1st Midterm for Linear Algebra and its application, 401, section 0301 on 2/28/2012

The test lasts 1 hour and 15 minutes. No documents are allowed. The use of a calculator, cell phone or other equivalent electronic device is not allowed.

1)(2pts) Give the expression of $(AB)^{-1}$ in terms of $A^{-1}$ and $B^{-1}$.

Solution. $(AB)^{-1} = B^{-1} A^{-1}$.

2)(2pts) Let $A$ be a $n \times p$ matrix and $x \in \mathbb{R}^p$. Give the general expression of the coefficient $i$ of $Ax$ in terms of the coefficients of $A$ and of $x$.

Solution.

$$(Ax)_i = \sum_{j=1}^{p} A_{ij} x_j.$$ 

3)(3pts) Give the $LU$ decomposition of the following matrix

$$\begin{bmatrix}
1 & 1 & 1 \\
2 & 3 & 5 \\
1 & 4 & 6
\end{bmatrix}.$$ 

Solution. We compute the elimination steps, first $L_2 \leftarrow L_2 - 2L_1$ and $L_3 \leftarrow L_3 - L_1$ to find

$$\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 3 & 5
\end{bmatrix},$$

which gives the two coefficients of $L$ on the first column: $L_{21} = 2$ and $L_{31} = 1$.

Finally we do the elimination step $L_3 \leftarrow L_3 - 3L_2$ to obtain

$$U = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 0 & -4
\end{bmatrix},$$

and which gives the final coefficient for $L$ or $L_{32} = 3$. Hence

$$L = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
1 & 3 & 1
\end{bmatrix}.$$
4) (1pt) Assume that $A$ is a square $n \times n$ matrix, which is already lower triangular (so there is no need for any elimination step). Assume that $A$ has only non-zero coefficients on the diagonal and the two subdiagonals. How many multiplications does one need to solve $Ax = b$? We count divisions as multiplications.

Solution. We solve starting from the first line which gives $x_1 = b_1/A_{11}$ or 1 multiplication. The equation for the second line is $x_2 = (b_2 - A_{21}x_1)/A_{22}$ or 2 multiplications. After the second line, the general formula becomes

$$x_i = (b_i - A_{i,i-2}x_{i-2} - A_{i,i-1}x_{i-1})/A_{ii},$$

or 3 multiplications for $n - 2$ lines. The total is $3(n - 2) + 2 + 1 = 3n - 3$.

5) (1pt) Is the following matrix a permutation matrix? Why?

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

Solution. No it is not a permutation. For instance if it applied to the vector $(x_1, x_2, x_3)$ the last coefficient becomes $x_2 + x_3$ which is not any permutation of the other coefficients.

6) (1pt) Let $A$ be a square $n \times n$ matrix. Define $B = A^T A$. Assume that for some $x$ in $\mathbb{R}^n$, $Bx = 0$. Prove that then $Ax = 0$.

Solution. Simply multiply to the left by $x^T$ to get $x^T A^T A x = 0$. This is equal to $(Ax)^T Ax = 0$ which is exactly $\|Ax\|^2 = 0$ (the square of the euclidian norm). That finally implies that $Ax = 0$. 

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