) = -2\int_0^\infty \int_{\mathbb{R}^{12}} \left[a\, r(u-v)^2 + b\frac{r^4}{|p|^3}\Big(p\cdot(u-v)\Big)^2\right] f$$
$$-\int_{\mathbb{R}^3} \nu|\nabla u|^2. \tag{3.5}$$
ener-cons5

There is only a contribution of positive sign in the right hand side of the above equality and we conclude the announced result. □

4 Existence of weak solutions to (2.14)-(2.13)

The analysis of the coupled system (2.13), (2.14) and (2.15) is hard to deal with. The most difficult terms probably are

$$\int_0^\infty \int_{\mathbb{R}^9} (u-v) r f \, dv\, dp\, dq\, dr,$$

and

$$\int_0^\infty \int_{\mathbb{R}^9} \frac{r^4}{|p|}\frac{p\otimes p}{|p|^2}(u-v) f \, dv\, dp\, dq\, dr.$$

In order to obtain weak solutions (in a sense left unprecise for the moment), one would indeed need to show, using only a priori estimates, that if $(f_n,\ u_n)$ is a sequence of solutions (for instance classical) converging in some sense to $(f,\ u)$ (typically weak for f_n and strong for u_n) then

$$\int_0^\infty \int_{\mathbb{R}^9} (u_n - v) r f_n \, dv\, dp\, dq\, dr \longrightarrow \int_0^\infty \int_{\mathbb{R}^9} (u-v) r f \, dv\, dp\, dq\, dr.$$

Assuming we have suitable compactness for u_n, this would require some control on the moments of f_n so that

$$\int_0^\infty \int_{\mathbb{R}^9} r\,(1,\ v)\, f_n \, dv\, dp\, dq\, dr \longrightarrow \int_0^\infty \int_{\mathbb{R}^9} r\,(1,\ v)\, f \, dv\, dp\, dq\, dr.$$

However with the type of a priori estimates that we detailed in the previous section, no control on the moments in q or p is available. Therefore in order to stabilize the system, we reintroduce in the fluid equation the correction term at order 2 and study instead of (NS 2.13)

$$
\begin{aligned}
\frac{\partial u}{\partial t} &+ u\cdot\nabla u - \nabla\Pi = \nu\Delta u - 2a\int_0^\infty\int_{\mathbb{R}^9}(u-v)rf\,dv\,dp\,dq\,dr\\
&- 2b\int_0^\infty\int_{\mathbb{R}^9}\frac{r^4}{|p|}\frac{p\otimes p}{|p|^2}(u-v)f\,dv\,dp\,dq\,dr\\
&+ 2c\eta\,\mathrm{curl}\,\Big[\int_0^\infty\int_{\mathbb{R}^9} r(p\wedge(\omega\wedge p-q))f\,dv\,dp\,dq\,dr\Big].
\end{aligned}
\tag{4.1}
$$
NS2

We may then get

Theorem 4.1. *Take any $u^0\in L^2(\mathbb{R}^3)$, and $f^0\in L^1(\mathbb{R}^{12}\times\mathbb{R}_+)$, $f^0\geq 0$ such that initially energy, mass and some higher moment in r are bounded*

$$
\int_{\mathbb{R}_+}\int_{\mathbb{R}^{12}} r^3\left(|v|^2+|p|^2+|q|^2+\Phi(p)+r^3\right)f^0(x,v,p,q,r)<\infty, \tag{4.2}
$$
initialenergy

and in addition the entropy is bounded

$$
\int_{\mathbb{R}^{12}}\int_{\mathbb{R}_+} r^3 f^0(x,v,p,q,r)\log f^0<\infty. \tag{4.3}
$$
initialentropy

Assume moreover that

$$
\begin{aligned}
&|B_1| << C\left(1+r^3+r^3|p|^2+r^3|q|^2\right)\ \text{as}\ |p|+|q|+|r|\to\infty,\\
&\Phi(p,r)=+\infty\ \text{if}\ r>|p|.
\end{aligned}
$$

Then there exist $u\in L^\infty(\mathbb{R}_+,\ L^2(\mathbb{R}^3))$ and $f\in L^\infty_{loc}(\mathbb{R}_+,\ L^1(\mathbb{R}^{12}\times\mathbb{R}_+))$, solutions in the sense of distributions to (Vlasov2 2.14) and (NS2 4.1), and satisfying

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla u|^2\,dx\in L^1(\mathbb{R}_+),\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}^{12}} r^3\left(|v|^2+|p|^2+|q|^2+\Phi(p)+r^3\right)f(t,x,v,p,q,r)\in L^\infty(\mathbb{R}_+),
\end{aligned}
\tag{4.4}
$$
energyestimate

and

$$
\int_{\mathbb{R}^{12}}\int_{\mathbb{R}_+} r^3 f(t,x,v,p,q,r)\log f\in L^\infty_{loc}(\mathbb{R}_+). \tag{4.5}
$$
entropy

existtheorem

Let us comment briefly some aspects concerning the hypothesis of Theorem 4.1. The notation $a << b$ simply means that $b/a \to +\infty$. The initial conditions on f^0 are fairly natural: number of particles + total mass + total energy bounded. The entropy condition (4.3) is of a more technical nature as entropy does not seem to play any particular role. It appears to be nevertheless rather necessary.

The assumption on Φ seems logical from the derivation of the model: It only forces the deformation (distance between the centers of the two spheres) to be larger than the radius. The assumption on B_1 is purely technical and it is needed in order to control the terms in the fragmentation kernel.

Sketch of the Proof. We only sketch the main steps that would be required to prove Theorem 4.1. For some complementary details we will address the reader to the references [7, 10, 17].

The idea is, as usual, to prove a weak stability result, *i.e.* to show that a sequence of solutions $u_n,\ f_n$ (satisfying the assumptions in the theorem) converges to another solution in the sense of distributions.

Step 1 : A priori estimates. First of all from the formal analysis in the third section, one deduces from the conditions on the initial data that indeed $u_n \in L^\infty(\mathbb{R}_+,\ L^2(\mathbb{R}^3))$ uniformly in n and that (4.4) holds (except the r^6 part), also uniformly in n. Now we also need to control the total number of particles. For that, simply integrate (2.14) in x, v, p, q and r to get formally

$$\frac{d}{dt}\int_{\mathbb{R}_+}\int_{\mathbb{R}^{12}} f_n = \frac{1}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^{18}} f_n(t,x,v,p,q,r)\, B_1(p,q,r).$$

Now, using the assumption on B_1 and the energy bound we have

$$\frac{d}{dt}\int_{\mathbb{R}_+}\int_{\mathbb{R}^{12}} f_n \le C + C\int_{\mathbb{R}_+}\int_{\mathbb{R}^{12}} f_n(t,x,v,p,q,r),$$

which shows that this integral is bounded, locally in time.

The bound on the entropy requires a more careful calculation. It is identical to that performed in [17], thus we do not reproduce it here. If by any chance

$$\int_{\mathbb{R}_+\times\mathbb{R}^{12}} r^\alpha\, f^0 < \infty,$$

for some $\alpha > 3$ (or any superlinear function of r^3) then this bound remains true for all time, which finishes the proof of (4.4).

Note that we also have additional estimates for the energy dissipation, namely (3.3) and

$$\int_0^\infty \int_{\mathbb{R}^{12}} r\,|w_n \wedge p - q|^2\, f_n \in L^1(\mathbb{R}_+). \tag{4.6}$$

ener-dissip

Those are not convex however and therefore they cannot be used directly.

Finally we will need some more precise estimate on u_n and $w_n = \nabla \wedge u_n$. We proceed in the usual manner for Navier–Stokes (only a sketch again). We know that

$$\frac{\partial u_n}{\partial t} + u_n \cdot \nabla u_n - \nabla \Pi_n = \nu \Delta u_n + G_n,$$

with G_n uniformly bounded in L^1 thanks to the energy dissipation, the control on the total number of particles and (4.4). First take the divergence of the previous equation to get

$$\Delta \Pi_n = -\nabla \cdot G_n + \nabla \cdot (u_n \cdot \nabla u_n) = \nabla H_n,$$

with H_n locally in L^1. By standard elliptic arguments this shows that Π_n is locally in any $W^{s,1}$ with $s < 1$. Introducing this estimate in the Navier-Stokes equation,

$$\partial_t u_n - \nu \Delta u_n = I_n,$$

with I_n locally in $W^{s,1}$ for any $s < 0$. Using the semi-group for the heat equation

$$u_n = S_t\, u_n^0 + \int_0^t \int_{\mathbb{R}^3} \frac{C}{(t-s)^{3/2}}\, e^{-\frac{\nu}{2}|x-y|^2/(t-s)}\, I_n(s,y)\, dy\, ds.$$

Consequently, and again locally, u_n is uniformly in $W^{s,1}$ for any $s < 2$ and so w_n is uniformly in $W^{s',1}$ for any $s' < 1$. This gives the compactness of the sequence w_n in $L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^3)$ for all $1 \le p < 2$.

Step 2 : Compactness of the moments of f_n. A necessary ingredient is the compactness of objects like

$$\int_{\mathbb{R}_+ \times \mathbb{R}^9} \psi(r, v, p, q)\, f_n,$$

for a regular and compactly supported ψ. For this averaging lemmas are classically used. However here, as for other kinetic models, one only has a $L \log L$ estimate on f_n instead of L^p. This would therefore require the use of more refined versions of averaging lemmas, like the one in [10].

Unfortunately the theorem as it is stated in [10] cannot handle the v derivative in Eq. (2.14). We briefly explain how this can be overcome. Fix $M>0$, a regular cut-off χ ($\chi(\xi)=1$ for $\xi\le 1$ and $\chi(\xi)=0$ for $\xi>2$) and define

$$f_M = f_n\,\chi(f_n/M).$$

Notice that simply multiplying (2.14) by $\chi'(f_n/M)$, an equation may be obtained for f_M, namely

$$\begin{aligned}\frac{\partial f_M}{\partial t} &+ v\cdot\nabla_x f_M + \nabla_v\cdot\left[\left(\frac{a}{r^2}(u-v) + b\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(u-v)\right) f_M\right] + q\cdot\nabla_p f_M \\ &+\nabla_q\cdot\left[\left(\frac{c}{r^2}(\omega\wedge p-q) - d\frac{r}{|p|}\frac{p\otimes p}{|p|^2}\cdot(\omega\wedge p - q)\right.\right. \\ &\left.\left.-\mu p-\varphi(p,r)\right)f_M\right] = \tfrac{\chi'(f_n/M)}{M}\,Q(f) + \left(\tfrac{3a}{r^2}+3b\tfrac{r}{|p|}+\tfrac{3c}{r^2}+\tfrac{3dr}{|p|}\right)\tfrac{\chi'}{M}\,f_n. \qquad (4.7)\end{aligned}$$

The right hand side is locally in L^1 (for r^3 times the Lebesgue measure) and f_M is now in any L^p, both uniformly in n (but not M of course). So applying standard averaging lemmas (see [7] for instance), one gets that

$$\int_{\mathbb{R}_+\times\mathbb{R}^9} r^3\,\psi(r,v,p,q)\,f_M \in W^{s,p}_{loc}(\mathbb{R}_+\times\mathbb{R}^3),$$

for some $s>0$ and some p between 1 and 2, and this uniformly in n. From the entropy estimate (4.5), it is then easy to deduce the compactness of

$$\int_{\mathbb{R}_+\times\mathbb{R}^9} r^3\,\psi(r,v,p,q)\,f_n$$

locally in L^1. Combining this with the energy estimate (4.4) and the L^1 bound on f_n, one may further obtain the compactness of all quantities like

$$\int_{\mathbb{R}_+\times\mathbb{R}^9} r^3\,f_n,\quad \int_{\mathbb{R}_+\times\mathbb{R}^{12}} r\,f_n,\quad \int_{\mathbb{R}_+\times\mathbb{R}^9} r\,v\,f_n,$$

and so on.

Step 3 : Passing to the limit. Extracting subsequences (still denoted by n for simplicity), one gets

$$\begin{aligned}&u_n \longrightarrow u \text{ in } L^2,\quad \nabla_x\wedge u_n = w_n \longrightarrow \nabla_x\wedge u = w \quad weak-*\,L^2,\\ &f_n\longrightarrow f\quad weak-L^1,\quad \int_{\mathbb{R}_+\times\mathbb{R}^9} r^3\,\psi\,f_n \longrightarrow \int_{\mathbb{R}_+\times\mathbb{R}^{12}} r^3\,\psi\,f \text{ in } L^1_{loc},\end{aligned}$$

for any regular $\psi(r,p,q,v)$ dominated by $1+|v|^2+|p|^2+|q|^2$. Moreover the limits u and f satisfy the bounds (4.4) and (4.5).

We now have to pass to the limit in the equation and that means taking the limit of terms like $r(w_n \wedge p)\, f_n$ (as always one takes r^3dr as measure for r). This is not possible with the bounds that we have for the moment and we would need to use renormalized solutions. So even though it is not a priori convex, one should try to use the dissipation bounds (3.3) and (4.6). Let us illustrate the way to proceed with (4.6), which is the most difficult.

Denote

$$A_n^M = \int_0^T \int_{\mathbb{R}_+\times\mathbb{R}^{12}} r\,|w_n \wedge p - q|^2\, f_n\, \mathbb{I}_{r+|x|+|p|+|q|+|v|\le M}\, \mathbb{I}_{|w_n|\le M}.$$

The sequence $|w_n \wedge p - q|^2 \mathbb{I}_{|w_n|\le M} \mathbb{I}_{r+|x|+|p|+|q|+|v|\le M}$ is compact in any L^p (thanks to the two cut-off) and in fact in any space "less" than L^∞. On the other hand $\int_{\mathbb{R}_+\times\mathbb{R}^9} \psi(p,q)\, r\, f_n\, \mathbb{I}_{r+|x|+|p|+|q|+|v|\le M}$ is compact in L^1 and in any space "between" L^1 and $L\log L$ (thanks to the cut-off again). Therefore it is possible to pass to the limit in A_n^M (with some easy extra technical work that we omit here) and to get

$$A^M = \int_0^T \int_{\mathbb{R}_+\times\mathbb{R}^{12}} r\,|w\wedge p-q|^2\, f\, \mathbb{I}_{r+|x|+|p|+|q|+|v|\le M}\, \mathbb{I}_{|w|\le M} \le \liminf A_n^M \le A^0.$$

Note that the sequence in M, $r\,|w\wedge p-q|^2\, f\, \mathbb{I}_{r+|x|+|p|+|q|+|v|\le M}\, \mathbb{I}_{|w|\le M}$ is non decreasing. As its integral is uniformly bounded in M, one may pass to the limit in M to finally obtain

$$\int_0^T \int_{\mathbb{R}_+\times\mathbb{R}^{12}} r\,|w\wedge p-q|^2\, f \le A^0 < \infty.$$

This estimate combined with the energy dissipation (4.6) and the entropy bound (4.5) is now enough to pass to the limit in terms like $(w_n\wedge p)\, f_n$, enabling us to derive the equation at the limit.

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