Final exam for PDE I, AMSC/Math 673: 1st part, home exercises due
Friday 12/12

Exercise 1.
Find the fundamental solution to the Stokes’ system, i.e. find a vector field $u(x)$ for which there exists a scalar field $p(x)$ satisfying
\[
- \Delta u = \nabla p + (\delta_0, 0, 0), \quad x \in \mathbb{R}^3, \\
\text{div } u = 0.
\]
You may use any method you wish (Fourier transform, spherical averages...) but you must explain every step.

Exercise 2. We wish to study the following equation on $\mathbb{R}$ on $u(t,x)$
\[
\partial_{tt} u - \partial_{xx} u + i \partial_{xxx} u + \partial_{xxxx} u = 0, \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R},
\]
with the initial conditions
\[
|t=0 = f, \quad \partial_t|_{t=0} = g.
\]
1. Assume $u \in L^\infty_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{R}))$ solves the equation. Define $\hat{u}$ as the Fourier transform of $u$ in $x$. Show that $\hat{u}$ solves
\[
\partial_{tt} \hat{u}(t,x) + m(\xi) \hat{u}(t,\xi) = 0,
\]
for some $m(\xi)$ that you should specify.
2. Prove that $m(\xi) \geq 0$ for every $\xi$.
3. Find an explicit formula for $\hat{u}$ in terms of $\hat{f}$ and $\hat{g}$.
4. Prove that there exists a unique solution $u \in L^\infty_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{R}))$.

Exercise 3. Find the explicit solution $u \geq 0$ to
\[
\partial_t u - x_2 \partial_{x_1} u + x_1 \partial_{x_2} u = \sqrt{u}, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}_+,
\]
in terms of the initial data $u_0$.

Exercise 4. Consider the following equation
\[
\partial_t u - \Delta u = f(u),
\]
for a given non linear function $f \in C^\infty$.
1. Assume that $f(0) = 0$. Show that if $u \in C^2$ solves the equation and if $u(t=0) > 0$ then $u$ remains positive at all times.
2. Consider $u_1, u_2 \in C^2$ two solutions to the equation s.t. $u_1(t=0, x) < u_2(t=0, x)$ for all $x$. Prove that $u_1 < u_2$ at any later time.