DiPerna-Lions flow for relativistic particles in an electromagnetic field

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Abstract

We show existence and uniqueness of a DiPerna-Lions flow for relativistic particles subject to a Lorentz force in an electromagnetic field. The electric and magnetic fields solve the linear Maxwell system in the vacuum but for singular initial conditions which are only in the physical energy space. As the corresponding force field is only in L^2, we have to perform a careful analysis of the cancellations over a trajectory.

1 Introduction

1.1 The model

We study the existence and stability of relativistic charged particles in given electro-magnetic field. Denoting by $X(t)$ the position of a particle and by

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$V(t)$ its velocity, one has to solve the system

$$\frac{d}{dt}X(t) = \frac{V(t)}{\sqrt{1+V^2(t)}}, \quad \frac{d}{dt}V(t) = E(t,X) + V \times B(t,X), \tag{1.1}$$

where $E(t,x)$ and $B(t,x)$ are the electric and magnetic fields and solve Maxwell's equations in the vacuum

$$\begin{aligned} &\operatorname{div} E = \rho(t,x) = 0, \quad \operatorname{div} B = 0,\\ &\partial_t B + \nabla \times E = 0, \quad \partial_t E - \nabla \times B = 0,\\ &E(t=0,x) = E^0(x), \quad B(t=0,x) = B^0(x). \end{aligned} \tag{1.2}$$

The initial fields E^0 and B^0 are given.
The system (1.1) coupled with (1.2) is relatively simple but already has many physical applications, for example to plasmas with densities that are low enough to neglect both the interactions of the particles (ions and electrons) and their influence on the electromagnetic fields. In that case the trajectory of the ions or the electrons is described by (1.1)-(1.2).
For plasmas with higher densities, the models should be modified: The Maxwell's equations on E and B would now be coupled to the particles' distribution of charges and current, collisions between particles should be taken into account... However in this more complicated framework, a good understanding of (1.1)-(1.2) is required (see the more technical part at the end of the introduction).
Well posedness is a basic but crucial question for (1.1)-(1.2) which is related to several natural physical issues; for instance is the trajectory stable, at least over a fixed time interval, if one perturbs the initial position and velocity?
The classical Cauchy-Lipschitz theory guarantees well posedness and stability through a Gronwall type estimate, provided that the force field is Lipschitz. This would require here that $E,\ B \in W^{1,\infty}_{loc}$. Of course Maxwell' equations (1.2) propagate any Sobolev H^k norms of the solutions so by taking E^0 and B^0 smooth enough, it would be possible to satisfy the previous Lipschitz condition.
Unfortunately, this precludes any fast oscillations in E or B, *i.e.* any high frequency fields which are nevertheless commonly found in applications. Instead a more physical assumption would just be that the electromagnetic energy associated with (1.2) be finite, namely

$$\int_{\mathbb{R}^3} (|E(t,x)|^2 + |B(t,x)|^2)\,dx = \int_{\mathbb{R}^3} (|E^0(x)|^2 + |B^0(x)|^2)\,dx.$$

The goal of this article is to give a proof of a notion of well posedness and stability (for the flow) under the only physical condition on E^0 and B^0 that they belong to $L^2(\mathbb{R}^3)$ and hence the electromagnetic energy is bounded.
Note finally that Maxwell equations (1.2) have simple explicit solutions in dimension 3 which can be obtained by choosing a gauge and solving two wave equations. In order to simplify the formulation (and the calculations), we will restrict ourselves to

$$E(t,x)=\partial_t\left(t\int_{S^2}E_0(x+t\,\omega)\,d\omega\right),\quad B(t,x)=\partial_t\left(t\int_{S^2}B_0(x+t\,\omega)\,d\omega\right),\tag{1.3}$$

without loss of generality as the other terms may be treated in the same manner. The explicit formulas (1.3) propagate the L^2 norms of E and B and the natural question is still whether it is possible to obtain stability and well posedness assuming only that $E^0,\ B^0\in L^2$.

1.2 The flow and renormalized solutions

In general if the force field is not Lipschitz, it is not realistic to expect well posedness for every trajectory. Instead one chooses a more statitiscal approach and defines the flow or the characteristics for the problem

$$\begin{aligned}&\frac{d}{dt}X(t,x,v)=\frac{V(t,x,v)}{\sqrt{1+V^2(t,x,v)}},\quad \frac{d}{dt}V(t,x,v)=E(t,X)+V\times B(t,X),\\ &X(0,x,v)=x,\qquad V(0,x,v)=v,\end{aligned}\tag{1.4}$$

where E and B still solve (1.2). As flows the solutions are required to satisfy

Property 1 *For any $t\in\mathbb{R}$ the application*

$$(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\mapsto(X(t,x,v),V(t,x,v))\in\mathbb{R}^3\times\mathbb{R}^3\tag{1.5}$$

is globally invertible and has Jacobian 1 at **almost every** $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$. *It also defines a semi-group*

$$\begin{aligned}&\forall s,t\in\mathbb{R},\qquad X(t+s,x,v)=X(s,X(t,x,v),V(t,x,v)),\\ &\textit{and}\qquad\quad V(t+s,x,v)=V(s,X(t,x,v),V(t,x,v)).\end{aligned}\tag{1.6}$$

Instead of looking for a solution for a particular initial data, we look for X and V satisfying Property 1 and (1.4) in the sense of distribution, *i.e.*

$$X(t,x,v) = x + \int_0^t \frac{V(s,x,v)}{\sqrt{1+V^2(s,x,v)}}\,ds \quad a.e. \text{ in } x,v,$$
$$V(t,x,v) = v + \int_0^s E(s,X(s)) + V(s)\times B(s,X(s))\,ds \quad a.e. \text{ in } x,v.$$

Similarly uniqueness means that if (X,V) and (Y,W) both solve (1.4) while satisfying Property 1 then $X=Y$ and $V=W$ *a.e.* in x,v.
The well posedness of (1.4) is closely connected to the question of renormalized solutions for the associated Liouville equation which is simply here

$$\begin{aligned}&\partial_t f + \frac{v}{\sqrt{1+|v|^2}}\cdot\nabla_x f + (E+v\times B)\cdot\nabla_v f = 0, \quad x,v\in\mathbb{R}^3,\\ &f(t=0,x,v) = f^0(x,v).\end{aligned} \tag{1.7}$$

The density in phase space $f(t,x,v)$ gives the probability of finding a particle with position x and velocity v at time t.
Eq. (1.7) is a particular case of the more general transport equation

$$\partial_t u + b(t,\xi)\cdot\nabla_\xi u = 0. \tag{1.8}$$

Indeed if $\xi=(x,v)$ and $b(t,\xi) = (v/\sqrt{1+v^2},\ E(t,x)+v\times B(t,x))$ then (1.8) is simply (1.7).
The concept of renormalized solution was introduced in [13] (we also refer to [11] for a very nice introduction to the theory).

Definition 1.1 *A solution $u\in L^\infty([0,\ T]\times\mathbb{R}^3)$ to* (1.8) *in the sense of distribution is said to be a renormalized solution iff any smooth non linear function $\beta(u)$ of u also solves* (1.8).
The field b satisfies the renormalization property if any solution u to (1.8) *in the sense of distribution is a renormalized solution.*

If the renormalization property holds for b, then it implies uniqueness of any weak solution to (1.8). Moreover in this general context, one may again define the flow as a system of characteristics, or

Definition 1.2 *A measurable function $\Xi(t,\xi)$ is a flow associated to* (1.8) *iff*

i. For any set A of measure 0 and time t, one has $|\Xi(t,A)|=0$.
ii. $\Xi(t,.)$ defines a group of transforms in space: $\Xi(t,\Xi(s,\xi))=\Xi(t+s,\xi)$.
iii. Ξ is a solution in the sense of distribution to

$$\partial_t \Xi(t,\xi)=b(t,\Xi(t,\xi)).$$

It can then be proved (see [11] for instance) that if b satisfies the renormalization property then there exists a unique flow associated to (1.8).
Note here that if there exists a flow Ξ then it is straightforward to build a solution u to (1.8) by $u=\Xi\#u^0$ and that solution is automatically renormalized. However without additional assumptions on b, it is not known whether the uniqueness of the flow would imply that b has the renormalization property.
As a partial conclusion, we now have two ways of obtaining well posedness: Either by proving directly existence and uniqueness of the flow or by proving that the force field satisfies the renormalization property.

1.3 Previous results, our contribution

It was first shown in [13] that the renormalization property holds provided $b\in L^p_t W^{1,p}_x$ with $p\geq 1$. This was extended to $b\in BV$ with $\operatorname{div} b\in L^1$, first in [4] for the special kinetic case, and then in [2] for the general case. We also refer to [11] for a very nice presentation of the main results and in particular more precise statements of the connections between renormalized solutions and the flow for (1.8).
In our context, Eq. (1.4) or (1.7), the force field is divergence free but the BV assumption would require that $E,\ B\in BV$. This is quite far from our goal of $E,\ B\in L^2$; and even more as (1.2) or (1.3) do not propagate BV bounds and $E,\ B\in BV$ would in fact require $E^0,\ B^0\in H^1$.
In the general framework of (1.8), the assumption $b\in BV$ is optimal in the sense that it is possible to find examples of fields b, "almost but not quite" in BV s.t. there is no uniqueness to (1.8), see [12].
However Eq. (1.4) or (1.7) is not posed in such a general framework but instead in the phase space. It has long been recognized that well posedness for kinetic equations is easier than for general transport equations. The BV case was obtained earlier in [4] and it was even improved in [19] to systems with a finite number of particles interacting through a kernel in $BV_{loc}(\mathbb{R}^d\setminus\{0\})$, possibly more singular at 0.

Here it is possible to use the additional phase space structure of our particular equations, Eq. (1.4) or (1.7), to improve on this BV assumptions. Applying the results of [7], one deduces that there exists a unique flow solving (1.4) and satisfying Property 1 provided $E,\ B \in H^{3/4}$. Observe that this does not imply the renormalization property but simply the well posedness of the flow. If $E,\ B$ satisfy (1.3), the $H^{3/4}$ requirement is equivalent to $E^0,\ B^0 \in H^{3/4}$ and slightly better than the previous condition, but it is unsatisfying and leaves a large gap with the desired physical assumption $E^0,\ B^0 \in L^2$.
Unfortunately an example given in [7] shows that the techniques in this article cannot in general be extended if the force field is less than $W^{1/2,1}$; still 1/2 a derivative more than available.
In this article, we use the same general framework as in [7] but we introduce a completely new way of handling the estimates with the force fields. We combine the dispersive properties of (1.4) with the dispersive structure of (1.3) (or (1.2) in general) to completely remove any regularity assumptions on E or B (see also a sketch of the proof in the second section). To our knowledge this is the first result in dimension larger than 1 (*i.e.* 2 of phase space) to completely replace regularity by structural assumptions on the forces.
We are thus able to prove well posedness in the critical physical space

Theorem 1.1 *Assume that* $E_0,\ B_0 \in L^1 \cap L^2(\mathbb{R}^3)$ *and* $f^0 \in L^1 \cap L^\infty(\mathbb{R}^6)$. *Then there exists a unique flow* $(X,\, V)$ *solving* (1.4) *with Property* (1) *and a unique solution* $f \in L^\infty(\mathbb{R}_+,\ L^p(\mathbb{R}^6))$ *to Eq.* (1.7) *satisfying*

$$f(t, X(t,x,v), V(t,x,v)) = f^0(x,v).$$

Remark that we do not prove that the renormalization property holds. Instead we prove well posedness directly on the flow.
A key ingredient in the proof is that there are different different speeds of propagation between the slow particles and the fields which propagate at the speed of light; thus yielding some regularizing properties on the wave equation. This kind of idea was already used, in a very different form, for the existence theory of strong solutions to the Vlasov-Maxwell system [21, 6, 17], also it is at the origin of the space resonance method used for instance in [16].
While this is perfectly satisfactory for the physical application we have in mind, one could nevertheless wish to study the interaction of particles with

fields that are propagated at several different speeds, possibly comparable to the particles' velocity.
We offer a partial answer in an essentially $1-d$ setting in x and now turn to

$$\begin{aligned}&\partial_t f+\alpha(v)\partial_x f+F(t,x)\cdot\nabla_v f=0,\quad x\in\mathbb{R},v\in\mathbb{R}^d,\\&f(t=0,x,v)=f^0(x,v).\end{aligned}\tag{1.9}$$

The function α is assumed to be Lipschitz: $\alpha\in W^{1,\infty}(\mathbb{R}^d)$ and satisfies a genuine non degenerescence assumption: There exists C such that for all $w\in\mathbb{R}$ and $\eta>0$, we have

$$|\{v,\ |\alpha(v)-w|\leq\eta\}|\leq C\eta.\tag{1.10}$$

The force field F is assumed to be given by

$$F(t,x)=\sum_n F^0(x-\xi_n\,t)\,\mu_n,\tag{1.11}$$

with $F^0\in L^\infty$.
Some additional decay assumption is also needed on the coefficients μ_n,

$$\exists\gamma>2,\quad\sum_n(1+n^\gamma)\,\mu_n<\infty.\tag{1.12}$$

It is of course necessary to have $\sum_n\mu_n<\infty$ in order for the definition of F to make sense. We need a slightly stronger decay for the well posedness of the flow though at this stage it is far from clear whether $\gamma>2$ is optimal.
Note that whereas x is necessarily 1 dimensional, there is no such constraint on v. So for instance, Eq. (1.7) in the radially symmetric case would fit in this reduced framework.
We have

Theorem 1.2 *Assume that F is given by* (1.11) *and satisfies* (1.12)*, then for any $f^0\in L^p(\mathbb{R}\times\mathbb{R}^d)$ for $p>1$, there exists a unique flow (X,V) and a unique solution $f\in L^\infty(\mathbb{R}_+,\ L^p(\mathbb{R}\times\mathbb{R}^d)$ to Eq.* (1.9) *which is constant along the flow*

$$f(t,X(t,x,v),V(t,x,v))=f^0(x,v).$$

In this 1 dimensional context, many results are already known. If $v \in \mathbb{R}$, $\alpha(v) = v$ and $F(t, x) = F^0(x)$ is autonomous then well posedness was already obtained in [5], with an extension when F^0 is only L^p in [18]. The key for both results is the Hamiltonian structure which implies the propagation of the total energy $v^2/2 + \Phi(x)$ with $-\Phi' = F^0$ which allows to compute v in terms of x (up to a sign).

This type of result was extended to general, autonomous transport equations in dimension 2 with a force field of bounded divergence, which is hence "close enough" in some sense to the Hamiltonian case.

An additional assumption of non characteristic curve is always needed. This is also the case here and it is ensured by (1.10), which is however much stronger than the assumptions in [9], [8] and [1] (which has the most general assumptions). It would be possible to weaken assumption (1.10) but it would then be necessary to make stronger (1.12) and in that sense (1.10) seems to be the price to pay to be able to handle some time dependence.

Eq. (1.9) is still a kinetic equation: Even though strictly speaking, we are not in a Hamiltonian case, it is very close to the earlier formulation of [5] or [18]. But Theorem 1.2 is not limited to autonomous F which is the real improvement here. Unfortunately it is still not as general as one would like as we still have to assume some structure on the time dependence of F given by (1.11)-(1.12).

1.4 Renormalized solutions for the Boltzmann-Vlasov-Maxwell system

This subsection presents some further motivation for our study. It is more technical and can be safely ignored by readers who are not concerned with this more specialized application of kinetic theory.

The Boltzmann-Vlasov-Maxwell system reads

$$\begin{aligned} &\partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = Q(f), \quad x, v \in \mathbb{R}^3, \\ &f(t=0, x, v) = f^0(x, v). \end{aligned} \tag{1.13}$$

Here, E and B are the electric and magnetic fields solving the Maxwell equations with charges and current sources given by f

$$\begin{aligned} &\operatorname{div} E = \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv, \quad \operatorname{div} B = 0,\\ &\partial_t B + \nabla\times E = 0, \quad \partial_t E - \nabla\times B = -j(t,x) = -\int v\, f(t,x,v)\,dv, \end{aligned} \tag{1.14}$$

while Q is the usual Boltzmann collision kernel

$$Q(f) = \int_{\mathbb{R}^3}\int_{S^2} b(\omega,\, v_*)\ \left(f(t,x,v'_*)\,f(t,x,v') - f(t,x,v_*)\,f(t,x,v)\right)\,d\omega\,dv_* \tag{1.15}$$

where b is a given cross section and the velocities satisfy

$$v' = v - (\omega\cdot(v-v_*))\,\omega, \quad v'_* = v_* + (\omega\cdot(v-v_*))\,\omega.$$

The two differences between the system (1.13)-(1.14) and (1.4) or (1.7) coupled with (1.2) are

- Maxwell equations are not solved in the vacuum anymore but are coupled to the density of charges and current created by the particles themselves.
- Collisions between particles are now taken into account.

Therefore System (1.13)-(1.14) can accurately describe the evolution of a plasma of charged particles at higher densities than (1.7)-(1.2) which can be seen as a limit of (1.13)-(1.14) for very low densities. Indeed, note that if f is very small then the collision term in (1.13) may be neglected and (1.13) reduces to (1.7); similarly the density of charges and current in (1.14) are very small and (1.14) would be reduced to (1.2).
The global existence of solutions to (1.13) is an important open problem in kinetic theory but it is notoriously hard as it combines the difficulties of the Boltzmann and Vlasov-Maxwell systems.
To better understand where the difficulty lies, let us briefly describe the standard theory of renormalized solutions on the Boltzmann equation which is simply

$$\partial_t f + v\cdot\nabla_x f = Q(f), \tag{1.16}$$

where Q is still defined by (1.15). Because Q is quadratic in f and we only have an $L\log L$ bound on f coming from the entropy inequality, the only available theory of global existence for (1.16) is the theory of renormalized solution initiated in [15]. Roughly speaking f is a renormalized solution to (1.16) if in the sense of distributions

$$\partial_t \log(1+f) + v\cdot\nabla_x \log(1+f) = \frac{Q(f,f)}{1+f}. \tag{1.17}$$

It is of course now much simpler to give a meaning to $Q(f,f)/(1+f)$. This strategy relies on the formal computation

$$\partial_t \log(1+f) + v\cdot\nabla_x \log(1+f) = \frac{\partial_t f + v\cdot\nabla_x f}{1+f} \tag{1.18}$$

This part is straightforward in the case of the Boltzmann equation as it is possible to use the characteristics for the transport operator which are here simply lines $x+v\,t$. This is also useful in order to pass to the limit in the above formulation and obtain a renormalized solution as a limit of a regularized problem. It is connected to the notion of mild solution to (1.16) or

$$f(t,x+v\,t,v) - f(s,x+v\,s,v) = \int_s^t Q(f(t,x+v\,r,v))\,dr. \tag{1.19}$$

Ideally one would like to apply the same strategy to (1.13). One key additional problem in extending (1.17) is to be able to justify, in the more complex case with an electro-magnetic field, a computation like (1.18) or define the characteristics to obtain a generalization of (1.19).
Therefore we would need to show that the system (1.7) coupled with (1.14) has the renormalization property or to obtain well posedness for (1.4) where E, B solve (1.14).
The only available global existence result for (1.7)-(1.14) is in [14]. The proof in [14] uses a weak compactness argument that only relies on averaging lemmas and the solutions constructed can only be weak solutions for which it is not possible to show the renormalization property.
Note that the system (1.13)-(1.14) has very few a priori estimates. In particular f is at most in $L^1\cap L\log L$ and the vector fields E and B are only known to be in $L^\infty(0,T;L^2(\mathbb{R}^3))$ and there is no other choice than to work in the critical space. In this sense, our present result can be seen as a first step (with (1.2) instead of (1.14)) in this program.

We are mostly interested here in the so-called cut-off case for the Boltzmann-Maxwell system; those models typically include a cut-off in the angular variable of the kernel which makes it integrable. Remark that existence of renormalized solutions can be proved for the Boltzmann-Maxwell system with long range interaction; this is usually called the non cut-off case in kinetic theory. One only has to renormalize the equation on a regularized approximation (before the passage to the limit). It is then possible to take advantage of the regularizing property of the equation and hence the strong convergence of $f^n(t,x,v)$ in all variables to pass to the limit and recover a renormalized solution, as in the announced result [3]. This is unfortunately not possible in the case with cut-off.

2 Proof of Theorem 1.1

2.1 Definition of the functional and reduction of the problem

The structure of E and B in (1.3) is essentially the same. To simplify the calculations, study the following equation

$$\partial_t f + \frac{v}{\sqrt{1+v^2}} \cdot \nabla_x f + F(t,x,v)\cdot\nabla_v f = 0, \tag{2.1}$$

or the characteristic system

$$\begin{aligned} &\frac{d}{dt}X(t,x,v) = \frac{V(t,x,v)}{\sqrt{1+V^2(t,x,v)}}, \quad \frac{d}{dt}V(t,x,v) = F(t,X), \\ &X(0,x,v) = x, \qquad V(0,x,v) = v. \end{aligned} \tag{2.2}$$

The force term F is given by

$$F(t,x,v) = \nu(v)\,\partial_t\left(t\int_{S^2} F_0(x+t\omega)\,d\omega\right), \tag{2.3}$$

with ν a C^∞ function of v. F now encompasses the possible form of both E and B.

Theorem 1.1 is then equivalent to

Theorem 2.1 *Assume that F satisfies Equation* (2.3) *with initial data* $F_0 \in L^1 \cap L^2(\mathbb{R}^3)$. *Then there exists a unique solution to* (2.2) *satisfying Property 1.*

Our strategy here is to derive explicit quantitative estimates on the trajectory. A functional was introduced in [10] for that (see also [20] for an extension). We use here the modified functional introduced in [7] specifically for kinetic equations: For any compact domain $\Omega \subset \mathbb{R}^6$ we look at

$$Q_\delta(T) = \int_\Omega \log\left(1 + \frac{1}{|\delta|^2}\left[\left(\sup_{0\le t\le T} |X(t,x,v) - X^\delta(t,x,v)|^2 \right.\right.\right.$$
$$\left.\left.\left. + \int_0^T |V(t,x,v) - V^\delta(t,x,v)|^2\,dt\right) \wedge 1\right]\right)\,dx\,dv$$

where (X, V) and (X^δ, V^δ) are both solutions to (2.2) satisfying Property 1 or (X, V) solves a regularized version of (2.2) and (X^δ, V^δ) satisfies

$$\begin{aligned} \exists(\delta_1,\delta_2) \in \mathbb{R}^6 \quad & \text{with } |(\delta_1,\delta_2)| \le \delta, \\ & (X^\delta, V^\delta)(t,x,v) = (X,V)(t,x+\delta_1,v+\delta_2). \end{aligned} \tag{2.4}$$

In the following, we will frequently abuse the notation δ for in fact $|\delta|$. Then Theorem 1.1 is implied by

Proposition 2.1 *For any Ω compact, any $F_0 \in L^1 \cap L^2(\mathbb{R}^3)$, there exists a function ψ depending on T, Ω and F^0, such that for any (X, V) solution to* (2.2) *with F given by Equation* (1.3)*, satisfying Property 1, and (X^δ, V^δ) satisfying* (2.4)*,*

$$Q_\delta(T) \le T\,\psi(-\log|\delta|),$$

with

$$\frac{\psi(\xi)}{\xi} \longrightarrow 0, \quad \text{as } \xi \to \infty.$$

We sketch the connection between Prop. 2.1 and Th. 2.1 in the following subsection, and give a sketch of the proof of Prop. 2.1 after that. The actual proof of Prop. 2.1 is performed in the third section.

2.2 Sketch of the connection between Prop. 2.1 and Theorem 2.1

The connection between Proposition 2.1 and Theorem 2.1 has already been established in similar frameworks and we only sketch the main steps refer to [10] or [7] for a detailed explanation. Note however that it is not possible to

obtain here a direct estimate on $\sup_t |X - X^\delta|^2 + |V - V^\delta|$ and it is necessary to distinguish between $X - X^\delta$ and $V - V^\delta$ as in the above definition of Q_δ.
• *Existence.* For any $F_0 \in L^1 \cap L^2$, first consider a sequence of smooth C^∞ approximations $F_{0,n}$. The corresponding force term F_n is now at least Lipschitz and there exists a unique solution X_n, V_n to

$$\begin{aligned} &\frac{d}{dt} X_n(t,x,v) = \frac{V_n(t,x,v)}{\sqrt{1+V_n^2(t,x,v)}}, \quad \frac{d}{dt} V_n(t,x,v) = F_n(t,X), \\ &X_n(0,x,v) = x, \qquad V_n(0,x,v) = v. \end{aligned} \tag{2.5}$$

Since F_n is uniformly bounded in L^2, the sequence X_n, V_n has uniform bounds in n. Here (see for instance subsection 3.1.1 for detailed calculation)

$$|X_n| \leq |x| + t, \quad \int_\Omega |V_n|^2(t,x,v)\,dx\,dv \leq C_t,$$

for any compactly supported set Ω.
If we are able to show that the sequences X_n and V_n are locally compact then by extracting converging subsequences and passing to the limit, we obtain X, V solution to (2.2). Compactness is hence the key part for existence and it is provided by Proposition 2.1.
Indeed for any δ we may apply Proposition 2.1 to X_n, V_n and to X_n^δ, V_n^δ. One finds a function ψ independent of n s.t.

$$Q_{\delta,\,n}(T) \leq T\,\psi(-\log|\delta|).$$

For any ϵ, denote $\Omega_\delta^\varepsilon$ the subset of $(x,\,v) \in \Omega$ s.t.

$$\sup_{t\leq T} |X_n(t,x,v) - X_n^\delta(t,x,v)|^2 + \int_0^T |V_n(t,x,v) - V_n^\delta(t,x,v)|^2\,dt \geq \varepsilon.$$

Then by the previous bound $|\Omega_\delta^\varepsilon| \leq T\,\frac{\psi(-\log|\delta|)}{\log(1+\varepsilon/|\delta|^2)}$ and consequently

$$\int_\Omega \int_0^T |V_n(t,x,v) - V_n^\delta(t,x,v)|^2\,dt\,dx\,dv \leq T\,\varepsilon + C_T\,\frac{\psi(-\log|\delta|)}{\log(1+\varepsilon/|\delta|^2)},$$

with a similar bound on $X_n - X_n^\delta$. As $\psi(\xi)/\xi \to 0$, one may optimize in ε to find that for some continuous Ψ with $\Psi(0) = 0$

$$\int_\Omega \int_0^T |V_n(t,x,v) - V_n(t,x+\delta_1,v+\delta_2)|^2\,dt\,dx\,dv \leq C_T\,\Psi(\delta),$$

with again a similar bound on $X_n - X_n^\delta$. This implies compactness of the sequence X_n, V_n in L^2. Extracting a converging subsequence $X_{\sigma(n)}$, $V_{\sigma(n)}$ to some X, V, we may deduce that (X, V) solves (2.2). Moreover since X_n, V_n satisfy Property 1 then so is the case for X, V.

• *Uniqueness.* This step is straightforward. Just consider two solutions X, V and $\tilde X$, $\tilde V$ to (2.2), satisfying both Property 1. Then apply Proposition 2.1 to X, V and $X^\delta = \tilde X$, $V^\delta = \tilde V$ for any δ. Assuming that $(X, V) \neq (\tilde X, \tilde V)$ (in the usual almost everywhere sense) then letting $|\delta| \to 0$ in Q_δ, one would have

$$Q_\delta(T) \sim -\log|\delta|,$$

which contradicts Proposition 2.1. One may hence deduce that $(X, V) = (\tilde X, \tilde V)$ *a.e.*

2.3 Sketch of the proof of Proposition 2.1

The beginning of the proof of Prop. 2.1 initially follows the step of [7]. More precisely the first two steps are

- The force field F does not belong to L^∞ and some trajectories may be unbounded or some velocities very close to the speed of light. The first step is to identify a subdomain $\Omega_K \subset \Omega$ s.t.

$$\int_0^T |F(t, X(t, x, v))|\,dt \le K, \quad \forall (x, v) \in \Omega_K,$$

  with the same bound on X^δ. Using L^2 bounds on F, it is easy to show that for large K, $\Omega \setminus \Omega_K$ is very small. By restricting Q_δ to Ω_K,

$$Q_\delta^K(T) = \int_{\Omega_K} \log\left(1 + \frac{1}{|\delta|^2}\left(\sup_{0\le t\le T} |X(t,x,v) - X^\delta(t,x,v)|^2 \right.\right.$$
$$\left.\left. + \int_0^T |V(t,x,v) - V^\delta(t,x,v)|^2\,dt\right)\right)\,dx\,dv,$$

  we can focus on Q_δ^K where all trajectories are bounded and the velocities away from the speed of light. Note that this step was not needed in [7].

- Calculate the time derivative of Q_δ^K. Using (2.2) it is easy to differentiate in time. Skipping the details of the calculation, one obtains

$$\begin{aligned} Q_\delta^K(T) \leq & C\,T + \tilde{Q}_\delta(T) \\ & + \frac{1}{2}\int_{\Omega_K}\int_0^T \frac{|V-V^\delta|^2}{|\delta|^2 + \sup|X-X^\delta|^2 + \int_0^t |V-V^\delta|^2\,ds}\,dt\,dx\,dv, \end{aligned}$$

with

$$\begin{aligned} \tilde{Q}_\delta(T) = 2\int_{\Omega_K}\int_0^T & \frac{V^\delta(t,x,v)-V(t,x,v)}{A_\delta(t,x,v)}\cdot \\ & \int_0^t (F(s,(X^\delta,V^\delta)(s,x,v) - F(s,(X,V)(s,x,v)))\,ds\,dt\,dx\,dv, \end{aligned}$$

where one defines

$$\begin{aligned} A_\delta(t,x,v) = & |\delta|^2 + \sup_{0\leq s\leq t}|X(s,x,v)-X^\delta(s,x,v)|^2 \\ & + \int_0^t |V(s,x,v)-V^\delta(s,x,v)|^2\,ds. \end{aligned}$$

The third term in the bound of Q_δ^K is the contribution from the free transport. This term can actually be bounded directly by Q_δ^K itself. The way the differences $|X-X^\delta|$ and $|V-V^\delta|$ are measured in Q_δ is important here. For instance the corresponding bound would not hold if one replaced $\int_0^t |V-V^\delta|^2\,ds$ by $(\int_0^t |V-V^\delta|\,ds)^2$in the definition of Q_δ.

Therefore the difficulty is in controlling $\tilde{Q}_\delta$, using (2.3). We now decouple the dynamics, meaning that the F appearing in $\tilde{Q}_\delta$ is not necessarily anymore the force field in (2.2). To make things clearer, we bound

$$\begin{aligned} \bar{Q}_\delta(T) = 2\int_{\Omega_K}\int_0^T & \frac{V^\delta(t,x,v)-V(t,x,v)}{A_\delta(t,x,v)}\cdot \\ \int_0^t (\nu(V^\delta)\,G(s,(X^\delta,V^\delta)(s,x,v) & -\nu(V)\,G(s,(X,V)(s,x,v)))dsdtdxdv, \end{aligned}$$

where G satisfies

$$G(s,x) = \partial_t\left(t\int_{S^2} G_0(x+t\omega)\,d\omega\right).$$

for some $G^0 \in L^2$. This will allow us to use linear interpolation on $\bar{Q}_\delta$ in G^0.

Notice that since ν is smooth then one may do one last simple reduction

$$\begin{aligned}\bar{Q}_\delta(T) \leq & OK + \int_{\Omega_K} \int_0^T \frac{V^\delta(t,x,v) - V(t,x,v)}{A_\delta(t,x,v)} \cdot (\nu(V) + \nu(V^\delta)) \\ & \int_0^t (\bar{G}(s, X^\delta(s,x,v)) - \bar{G}(s, XV(s,x,v)))\, ds\, dt\, dx\, dv.\end{aligned}$$

This leads to the main part of the proof, which diverges completely from [7]. In that former article, one would use Fourier transform and integrate by part in time over the trajectories to make explicit some cancellations.
Instead here the key ingredient is a change of variable mixing the particles trajectory with the spherical structure of G, coupled with a clever use of appropriately defined maximal operators (like the spherical maximal operator). By using the definition of G

$$\begin{aligned}\int_0^t \big(G(s, X_s^\delta) - G(s, X_s)\big)\, ds = & \int_0^t \int_{S^2} \big(G^0(X_s - \omega s) - G^0(X_s^\delta - \omega s)\big)\, d\omega ds \\ & + \int_0^t \int_{S^2} (\omega \cdot \nabla_x G^0(X_s - \omega s) - \omega \cdot \nabla_x G^0(X_s^\delta - \omega s))\, s\, d\omega ds.\end{aligned}$$

Now introduce the two changes of variables

$$\Phi_X(s,\omega) = X_s - s\omega, \quad \Phi_{X^\delta}(s,\omega) = X_s^\delta - s\omega.$$

The Jacobian of the transform is

$$J_X = C\, s^2\, |\dot{X}_s \cdot \omega - 1|,$$

and the corresponding formula for J_{X^δ}.
It is crucial to notice that because $(x,b) \in \Omega_K$ then $|\dot{X}_s| \leq 1 - C/K$ and therefore the Jacobian of this transform is always bounded from above and below by s^2. This is where the difference in speed of propagation between the particles and the wave is used.
Denote $(s_X, \omega_X)(z)$ the inverse of Φ_X, namely $z = X_{s_X(z)} - s_X(z)\omega_X(z)$ and $O_X^t = \bigcup_{s\in[0,\ t],\omega\in S^2} \Phi_X(s,\omega)$ the image domain. Because of the previous

remark on the Jacobian, we may control the regularity of s_X and ω_X and we for instance derive the estimate

$$|s_X - s_{X^\delta}| \leq C\,K \max_{s\leq \inf(s_X, s_{X^\delta})} |X_s - X^\delta_s|.$$

Still because the velocity is less than the speed of light, one may see that $O^t_X = B(X(t,x,v),t)$.
Introducing this transform in G and integrating by parts if needed the terms with $\omega\cdot\nabla$, one obtains five terms

$$\left|\int_0^t \left(G(s,X^\delta_s) - G(s,X_s)\right)\right| ds \leq A + |B| + |C| + D + |E|,$$

where for instance

$$A = \int_{O^t_X\setminus O^t_{X^\delta}} |G^0(z)|\,\frac{C\,dz}{s_X^2\,|\dot X_s\cdot\omega_X - 1|} + \int_{O^t_{X^\delta}\setminus O^t_X} |G^0(z)|\,\frac{C\,dz}{s_{X^\delta}^2\,|\dot X^\delta_s\cdot\omega_{X^\delta} - 1|},$$

and the formulas for the other terms are of course given later in the proof itself. Instead we just sketch here how to control A which (with B) is by far the simplest term.
With clever uses of the triangle inequality, one has that for $z\in O^t_X\setminus O^t_{X^\delta}$

$$s_X(z) \geq \frac{t}{CK}, \qquad t - |X_t - X^\delta_t| \leq |z - X_t| \leq t.$$

This implies that

$$A \leq \frac{C\,K^2}{t^2}\int_{t-|X_t-X^\delta_t|\leq|z-X_t|\leq t} |G^0(z)|\,dz + symmetric.$$

This leads us to the following maximal operator

$$\tilde M g(x) = \sup_{\eta\leq\varepsilon} \frac{1}{\varepsilon^2\,\eta}\int_{\varepsilon-\eta\leq|z-x|\leq\varepsilon} |g(z)| dz,$$

and the now obvious bound

$$A \leq C\,K^2\,|X_t - X^\delta_t|\,(\tilde M G^0(X_t) + \tilde M G^0(X^\delta_t)).$$

As consequence with a simple Cauchy-Schwartz and the change of variable $(x,v)\to(X_t,V_t)$ (or its symmetric) which has Jacobian 1

$$\bar Q_\delta(T)\leq OK+C(K)\,\sqrt{|\log\delta|}\,\left(\int_{|x|\leq C\,K}|\tilde M G^0(x)|^2\,dx\right)^{1/2}+\textit{other terms},$$

where *other terms* refer to the contributions of B, C, D and E, and $C(K)$ denotes a function of K.

It turns out that $\tilde M$ can be controlled by the maximal operator on the sphere M_S and is hence bounded on L^2, which allows to conclude for A.

Each of the five terms has to be controlled in a specific manner, using cleverly the properties of the changes of variable Φ_X, Φ_{X^δ}, and of maximal operators of spherical type (or wave operators). With careful estimates, one may obtain a bound of the type

$$\bar Q_\delta\leq OK+C(K)\,|\log\delta|\,(\|G^0\|_{L^1}+\|G^0\|_{L^2})\,(1+\|F^0\|_{L^2}).\tag{2.6}$$

Note that the bound includes a full $|\log\delta|$ factor (contrary to the contribution from A with only a square root). Therefore it cannot be used directly. Instead a final step is needed.

Observe that $\bar Q_\delta\leq C(K)\,\|G^0\|_{H^k}$ for k large enough so that G is Lipschitz. Therefore one may do some linear interpolation in $\bar Q_\delta$ in G^0. This actually implies that for any fixed $G^0\in L^1\cap L^2$, the behavior in $|\delta|$ is better than what is given in (2.6), thus yielding Prop. 2.1.

3 Proof of Proposition 2.1

3.1 First steps

We perform here some straightforward but necessary steps: Reducing the domain Ω to eliminate large velocities and controlling the contribution of free transport.

3.1.1 Truncation of large velocities

The aim of this subsection is to define a subset $\Omega_K\subset\Omega$ s.t. the velocity of any trajectory starting from Ω_K is bounded while at the same time $\Omega\setminus\Omega_K$ is suitably small.

First note that, by the usual estimates on solutions to wave equations, since F_0 is in $L^1\cap L^2$ then so is $F(t,x)$ and B. Indeed

$$F(t,x)=\int_{S^2} F_0(x+t\omega)\,d\omega+t\int_{S^2}\omega\cdot\nabla_x F_0(x+t\omega)\,d\omega.$$

The bound is obviously true for the first term. As for the second, applying Fourier transform in x yields

$$\mathcal{F}t\int_{S^2}\omega\cdot\nabla_x F_0(.+t\omega)\,d\omega=\hat{F}_0(\xi)\,t\int_{S^2}e^{it\,\xi\cdot\omega}\omega\cdot\xi\,d\omega=\hat{F}_0(\xi)\,M_t(\xi).$$

The multiplier $M_t(\xi)$ is bounded uniformly in t.
This means that for any K,

$$\begin{aligned}\int_\Omega \mathbb{I}_{\{(x,v)|\,\int_0^T|F(t,X(t,x,v))|\,dt\geq K\}}\,dx\,dv&\\ \leq\frac{1}{K^2}\int_\Omega\int_0^T|F(t,X(t,x,v))|^2\,dt\,dx\,dv\leq C\,\frac{T}{K^2}.\end{aligned}$$

Therefore denoting by Ω_K

$$\begin{aligned}\Omega_K=\Bigg\{(x,v)\in\Omega\ s.t.\quad&\int_0^T|F(t,X(t,x,v))|\,dt\leq K\\ \text{and}\quad&\int_0^T|F(t,X^\delta(t,x,v))|\,dt\leq K\Bigg\},\end{aligned}\tag{3.1}$$

one deduces that $|\Omega\setminus\Omega_K|\leq CT/K^2$ and hence

$$Q_\delta(T)\leq\frac{CT}{K^2}\,|\log\delta|+Q_\delta^K(T),\tag{3.2}$$

with

$$\begin{aligned}Q_\delta^K(T)=\int_{\Omega_K}\log\Bigg(1+\frac{1}{|\delta|^2}\Bigg(&\sup_{0\leq t\leq T}|X(t,x,v)-X^\delta(t,x,v)|^2\\ &+\int_0^T|V(t,x,v)-V^\delta(t,x,v)|^2\,dt\Bigg)\Bigg)\,dx\,dv.\end{aligned}$$

Finally note that for (x,v) in Ω_K

$$|V(t,x,v)|\leq|v|+K,\quad |X(t,x,v)|\leq|x|+t\,|v|+t\,K.$$

As $(x,v)\in\Omega$ which is compact then for some constant C, $|V|+|X|\leq C\,K$ and the same is of course true for X^δ and V^δ.

3.1.2 The free transport contribution

The next step is to differentiate in time Q_δ^K, making explicit the contributions from free transport (straightforward to bound) and from the force term. Now let

$$\begin{aligned} A_\delta(t,x,v) =& |\delta|^2 + \sup_{0\le s\le t} |X(s,x,v) - X^\delta(s,x,v)|^2 \\ &+ \int_0^t |V(s,x,v) - V^\delta(s,x,v)|^2\, ds. \end{aligned}$$

Compute

$$\begin{aligned} &\frac{d}{dt} \log\Bigg(1 + \frac{1}{|\delta|^2}\Big(\sup_{0\le s\le t} |X(s,x,v) - X^\delta(s,x,v)|^2 \\ &\qquad\qquad + \int_0^t |V(s,x,v) - V^\delta(s,x,v)|^2\, ds\Big)\Bigg) \\ &= \frac{2}{A_\delta(t,x,v)}\Bigg(\frac{d}{dt}\left(\sup_{0\le s\le t} |X(s,x,v) - X^\delta(s,x,v)|^2\right) + (V(t) - V^\delta(t)) \\ &\quad \times \int_0^t (F(s,(X,V)(s,x,v)) - F(s,(X^\delta,V^\delta)(s,x,v)))\, ds\Bigg). \end{aligned}$$

Recall that for any $f \in BV(0,T)$, we have

$$\frac{d}{dt}\left(\max_{0\le s\le t} f(s)^2\right) \le 2|f(t)f'(t)| \le 4|f(t)|^2 + \frac{1}{2}|f'(t)|^2.$$

And in addition

$$|\partial_t X - \partial_t X^\delta|^2 = |V/\sqrt{1+|V|^2} - V^\delta/\sqrt{1+|V^\delta|}|^2 \le 4|V - V^\delta|^2.$$

Hence we deduce from the previous computation that

$$\begin{aligned} Q_\delta^K(T) &\le \int_{\Omega_K}\int_0^T \frac{8\,|X - X^\delta|^2 + |V - V^\delta|^2/2}{A_\delta(t,x,v)}\, dt\, dx\, dv + \tilde{Q}_\delta(T) \\ &\le 8|\Omega|(1+T) + \tilde{Q}_\delta(T) + \frac{1}{2}\int_{\Omega_K}\int_0^T \frac{|V - V^\delta|^2}{A_\delta(t,x,v)}\, dt\, dx\, dv \end{aligned}$$

where,

$$\tilde{Q}_\delta(T) = 2\int_{\Omega_K}\int_0^T \frac{V^\delta(t,x,v) - V(t,x,v)}{A_\delta(t,x,v)} \cdot \int_0^t (F(s,(X^\delta,V^\delta)(s,x,v) - F(s,(X,V)(s,x,v)))\, ds\, dt\, dx\, dv.$$

Remark that

$$\begin{aligned}\int_{\Omega_K}\int_0^T \frac{|V-V^\delta|^2}{A_\delta(t,x,v)}\, dt\, dx\, dv &\leq \int_{\Omega_K}\int_0^T \frac{\partial_t A_\delta(t,x,v)}{A_\delta(t,x,v)}\, dt\, dx\, dv \\ &\leq \int_{\Omega_K} \log\left(\frac{A_\delta(T,x,v)}{|\delta|^2}\right)\, dx\, dv \\ &\leq Q_\delta^K(T),\end{aligned}$$

where we recall that $A_\delta(0,x,v) = \delta^2$. Therefore, we have

$$Q_\delta^K(T) \leq 8|\Omega|(1+T) + 2\,\tilde{Q}_\delta(T),$$

and it is enough to bound $\tilde{Q}_\delta(T)$.
For technical reasons related to some interpolations that will be explained later, we will bound a more general term than $\tilde{Q}_\delta$, namely

$$\bar{Q}_\delta(T) = 2\int_{\Omega_K}\int_0^T \frac{V\delta(t,x,v) - V(t,x,v)}{A_\delta(t,x,v)} \cdot \int_0^t (\nu(V_s^\delta)\, G(s,X^\delta(s,x,v)) - \nu(V_s)\, G(s,X(s,x,v)))\, ds\, dt\, dx\, dv,$$

where we assume that G satisfies the same assumption as F, namely

$$G(t,x) = \partial_t\left(t\int_{S^2} G^0(x-\omega t)\, d\omega\right). \tag{3.3}$$

In the term $\bar{Q}_\delta$ we decouple the connection between the dynamics of (X,V) and (X^δ,V^δ) which is related to F and the G function which appears in $\bar{Q}_\delta$. This means that $\bar{Q}_\delta$ is now linear in G^0 which will later allow us to use interpolation theory.

Remark that $\bar{Q}_\delta$ can be simplified right away as

$$\begin{aligned}
&(\nu(V_s^\delta)\, G(s, X^\delta(s,x,v)) - \nu(V_s)\, G(s, X(s,x,v))) \\
&\quad = \frac{1}{2}(\nu(V) + \nu(V^\delta))\,(G(s, X_s^\delta) - G(s, X_s)) \\
&\quad\quad + \frac{1}{2}(\nu(V) - \nu(V^\delta))\,(G(s, X_s^\delta) + G(s, X_s))
\end{aligned}$$

so that

$$\begin{aligned}
\bar{Q}_\delta(T) \le \int_{\Omega_K} \int_0^T (\nu(V) + \nu(V^\delta))\, &\frac{V(t,x,v) - V^\delta(t,x,v)}{A_\delta(t,x,v)} \\
&\int_0^t (G(s, X_s^\delta) - G(s, X_s))\, ds\, dx\, dv\, dt \\
+ \int_{\Omega_K} \int_0^T (\nu(V_t) - \nu(V_t^\delta))\, &\frac{V(t,x,v) - V^\delta(t,x,v)}{A_\delta(t,x,v)} \\
&\int_0^t (G(s, X_s^\delta) + G(s, X_s))\, ds\, dx\, dv\, dt.
\end{aligned}$$

As ν is Lipschitz the second term may be directly bounded by

$$\begin{aligned}
C_K \int_{\Omega_K} \int_0^T &\frac{|V(t,x,v) - V^\delta(t,x,v)|^2}{A_\delta(t,x,v)} \\
&\int_0^t (|G(s, X_s^\delta)| + |G(s, X_s)|)\, ds\, dx\, dv\, dt,
\end{aligned}$$

where $C_K = \max_{B(0, C\,K)} |\nabla \nu(K)|$. Note that

$$\int_s^T \frac{|V(t,x,v) - V^\delta(t,x,v)|^2}{A_\delta(t,x,v)}\, dt \le \int_s^T \frac{\partial_t A_\delta}{A_\delta}\, dt \le -C\, \log|\delta|.$$

Hence by Fubini

$$\bar{Q}_\delta(T) \le 2\, I_\delta(T) + C_K\, T\, |\log|\delta||\, (\|G^0\|_{L^1} + \|G^0\|_{L^2}), \tag{3.4}$$

with

$$\begin{aligned}
I_\delta = \nu(K) \int_{\Omega_K} \int_0^T &\frac{|V(t,x,v) - V^\delta(t,x,v)|}{A_\delta(t,x,v)} \\
&\left| \int_0^t (G(s, X_s^\delta) - G(s, X_s))\, ds \right| dx\, dv\, dt.
\end{aligned}$$

3.2 Proof of Proposition 2.1: The main bound

The term I_δ is quite technical to bound and we hence summarize the computations in the following lemma

Lemma 3.1 *For any G satisfying* (3.3) *with $G_0 \in L^1 \cap L^2(\mathbb{R}^3)$ and any (X, V) solution to* (2.2) *with* (1), *any (X^δ, V^δ) satisfying* (2.4), *there exists a constant C depending only on T, s.t.*

$$I_\delta \leq C\,\nu(K)\,K^{10}\,|\log|\delta||\,(\|G^0\|_{L^1} + \|G^0\|_{L^2})\,(1 + \|F^0\|_{L^2}).$$

The proof of this lemma is the main difficulty of the article and will be carried over in several step. First of all we introduce the main change of variable mixing the characteristics with the spherical averages from the wave equation.

Beginning of the proof of Lemma 3.1. Write

$$\int_0^t \big(G(s, X_s^\delta) - G(s, X_s)\big)\,ds = \int_0^t \int_{S^2} \big(G^0(X_s - \omega s) - G^0(X_s^\delta - \omega s)\big)\,d\omega ds$$
$$+ \int_0^t \int_{S^2} (\omega \cdot \nabla_x G^0(X_s - \omega s) - \omega \cdot \nabla_x G^0(X_s^\delta - \omega s))\,s\,d\omega ds.$$

Introduce the two changes of variables

$$\Phi_X(s, \omega) = X_s - s\omega, \quad \Phi_{X^\delta}(s, \omega) = X_s^\delta - s\omega.$$

The Jacobian of the transform is

$$J_X = C\,s^2\,|\dot{X}_s \cdot \omega - 1|,$$

and the corresponding formula for J_{X^δ}. Denote $(s_X, \omega_X)(z)$ the inverse of Φ_X, namely $z = X_{s_X(z)} - s_X(z)\omega_X(z)$ and $O_X^t = \bigcup_{s\in[0,\ t],\omega\in S^2} \Phi_X(s, \omega)$. One can easily prove that

$$O_X^t = B(X(t, x, v), t). \tag{3.5}$$

One inclusion is indeed obvious and as for the other one, note that

$$|\Phi_X(s, \omega) - X(t)| \leq s + |X(s) - X(t)| \leq s + |t - s|,$$

as $|\dot{X}| < 1$.

One obtains

$$|\int_0^t (G(s,X_s^\delta)-G(s,X_s))\,ds| \leq A+|B|+|C|+D+|E|,$$

with

$$\begin{aligned}
A=&\int_{O_X^t\setminus O_{X^\delta}^t} |G^0(z)|\,\frac{C\,dz}{s_X^2|\dot X_s\cdot\omega_X-1|}\\
&\quad+\int_{O_{X^\delta}^t\setminus O_X^t} |G^0(z)|\,\frac{C\,dz}{s_{X^\delta}^2|\dot X_s^\delta\cdot\omega_{X^\delta}-1|}\\
B=&\int_{O_X^t\cap O_{X^\delta}^t} G^0(z)\,\left(\frac{C}{s_X^2|\dot X_{s_X}\cdot\omega_X-1|}-\frac{C}{s_{X^\delta}^2|\dot X^\delta_{s_{X^\delta}}\cdot\omega_{X^\delta}-1|}\right)\,dz,
\end{aligned}\tag{3.6}$$

which correspond to the term without derivative,

$$\begin{aligned}
C=&\int_{\partial B(X(t),t)} G^0(z)\,\frac{C}{s_X|\dot X_{s_X}\cdot\omega_X-1|}\,dS(z)\\
&\quad-\int_{\partial B(X^\delta(t),t)} G^0(z)\,\frac{C}{s_{X^\delta}|\dot X^\delta_{s_{X^\delta}}\cdot\omega_{X^\delta}-1|}\,dS(z),\\
D=&\int_{O_X^t\setminus O_{X^\delta}^t} |G^0(z)|\,\left|\nabla_z\cdot\left(\omega_X\,\frac{C}{s_X|\dot X_s\cdot\omega_X-1|}\right)\right|\,dz\\
&\quad+\int_{O_{X^\delta}^t\setminus O_X^t} |G^0(z)|\,\left|\nabla_z\cdot\left(\omega_{X^\delta}\,\frac{C}{s_{X^\delta}^2|\dot X_s^\delta\cdot\omega_{X^\delta}-1|}\right)\right|\,dz,
\end{aligned}\tag{3.7}$$

and finally

$$\begin{aligned}
E=\int_{O_X^t\cap O_{X^\delta}^t} G^0(z)\,\Bigg(&\nabla_z\cdot\left(\frac{C\,\omega_X}{s_X|\dot X_{s_X}\cdot\omega_X-1|}\right)\\
&-\nabla_z\cdot\left(\frac{C\,\omega_{X^\delta}}{s_{X^\delta}|\dot X^\delta_{s_{X^\delta}}\cdot\omega_{X^\delta}-1|}\right)\Bigg)\,dz.
\end{aligned}\tag{3.8}$$

Note that C is a boundary term coming from the integration by parts. We hope there is no confusion since C is used for constants that may change from one line to the other. We denote by $I_A,\ldots,I_E$ the integrals, over $\Omega_K\times[0,\ T]$ of the previous quantities multiplied by

$$(\nu(V_t)+\nu(V_t^\delta))\,\frac{|V(t,x,v)-V^\delta(t,x,v)|}{A_\delta(t,x,v)}.$$

3.2.1 Bound for I_B

The bound for I_B uses estimates on the differences $s_X - s_{X^\delta}$ and $\omega_X - \omega_{X^\delta}$. After changing variables back, it is possible to reduce the problem to a bound on spherical averages of G^0. No maximal function is required here.
As part of the bound for I_E uses the same steps, we prove here a more general result on quantities like I_B. We call them I_{Bmod}.

Lemma 3.2 *Assume that $H \in W^{1,\infty}$ and that $4/3 < p < 2$, then one has for some constant C depending on p and the norm of H and for $k = 1, 2$,*

$$\begin{aligned} I_{Bmod} :=& \nu(K) \int_0^T \int_{\Omega_K} \frac{|V_t - V_t^\delta|}{A_\delta(t,x,v)} \int_{O_X^t \cap O_{X^\delta}^t} |G^0(z)| \left| \frac{H(s_X, \omega_X)}{s_X^2 \, |\dot{X}^\delta_{s_X} \cdot \omega_X - 1|^k} \right. \\ & \left. - \frac{H(s_{X^\delta}, \omega_{X^\delta})}{s_{X^\delta}^2 \, |\dot{X}_{s_{X^\delta}} \cdot \omega_{X^\delta} - 1|^k} \right| dz \, dx \, dv \, dt \\ \leq& \, C \, K^{5+k} \, \nu_K \sqrt{-\log|\delta|} \, (\|G^0\|_{L^1} + \|G^0\|_{L^p}). \end{aligned}$$

Lemma 3.2 with $k = 1$ and H constant implies that for $4/3 < p \leq 2$, we have

$$I_B \leq C \, \nu(K) \, K^6 \sqrt{-\log|\delta|} \, (\|G^0\|_{L^1} + \|G^0\|_{L^p}). \tag{3.9}$$

Proof of Lemma 3.2. Denote

$$Bmod = \int_{O_X^t \cap O_{X^\delta}^t} |G^0(z)| \left| \frac{H(s_X, \omega_X)}{s_X^2 \, |\dot{X}_{s_X} \cdot \omega_X - 1|^k} - \frac{H(s_{X^\delta}, \omega_{X^\delta})}{s_{X^\delta}^2 \, |\dot{X}^\delta_{s_{X^\delta}} \cdot \omega_{X^\delta} - 1|^k} \right| dz.$$

Recall that $\dot{X}_s = V_s/\sqrt{1+V_s^2}$ and that $|V_s| \leq C\,K$, so that $|\dot{X}_s| \leq /1 - 1/(CK)$ and $|\dot{X}_s \cdot \omega - 1| \geq 1/(CK)$. Then

$$\begin{aligned} \left| \frac{H(s_X, \omega_X)}{s_X^2 |\dot{X}_s \cdot \omega - 1|^k} - \frac{H(s_{X^\delta}, \omega_{X^\delta})}{s_{X^\delta}^2 |\dot{X}_s^\delta \cdot \omega_{X^\delta} - 1|^k} \right| \leq& C\,K^k \left(|s_X - s_{X^\delta}| \left(\frac{1}{s_X^3} + \frac{1}{s_{X^\delta}^3} \right) \right. \\ & \left. + K \, |\omega_X - \omega_{X^\delta}| \left(\frac{1}{s_X^2} + \frac{1}{s_{X^\delta}^2} \right) + K \frac{|V_{s_X} - V^\delta_{s_X}|}{s_X^2} + K \, \frac{|V_{s_{X^\delta}} - V^\delta_{s_{X^\delta}}|}{s_{X^\delta}^2} \right). \end{aligned}$$

On the other hand, by definition $X_{s_X} - s_X \, \omega_X = X^\delta_{s_{X^\delta}} - s_{X^\delta} \, \omega_{X^\delta}$, so

$$\Phi_X(s_X, \omega_X) - \Phi_X(s_{X^\delta}, \omega_{X^\delta}) = X_{s_X} - X_{s_{X^\delta}} - s_X \, \omega_X + s_{X^\delta} \, \omega_{X^\delta} = X^\delta_{s_{X^\delta}} - X_{s_{X^\delta}}. \tag{3.10}$$

Recall that $|\omega_X| = |\omega_{X^\delta}| = 1$. Therefore $|s_X\,\omega_X - s_{X^\delta}\,\omega_{X^\delta}| \geq |s_X - s_{X^\delta}|$ and as $|X_{s_X} - X_{s_{X^\delta}}| \leq (1 - 1/(CK))|s_X - s_{X^\delta}|$,

$$|s_X - s_{X^\delta}| \leq \max_{s \leq inf(s_X, s_{X^\delta})} |X_s - X_s^\delta| + (1 - 1/(CK))|s_X - s_{X^\delta}|,$$

which implies

$$|s_X - s_{X^\delta}| \leq CK \max_{s \leq inf(s_X, s_{X^\delta})} |X_s - X_s^\delta|.$$

Using this estimate in (3.10), one concludes that

$$\begin{aligned} &|s_X - s_{X^\delta}| \leq C\,K \max_{s \leq inf(s_X, s_{X^\delta})} |X_s - X_s^\delta|, \\ &|\omega_X - \omega_{X^\delta}| \leq C\,K \max_{s \leq inf(s_X, s_{X^\delta})} |X_s - X_s^\delta| \left(\frac{1}{s_X} + \frac{1}{s_{X^\delta}} \right). \end{aligned} \tag{3.11}$$

Inserting this in the term I_{Bmod} enables to bound it by

$$\begin{aligned} I_{Bmod} \leq &C\,K^{k+2}\,\nu_K \int_0^T \int_{\Omega_K} \frac{|V(t,x,v) - V^\delta(t,x,v)|}{A_\delta(t,x,v)} \\ &\left(\int_{O_X^t \cap O_{X^\delta}^t} |G^0(z)| \max_{s \leq \inf(s_X, s_{X^\delta})} |X_s - X_s^\delta| \left(\frac{1}{s_X^3} + \frac{1}{s_{X^\delta}^3} \right) dz \right. \\ &\left. + \int_{O_X^t \cap O_{X^\delta}^t} |G^0(z)| \left(\frac{|V_{s_X} - V_{s_X}^\delta|}{s_X^2} + \frac{|V_{s_{X^\delta}} - V_{s_{X^\delta}}^\delta|}{s_{X^\delta}^2} \right) dz \right) dx\,dv\,dt, \end{aligned}$$

with $\nu_K = \max_{B(0,K)} |\nu(v)|$.
Changing back variables to s, ω, one eventually finds

$$\begin{aligned} I_{Bmod} \leq C\,K^{k+2}\,\nu_K &\int_0^T \int_{\Omega_K} \frac{|V(t,x,v) - V^\delta(t,x,v)|}{A_\delta(t,x,v)} \\ &\left(\int_0^t \frac{\max_{r \leq s} |X_s - X_s^\delta|}{s} \int_{S^2} |G^0(X_s + s\omega)|\,d\omega\,ds \right. \\ &\left. + \int_0^t \int_{S^2} |G^0(X_s + s\omega)|\,|V_s - V_s^\delta|\,d\omega\,ds + symmetric\ terms \right) dx\,dv\,dt. \end{aligned} \tag{3.12}$$

Note that

$$\max_{r\le s}|X_s - X_s^\delta| \le \sqrt{s}\,\left(\int_0^s |V_r - V_r^\delta|^2\,dr\right)^{1/2} \le \sqrt{s}\,\sqrt{A_\delta(t,x,v)}.$$

Hence by the definition of A_δ

$$\int_0^T\int_{\Omega_K}\frac{|V-V^\delta|(t,x,v)}{A_\delta(t,x,v)}\int_0^t\frac{\max_{r\le s}|X_s-X_s^\delta|}{s}\int_{S^2}|G^0(X_s+s\omega)|\,d\omega\,ds$$
$$\le \int_0^T s^{-1/2}\int_{\Omega_K}\int_{S^2}|G^0(X_s+s\omega)|\,d\omega\int_s^T\frac{|V-V^\delta|(t,x,v)}{\sqrt{A_\delta(t,x,v)}}\,dt\,dx\,dv\,ds.$$

However as $|V-V^\delta|^2\le\partial_t A_\delta$

$$\begin{aligned}\int_s^T\frac{|V-V^\delta|(t,x,v)}{\sqrt{A_\delta(t,x,v)}}\,dt &\le \sqrt{T}\,\left(\int_s^T\frac{|V-V^\delta|^2(t,x,v)}{A_\delta(t,x,v)}\,dt\right)^{1/2} \\ &\le C\,\sqrt{-\log|\delta||},\end{aligned}\tag{3.13}$$

one deduces that

$$\int_0^T\int_{\Omega_K}\frac{|V-V^\delta|(t,x,v)}{A_\delta(t,x,v)}\int_0^t\frac{\max_{r\le s}|X_s-X_s^\delta|}{s}\int_{S^2}|G^0(X_s+s\omega)|\,d\omega\,ds$$
$$\le C\,\sqrt{-\log|\delta||}\,\int_0^T s^{-1/2}\int_{\Omega_K}\int_{S^2}|G^0(X_s+s\omega)|\,d\omega\,dx\,dv$$
$$\le C\,K^3\,\sqrt{-\log|\delta|}\,\|G^0\|_{L^1},$$

by a change of variables.
Let us turn to the second term in(3.12). Denote

$$\tilde G(s,x)=\int_{S^2}|G^0(x+s\omega)|\,d\omega.$$

By Cauchy-Schwartz, and since $A_\delta(t)\ge\int_0^t|V-V^\delta|^2$

$$\int_0^T\int_{\Omega_K}\frac{|V-V^\delta|(t,x,v)}{A_\delta(t,x,v)}\int_0^t\int_{S^2}|G^0(X_s+s\omega)|\,|V_s-V_s^\delta|\,d\omega\,ds\,dx\,dv\,dt$$
$$\le C\int_{\Omega_K}\left(\int_0^T|\tilde G(s,X_s)|^2\,ds\right)^{1/2}\int_0^T\frac{|V-V^\delta|(t,x,v)\,dt}{\sqrt{A_\delta(t,x,v)}}\,dx\,dv.$$

Hence

$$\int_0^T \int_{\Omega_K} \frac{|V-V^\delta|(t,x,v)}{A_\delta(t,x,v)} \int_0^t \int_{S^2} |G^0(X_s+s\omega)|\,|V_s-V_s^\delta|\,d\omega\,ds\,dx\,dv\,dt$$
$$\le C\sqrt{-\log|\delta|}\int_{\Omega_K}\left(\int_0^T |\tilde G(s,X_s)|^2\,ds\right)^{1/2} dx\,dv$$
$$\le C\,|\Omega|\sqrt{-\log|\delta|}\,K^3\int_0^T\int_{\mathbb{R}^3}|\tilde G(s,X_s)|^2\,dx\,ds.$$

By the usual estimates on solutions to wave equations (see for instance [23], chapter. 8, 5.21), one has

Lemma 3.3 *For any $4/3<p<2$, there exists $C<\infty$ such that for all $G^0\in L^p\cap L^1$, we have*

$$\left\|\int_{S^2} G^0(x+s\omega)\,d\omega\right\|_{L^2} \le C\,s^{3/2-3/p}\left(\|G^0\|_{L^1}+\|G^0\|_{L^p}\right).$$

Combining all estimates and using Lemma 3.3 (note that $3-6/p>-1$ if $p>4/3$), one finally obtains

$$I_{Bmod}\le C\,K^{5+k}\,\nu_K\sqrt{-\log|\delta|}\,\left(\|G^0\|_{L^1}+\|G^0\|_{L^p}\right). \tag{3.14}$$

3.2.2 Bound on I_A

It is straightforward to control the volume of $O_X^t - O_{X^\delta}^t$. We can then conclude the calculation on I_A with the help of appropriate spherical maximal operators. Those operators are dominated by the classical spherical maximal operator.
Note that $O_X^t\setminus O_{X^\delta}^t = B(X_t,t)\setminus B(X_t^\delta,t)$. Since $|\dot X_s|\le 1-1/CK$ with the same for X^δ, one has that for any $\omega\in S^2$

$$|X_t^\delta - X_s - s\omega|\le |X_t^\delta - x| + |X_s - x| + s \le t\,(1-1/CK)+2s<t,$$

if $s<t/(2CK)$. Therefore

$$\forall z\in O_X^t\setminus O_{X^\delta}^t,\quad s_X(z)\ge \frac{t}{CK},$$

$$A \le \frac{C\,K^2}{t^2}\int_{O^t_X\setminus O^t_{X^\delta}} |G^0(z)|\,dz + \frac{C\,K^2}{t^2}\int_{O^t_X\setminus O^t_{X^\delta}} |G^0(z)|\,dz.$$

Now we introduce the following modified maximal operator

$$\tilde M g(x) = \sup_{\eta\le\varepsilon} \frac{1}{\varepsilon^2\,\eta}\int_{\varepsilon-\eta\le|z-x|\le\varepsilon} |g(z)|\,dz.$$

Note that for example if $z\in O^t_X\setminus O^t_{X^\delta}$ then $|z-X_t|\le t$ and $t\le |z-X^\delta_t|$. Using that $|z-X^\delta_t|\le |z-X_t|+|X_t-X^\delta_t|$, we deduce that

$$t-|X_t-X^\delta_t|\le |z-X_t|\le t.$$

Hence, one controls A with $\tilde M G^0$ as

$$A\le C\,K^2\,|X_t-X^\delta_t|\,(\tilde M G^0(X_t)+\tilde M G^0(X^\delta_t)).$$

This allows for an easy bound on I_A in terms of the L^1 norm of $\tilde M G^0$

$$\begin{aligned}
I_A &= \int_0^T\int_{\Omega_K} (\nu(V_t)+\nu(V^\delta_t))\,\frac{|V_t-V^\delta_t|}{A_\delta(t,x,v)}\,A\,dx\,dv\,dt\\
&\le C\,K^2\,\nu(K)\int_0^T\int_{\Omega_K}\frac{|V_t-V^\delta_t|}{\sqrt{A_\delta(t,x,v)}}\,(\tilde M G^0(X_t)+\tilde M G^0(X^\delta_t))\,dx\,dv\,dt\\
&\le C\,K^5\,\nu(K)\,\sqrt{|\log\delta|}\,\left(\int_{|x|\le C\,K}|\tilde M G^0(x)|^2\right)^{1/2},
\end{aligned}$$

by the change of variable $(x,v)\to(X_t,V_t)$ (or (X^δ_t,V^δ_t)) and (3.13).
We hence need to estimate $\tilde M G^0$. As it is defined, it turns out that it is of the same order as the spherical maximal operator for which bounds are well known

Lemma 3.4 *$\forall p>3/2$, $\exists C>0$ s.t. for any smooth function g*

$$\|\tilde M g\|_{L^p(\mathbb{R}^3)}\le C\,\|g\|_{L^p}.$$

Proof of Lemma 3.4. Simply note that

$$\int_{\varepsilon-\eta\le|z-x|\le\varepsilon}|g(z)|\,dz\le\int_{\varepsilon-\eta}^{\varepsilon}\int_{S(x,r)}|g(z)|\,dS(z)\,dr\le C\,\eta\,\varepsilon^2\,M_S\,g(x),$$

where M_S is defined by

$$M_S g(x) = \sup_r \oint_{S(x,r)} |g(z)|\, dS(z).$$

It is proved (see [23]) that in dimension 3, M_S is bounded on L^p for any $p > 3/2$ (the limit exponent is sharp) which easily implies the lemma.
Note that this exponent is also sharp for $\tilde M$ as obviously

$$\sup_{\eta\le\varepsilon} \frac{1}{\eta} \int_{\varepsilon-\eta\le|z-x|\le\varepsilon} |g(z)|\, dz \ge C \int_{S(x,\varepsilon)} |g(z)|\, dS(z).$$

And so $\tilde M g \ge C\, M_S g$. □
Coming back to I_A one concludes that

$$I_A \le C\, K^{8-3/2}\, \nu(K)\, \sqrt{|\log\delta|}\, \|G^0\|_{L^2}. \tag{3.15}$$

3.2.3 Bound for I_C

This term is the more delicate to bound as one needs to handle precisely the cancellations. This is mostly performed here by the use of a modified wave operator which still enjoys adequate L^2 estimates and a maximal operator defined along characteristics.
Note that on $\partial B(X_t, t)$, one obviously has $s_X = t$ and $\omega_X = (z - X_t)/t$. Hence

$$\int_{\partial B(X_t,t)} G^0(z)\, \frac{dS(z)}{s_X\, |\dot X_{s_X}\cdot\omega_X - 1|} = C\, t \int_{S^2} \frac{G^0(X_t + t\,\omega)}{|\dot X_t\cdot\omega - 1|}\, d\omega.$$

One finds that

$$\begin{aligned} I_C &\le C\, K^2\, \nu(K) \int_{\Omega_K}\int_0^T \frac{|V_t - V_t^\delta|^2}{A_\delta(t)}\, t \int_{S^2} |G^0(X_t + t\omega)|\, d\omega\, dx\, dv\, dt \\ &+ C\,\nu(K) \int_{\Omega_K}\int_0^T \frac{|V_t - V_t^\delta|}{A_\delta(t)}\, t \left|\int_{S^2} \frac{G^0(X_t + t\,\omega) - G^0(X_t^\delta + t\omega)}{|\dot X_t^\delta\cdot\omega - 1|}\, d\omega\right| dx\, dv\, dt, \\ &= I_C^1 + I_C^2 \end{aligned}$$

Let us deal first with I_C^1. Noticing as before that

$$\frac{|V_t - V_t^\delta|^2}{A_\delta(t)} \le \partial_t \log A_\delta(t).$$

Then denoting by W the usual wave operator

$$Wg(t,x) = t \int_{S^2} g(x+t\omega)\,d\omega,$$

one finds, after integration by parts in time, that

$$\begin{aligned} I_C^1 \leq &-C\,K^2\,\nu(K) \int_0^T \int_{\Omega_K} \log A_\delta\; \partial_t [W|G^0|(t,X_t)] \\ &+ C\,K^2 \int_{\Omega_K} \log A_\delta(T,x,v)\,W|G^0|(T,X_T) \\ \leq &\; C\,K^{13/2}\,\nu(K)\,|\log|\delta||\,(\|\partial_t W|G^0|\|_{L^2} + \|W|G^0|\|_{L^2}). \end{aligned}$$

Of course $(\partial_t, \nabla_x)W|G^0|$ is bounded in L^2 by the norm of G^0 in L^2 as can be seen by taking the Fourier transform

$$\hat{W}g(t,\xi) = \hat{g}(\xi)\,t \int_{S^2} e^{-it\xi\cdot\omega}\,d\omega = \hat{g}(\xi)\,\frac{4\pi\,\sin(|\xi|\,t)}{|\xi|}.$$

Consequently

$$I_C^1 \leq C\,K^{13/2}\,\nu(K)\,|\log|\delta||\,\|G^0\|_{L^2}. \tag{3.16}$$

Let us now turn to I_C^2. We have to define the modified wave operator

$$\tilde{W}_t g(x,v) = t \int_{S^2} \frac{g(x+t\omega)}{|\frac{v}{\sqrt{1+v^2}}\cdot\omega - 1|}\,d\omega. \tag{3.17}$$

Note that $\tilde{W}_t$ enjoys the same regularizing properties as W. In particular

$$\mathcal{F}\nabla_v^k \tilde{W}_t g(\xi,v) = \hat{g}(\xi)\,t \int_{S^2} \frac{\Phi_k(v,\omega)\,e^{-it\xi\cdot\omega}}{|\frac{v}{\sqrt{1+v^2}}\cdot\omega - 1|^{k+1}}\,d\omega,$$

for some smooth function Φ_k of v and ω. Therefore for any k

$$\|\nabla_x \nabla_v^k \tilde{W}_t g\|_{L^2_{xv}} \leq C_k\,K^{k+7/2}\,\|g\|_{L^2}, \tag{3.18}$$

where the L^2_{xv} norm is taken over any regular compact subset of $\mathbb{R}^6$ included in $B(0,K)$. By taking k large enough ($k=2$ for instance) and by Sobolev embedding, one may conclude that

$$\|\nabla_x \tilde{W}_t g\|_{L^2_x(L^\infty_v)} \leq C\,K^{15/2}\,\|g\|_{L^2}. \tag{3.19}$$

Notice now that

$$\left|t\int_{S^2}\frac{G^0(X_t+t\,\omega)-G^0(X_t^\delta+t\omega)}{|\dot X_t^\delta\cdot\omega-1|}\,d\omega\right| = |\tilde W_t G^0(X_t,V_t^\delta)-\tilde W_t G^0(X_t^\delta,V_t^\delta)|$$
$$\leq |X_t-X_t^\delta|\,(M_t|\nabla_x \tilde W_t G^0(X_t^\delta,V_t^\delta)|+M_t|\nabla_x\tilde W_t G^0(X_t,V_t^\delta)),$$

by applying Lemma 3.1 in [20], where we defined the modified maximal operator

$$M_s g(x)=\frac{1}{|\delta|+\max_{r\leq s}|X_r-X_r^\delta|}\int_{B(x,\max_{r\leq s}|X_r-X_r^\delta|)}\frac{g(z)\,dz}{|z-x|^2},$$

one has

$$I_C^2\leq C\nu(K)\int_{\Omega_K}\int_0^T\frac{|V_t-V_t^\delta|}{\sqrt{A_\delta(t)}}\Big(M_t|\nabla\tilde W_t G^0(X_t^\delta,V_t^\delta)|+M_t|\nabla\tilde W_t G^0(X_t,V_t^\delta)|\Big)$$
$$=I_C^{21}+I_C^{22}.$$

The first term can be easily bounded as it is symmetric. By Cauchy-Schwartz, we have

$$I_C^{2,1}\leq C\,\nu(K)\,K^3\,\sqrt{\log\delta}\left(\left(\int_{\Omega_K}\int_0^T(M_t|\nabla_x\tilde W_t G^0(X_t^\delta,V_t^\delta)|)^2\,dt\,dx\,dv\right)^{1/2}.\right.$$

Now by Fubini and a change of variable

$$\int_{\Omega_K}\int_0^T((M_t|\nabla_x\tilde W_t G^0(X_t^\delta,V_t^\delta)|)^2\,dt\,dx\,dv$$
$$\leq\int_0^T\int_{B(0,C\,K)}(M_t|\nabla W_t G^0(x,v)|)^2\,dx\,dv\,dt$$
$$\leq C\int_0^T\int_{B(0,C\,K)}|\nabla_x\tilde W_t G^0(x,v)|^2\,dx\,dv\,dt,$$

by the continuity of M_t on L^2. By the bound (3.18), one may bound

$$I_C^{2,1}\leq C\,\nu(K)\,K^{11/2}\,\|G^0\|_{L^2}. \tag{3.20}$$

The other term is not symmetric, as it mixes X_t^δ and V_t. It is hence more complicated but it can still be handled in a roughly similar way

$$I_C^{2,2}\leq C\,\nu(K)\,K^3\,\sqrt{|\log\delta|}\,\left(\int_0^T\int_{\Omega_K}M_t(|\nabla_x\tilde W G^0(X_t,V_t^\delta)|^2)\,dx\,dv\,dt\right)^{1/2}.$$

Now note that

$$M_t g(x) \le \int_{B(0,K)} \frac{|g(z)|\,dz}{(|\delta| + |z-x|)\,|z-x|^2}.$$

Hence one gets

$$I_C^{2,2} \le C\,\nu(K)\,K^3\,|\log\delta|^{1/2} \left(\int_0^T \int_{(B(0,K))^3} \frac{|\nabla_x \tilde W G^0(z, V_t^\delta)|^2}{(|\delta| + |z - X_t|)\,|z - X_t|^2}\,dz\,dx\,dv\,ds\right)^{1/2}.$$

Therefore

$$I_C^{2,2} \le C\,\nu(K)\,K^3\,|\log\delta|^{1/2} \left(\int_0^T \int_{(B(0,K))^3} \frac{\sup_w |\nabla_x \tilde W G^0(z, w)|^2}{(|\delta| + |z - X_t|)\,|z - X_t|^2}\,dz\,dx\,dv\,ds\right)^{1/2}.$$

Now by the usual change of variables, we get

$$I_C^{2,2} \le C\,\nu(K)\,K^{9/2}\,|\log\delta|\,\|\nabla_x \tilde W G^0\|_{L^2_x(L^\infty_v)}.$$

Using (3.19), one deduces that

$$I_C^{2,2} \le C\,\nu(K)\,K^{12}\,|\log\delta|\,\|G^0\|_{L^2}.$$

Therefore, combining with (3.16) and (3.20), one finally concludes that

$$I_C \le C\,K^{12}\,\nu(K)\,|\log\delta|\,\|G^0\|_{L^2}. \tag{3.21}$$

We did not deal with this term in a very subtle manner here. However to improve the result, one would need to do L^1 or at least L^p bounds (instead of L^2). Note as well that other terms anyway impose the use of the L^2 norm.

3.2.4 Bound for I_D

This bound essentially follows the line of I_A in a slightly more complicated way.
First of all, one may easily compute the z-derivative as

$$\nabla_z s_X = \frac{\omega_X}{\omega_X \cdot \dot X_{s_X} - 1}, \quad \nabla_z \omega_X = \frac{1}{s_X}\left(\frac{\omega_X \otimes (\dot X_{s_X} - \omega)}{\omega_X \cdot \dot X_{s_X} - 1} - I\right), \tag{3.22}$$

and the corresponding formulas for s_{X^δ} and ω_{X^δ}. Hence

$$\left|\nabla_z \cdot \frac{C\omega_X}{s_X\,|\dot X_{s_X}\cdot\omega_X - 1|}\right| \leq \frac{C\,|1+\dot V_{s_X}|}{s_X^2\,|\dot X_{s_X}\cdot\omega_X-1|^2} \leq \frac{C\,|1+F(s_X,X_{s_X})|}{s_X^2\,|\dot X_{s_X}\cdot\omega_X-1|^2}.$$

Inserting the corresponding terms in I_D, one finds that

$$I_D \leq C\,K^2\,\nu(K)\int_{\Omega_K}\int_0^T \frac{|V_t-V_t^\delta|}{A_\delta(t)}\int_{O_X^t\setminus O_{X^\delta}^t}\frac{|G^0(z)|}{s_X^2}\,|1+F(s_X,X_{s_X})| + \textit{symmetric term}.$$

The only difference with I_A is the additional term $F(s_X,X_{s_X})$. Now, note that as before

$$O_X^t\setminus O_{X^\delta}^t \subset \{t-|X_t-X_t^\delta|\leq |z-X_t|\leq t\}.$$

Hence, using spherical coordinates, one gets

$$\begin{aligned} I_D \leq\, & C\,K^2\,\nu(K)\int_{\Omega_K}\int_0^T \frac{|V_t-V_t^\delta|}{A_\delta(t)}\int_{t-|X_t-X_t^\delta|}^t |1+F(s,X_s)| \\ & \int_{S^2}|G^0(X_t+s\,\omega)|\,d\omega\,ds\,dt\,dx\,dv + \textit{symmetric} \\ \leq\, & C\,K^2\,\nu(K)\int_{\Omega_K}\int_0^T \frac{|V_t-V_t^\delta|}{A_\delta(t)}M_S G^0(X_t)\int_{t-|X_t-X_t^\delta|}^t |1+F(s,X_s)|\,ds \\ & + \textit{symmetric}, \end{aligned}$$

with M_S as before the spherical maximal function. By Cauchy-Schwartz

$$\begin{aligned} I_D \leq\, & C\,K^2\,\nu(K)\Bigg(\int_{\Omega_K}\int_0^T \frac{|V_t-V_t^\delta|^2}{A_\delta(t)}\,(M_S G^0(X_t))^2\,dt\,dx\,dv \\ & \int_{\Omega_K}\int_0^T \frac{|X_t-X_t^\delta|}{A_\delta(t)}\int_{t-|X_t-X_t^\delta|}^t (1+F(s,X_s))^2\,ds\,dt\,dx\,dv\Bigg)^{1/2}. \end{aligned}$$

The second term is bounded, using Fubini and then changing variables to (x,v) from (X_s,V_s), and is less than

$$C\,K^6\,\|F\|_{L^2}^2 \leq C\,K^6\,T\,\|F^0\|_{L^2}^2,$$

as the wave operator propagates the L^2 norm.
As for the first term, change variables to (x,v) from (X_t, V_t) to bound it by

$$\int_{B(0,K)} (M_S G^0(x))^2 \int_0^T \frac{|v - V_t^\delta(V_t^{-1}(x,v))|^2}{\delta^2 + \int_0^t |V_s(V_t^{-1}(x,v)) - V_s^\delta(V_t^{-1}(x,v))|^2\, ds}\, dt\, dx\, dv$$

Note that by the semi-group property, one still has that

$$|v - V_t^\delta(V_t^{-1}(x,v))|^2 = \partial_t \int_0^t |V_s(V_t^{-1}(x,v)) - V_s^\delta(V_t^{-1}(x,v))|^2\, ds,$$

and hence the previous term is bounded by

$$\sqrt{-\log|\delta|} \int_{B(0,K)} (M_S G^0(x))^2\, dx\, dv \leq C\, K^3 \sqrt{-\log|\delta|}\, \|G^0\|_{L^2},$$

as the spherical maximal function is bounded on L^2.
Combining all the estimates, one eventually finds that

$$I_D \leq C\, K^{11/2}\, \nu(K) \sqrt{|\log \delta|}\, \|G^0\|_{L^2}\, \|F^0\|_{L^2}. \tag{3.23}$$

Like I_C this term requires the L^2 norm of G^0. Contrary to I_C though, the computation here naturally produces a quadratic term in G^0 and F^0 and one does not see very well how to improve on it.

3.2.5 Bound for I_E

Applying formula (3.22), and recalling that $\phi \leq 1$, we can decompose E into

$$\begin{aligned}|E| \leq & C \int_{O_X^t \cap O_{X^\delta}^t} |G^0(z)| \left| \frac{H(s_X, \omega_X)}{s_X^2\, |\dot X_{s_X} \cdot \omega_X - 1|^2} - \frac{H(s_{X^\delta}, \omega_{X^\delta})}{s_{X^\delta}^2\, |\dot X_{s_{X^\delta}} \cdot \omega_{X^\delta} - 1|^2} \right| dz \\ & + \left| \int_{O_X^t \cap O_{X^\delta}^t} G^0(z) \left(\frac{\dot V_{s_X} \cdot \omega_X}{s_X (\dot X_{s_X} \cdot \omega_X - 1)^3} \right.\right. \\ & \qquad \left.\left. - \frac{\dot V_{s_{X^\delta}} \cdot \omega_{X^\delta}}{s_{X^\delta} (\dot X_{s_{X^\delta}} \cdot \omega_{X^\delta} - 1)^3} \right) dz \right| = E^1 + E^2.\end{aligned}$$

for some explicit smooth function H whose exact expression is unimportant here.

The term I_E^1 is bounded by a direct application of Lemma 3.2

$$I_E^1 \leq C\,\nu(K)\,K^7\,\sqrt{-\log|\delta|}\,(\|G^0\|_{L^1}+\|G^0\|_{L^p}). \tag{3.24}$$

As for E^2, we change back variables to find

$$E^2 \leq \left|\int_0^t (\dot V_s - \dot V_s^\delta)\phi(s)\,ds\right| + \left|\int_0^t \dot V_s\, s\,\int_{S^2}\left(\frac{\omega\,G^0(X_s+s\omega)}{(1-\dot X_s\cdot\omega)^2} - \frac{\omega\,G^0(X_s^\delta+s\omega)}{(1-\dot X_s^\delta\cdot\omega)^2}\right)d\omega\right| = E^{2,1}+E^{2,2},$$

with

$$\phi(s) = s\int_{S^2} G^0(X_s+s\omega)\,\frac{\omega}{(1-\dot X_s\cdot\omega)^2}\,d\omega.$$

The term $E^{2,2}$ is treated in a similar manner as the previous ones (note in particular that $\dot V_s$ is bounded in L^2 by $\|F^0\|_{L^2}$). We instead focus on $E^{2,1}$ and remark that by integration by part

$$I_E^{2,1} \leq \int_{\Omega_K}\int_0^T \frac{|V_t-V_t^\delta|^2}{A_\delta(t)}\,|\phi(t,x,v)|\,dx\,dv\,dt + \int_{\Omega_K}\int_0^T \frac{|V_t-V_t^\delta|}{A_\delta(t)}\int_0^t |V_s-V_s^\delta|\,|\partial_s\phi(s,x,v)|\,ds\,dt\,dx\,dv.$$

Note that

$$\begin{aligned}\int_{\Omega_K}\int_0^T &\frac{|V_t-V_t^\delta|^2}{A_\delta(t)}\,|\phi(t,x,v)|\,dx\,dv\,dt\\ &\leq C\,K^2\int_{\Omega_K}\int_0^T \partial_t\log(A_\delta(t)|)\,t\int_{S^2}|G^0(X_t+t\omega)|\,d\omega\\ &\leq C\,K^2\,|\log|\delta||\int_{\Omega_K}\int_0^T\left|\partial_t\left(t\int_{S^2}|G^0(X_t+t\omega)|\,d\omega\right)\right|\,dt\,dx\,dv.\end{aligned}$$

We remark that

$$\left|\partial_t\left(t\int_{S^2} g(X_t+t\omega)\,d\omega\right)\right| \leq C\,K\left|t\int_{S^2}(1,\omega)\,\nabla g(X_t+t\omega)\,d\omega\right| + \int_{S^2}|g(X_t+t\omega)|\,d\omega,$$

which implies that this term is bounded in L^2 by the L^2 norm of g. Consequently

$$\int_{\Omega_K}\int_0^T \frac{|V_t - V_t^\delta|^2}{A_\delta(t)}\,|\phi(t,x,v)|\,dx\,dv\,dt \le C\,K^2\,|\log|\delta||\,\|G^0\|_{L^2}.$$

As for the other term in $E^{2,1}$, compute

$$|\partial_s\phi| \le \int_{S^2}\frac{|G^0(X_s+s\omega)|}{(1-\dot X_s\cdot\omega)^2} + s\left|\int_{S^2}\frac{\omega\cdot\nabla G^0(X_s+s\omega)\,\omega}{(1-\dot X_s\cdot\omega)^2}\,ds\right| + C\,K^3\,|\dot V_s|\int_{S^2}|G^0(X_s+s\omega)|\,ds.$$

The first two terms are treated similarly as before. For the last term note that by Cauchy-Schwartz

$$\begin{aligned}
&\int_{\Omega_K}\int_0^T \frac{|V_t - V_t^\delta|}{A_\delta(t)}\int_0^t |V_s - V_s^\delta|\,|\dot V_s|\int_{S^2}|G^0(X_s+s\omega)|\,ds\,dt\,dx\,dv\\
&\quad\le \left(\int_{\Omega_K}\int_0^T\int_0^t |\dot V_s|^2\,\frac{|V_t - V_t^\delta|^2}{A_\delta(t)}\,ds\,dt\,dx\,dv\right)^{1/2}\\
&\qquad\left(\int_{\Omega_K}\int_0^T |M_S G^0(X_s)|^2\,\frac{\int_0^t |V_s - V_s^\delta|^2\,ds}{A_\delta(t)}\,ds\,dt\,dx\,dv\right)^{1/2},
\end{aligned}$$

with as always M_S the spherical maximal operator. Since $G^0\in L^2$, M_S is bounded on L^2 and $\dot V_s$ is also bounded in L^2 by the L^2 norm of G^0,

$$\begin{aligned}
&\int_{\Omega_K}\int_0^T \frac{|V_t - V_t^\delta|}{A_\delta(t)}\int_0^t |V_s - V_s^\delta|\,|\dot V_s|\int_{S^2}|G^0(X_s+s\omega)|\,ds\,dt\,dx\,dv\\
&\quad\le C\,K^3\sqrt{-\log|\delta|}\,\|G^0\|_{L^2}^2.
\end{aligned}$$

We conclude that

$$I_E \le C\,\nu(K)\,K^7\sqrt{-\log|\delta|}\,(\|G^0\|_{L^1} + \|G^0\|_{L^2} + \|G^0\|_{L^2}\,\|F^0\|_{L^2}). \tag{3.25}$$

3.2.6 Conclusion of the proof of Lemma 3.1

We combine estimates (3.9), (3.15), (3.21), (3.23) and (3.25). Taking the worst terms, one finds the estimate in Lemma 3.1.

3.3 Conclusion of the proof of Proposition 2.1

By Lemma 3.1, one has that

$$\bar{Q}_\delta(T) \le C_K\, T\, |\log|\delta||\,(\|G^0\|_{L^1} + \|G^0\|_{L^2})\,(1+\|F^0\|_{L^2}),$$

for a constant C_K increasing with K (which could be made explicit for a given ν).
On the other hand, the definition of $\bar{Q}_\delta$ yields the very obvious bound

$$\bar{Q}_\delta(T) \le C_K\, T\, \sqrt{-\log|\delta|}\, \|G^0\|_{W^{1,\infty}}.$$

We now use the linear dependence of $\bar{Q}_\delta(T)$ on G^0 to conclude by interpolation. For clarity, we now denote $\bar{Q}_\delta(T,\, G^0)$ making the dependence on G^0 explicit.
For any fixed $G^0 \in L^2 \cap L^1$, denote G^0_ε a sequence of C^∞_c functions converging to G^0 in $L^2 \cap L^1$. Just estimate by linearity

$$\begin{aligned}\bar{Q}_\delta(T,\, G^0) &= \bar{Q}_\delta(T,\, G^0_\varepsilon) + \bar{Q}_\delta(T,\, G^0 - G^0_\varepsilon)\\ &\le C_K\, T\, \left(\sqrt{-\log|\delta|}\, \|G^0_\varepsilon\|_{W^{1,\infty}} + |\log|\delta||\, \|G^0 - G^0_\varepsilon\|_{L^1\cap L^2}\right).\end{aligned}$$

Denote

$$\psi_0(\xi) = \inf_\varepsilon \sqrt{\xi}\, \|G^0_\varepsilon\|_{W^{1,\infty}} + \xi\, \|G^0 - G^0_\varepsilon\|_{L^1\cap L^2},$$

and note that ψ_0 is increasing and

$$\frac{\psi_0(\xi)}{\xi} \longrightarrow 0, \quad \text{as } \xi \to \infty.$$

Thus

$$\bar{Q}_\delta(T) \le C_K\, T\, \psi_0(-\log|\delta|).$$

Note that ψ_0 depends intrinsically on $\|G^0\|_{L^2}$ and on the equi-integrability of $\hat{G}^0$ in L^2. As $F^0 \in L^2$, we finally use this estimate for $G^0 = F^0$ and get

$$Q_\delta(T) \le \frac{C\,T}{K^2}\, |\log|\delta|| + C_K\, T\, \psi_0(-\log|\delta|).$$

It only remains to optimize in K by defining

$$\psi(\xi) = \inf_K \left(\frac{C\,\xi}{K^2} + C_K\, \psi_0(\xi)\right).$$

One still has that
$$\frac{\psi(\xi)}{\xi} \longrightarrow 0, \quad \text{as } \xi \to \infty,$$
and
$$Q_\delta(T) \le T\,\psi(-\log|\delta|),$$
which concludes the proof of Prop. 2.1.

4 Proof of Theorem 1.2

4.1 Well posedness for the corresponding ODE

We follow the same steps as for the proof of Th. 1.1. We study the ODE
$$\begin{aligned}&\frac{d}{dt}X(t,x,v) = \alpha(V(t,x,v)), \quad \frac{d}{dt}V(t,x,v) = F(t,X(t,x,v),\\ &X(0,x,v) = x, \qquad V(0,x,v) = v.\end{aligned} \tag{4.1}$$

As flows the solutions are again required to satisfy

Property 2 *For any $t \in \mathbb{R}$ the application*
$$(x,v) \in \mathbb{R} \times \mathbb{R}^d \mapsto (X(t,x,v), V(t,x,v)) \in \mathbb{R} \times \mathbb{R}^d \tag{4.2}$$
is globally invertible and has Jacobian 1 at any $(x,v) \in \mathbb{R}\times\mathbb{R}^d$. It also defines a semi-group
$$\begin{aligned}&\forall s,t \in \mathbb{R}, \qquad X(t+s,x,v) = X(s,X(t,x,v),V(t,x,v)),\\ &\textit{and} \qquad\qquad V(t+s,x,v) = V(s,X(t,x,v),V(t,x,v)).\end{aligned} \tag{4.3}$$

We look again at
$$R_\delta(T) = \log\left(1 + \frac{1}{\bar{\delta}(T,x,v,|\delta|)^2}\left(\sup_{0\le t\le T}|X(t,x,v) - X^\delta(t,x,v)|^2 \right.\right.$$
$$\left.\left. + \int_0^T |V(t,x,v) - V^\delta(t,x,v)|^2\,dt\right)\right)$$

for any $(x,v)\in\Omega$ a subset of $\mathbb{R}^{d+1}$, and for a function $\bar\delta(t,x,v)=\bar\delta(t,x,v,|\delta|)$ to be determined later. (X,V) is a solution to (4.1) satisfying Property 2 (or a regularized problem) and X^δ, V^δ satisfies:

$$
\begin{aligned}
&\text{Either } (X^\delta,\,V^\delta) \text{ is a solution to (4.1) (or a regularized version)}\\
&\qquad\qquad\qquad \text{satisfying Property 2,}\\
&\text{or } \exists(\delta_1,\delta_2)\in\mathbb{R}^{1+d} \quad \text{with } |(\delta_1,\delta_2)|\le\delta, \text{ such that}\\
&\qquad\qquad\qquad (X^\delta,V^\delta)(t,x,v)=(X,V)(t,x+\delta_1,v+\delta_2).
\end{aligned}
\tag{4.4}
$$

Theorem 1.2 is a consequence of

Proposition 4.1 *Assume $F^0\in L^\infty$ and (1.12). For any Ω compact, any F^0 bounded, there exists a function $\bar\delta(t,x,v,|\delta|)$, increasing in time with $\bar\delta(0,x,v,|\delta|)=|\delta|$, such that the following holds: For any solution (X,V) to (4.1) with F given by Eq. (1.11), satisfying Property 2, and (X^δ,V^δ) satisfying (4.4) and for all $T>0$, we have*

$$R_\delta(T)\le C\,T\,\|F^0\|_\infty\,(\log(1/|\delta|))^{3/4},\ \bar\delta(T,x,v,|\delta|)\longrightarrow 0 \ \textit{as}\ |\delta|\to 0,\ \textit{a.e.}(x,v).$$

First of all let us order the μ_n decreasingly

$$|\mu_0|\ge|\mu_1|\ge\ldots|\mu_n|\ldots$$

Note then that by (1.12)

$$\|F(t,x)\|_\infty\le\|F_0\|_\infty\sum_n|\alpha_n|\le C\|F^0\|_\infty.$$

Therefore defining $\tilde\Omega=\Omega+B(0,T\,\|F_0\|_\infty\sum_n|\alpha_n|)$, one has that $(X,V)\in\tilde\Omega$ for any $(x,v)\in\Omega$ and $t\in[0,\ T]$. The same is of course true for (X^δ,V^δ).
As before problems occur when the velocity of the particle is close to one of the propagation velocities ξ_n. So first of all it is necessary to control the time that each trajectory spends near one of those points.
Denote

$$
\begin{aligned}
&\omega(w,\eta,K)=\{(x,v)\in\Omega\ s.t.\ |\{t,\ |\alpha(V(t,x,v))-w|\le\eta\}|\ge K\,\eta\},\\
&\omega^\delta(w,\eta,K)=\{(x,v)\in\Omega\ s.t.\ |\{t,\ |\alpha(V^\delta)(t,x,v)-w|\le\eta\}|\ge K\,\eta\}.
\end{aligned}
$$

The parameter η will be chosen later but will tend to 0 as $|\delta|\to 0$. Then

Lemma 4.1 *There exists a constant C depending on $\|F^0\|_\infty$ and $\sum_n |\mu_n|$ such that*
$$|\omega(w,\eta,K)| \leq \frac{C}{K}.$$

Proof. Simply write that

$$\int_0^T \int_{\omega(w,\eta,K)} \mathbb{I}_{\{|\alpha(V(t,x,v))-w|\leq\eta\}}\,dx\,dv\,dt \geq K\,\eta\,|\omega(w,\eta,K)|.$$

On the other hand, using Property 4.2 and the assumption (1.10) on $\alpha(v)$, we have

$$\int_0^T \int_{\omega(w,\eta,K)} \mathbb{I}_{\{|\alpha(V(t,x,v))-w|\leq\eta\}}\,dx\,dv\,dt \leq \int_0^T \int_{\tilde\Omega} \mathbb{I}_{\{|\alpha(v)-w|\leq\eta\}}\,dx\,dv\,dt$$
$$\leq C\,|\tilde\Omega|\,\eta\,T,$$

for some constant C.
Finally one concludes that

$$|\omega(w,\eta,K)| \leq C/K.$$

For X^δ, V^δ, one uses (4.4) and either they also satisfy (4.2) in which case the proof is identical. Or one just has to shift (x,v) by δ and still follow the same steps. $\square$
Now for a_n (to be fixed later), we define

$$O_n = \{v,\ |\alpha(v)-\xi_n| \leq a_n\,\eta\}.$$

Then for any (x,v), we decompose the time interval $[0,\ T]$ into $I_{x,v}$ and the union $\bigcup_n [t_0^n,\ s_0^n] \cup \ldots \cup [t_{k_n}^n,\ s_{k_n}^n]$ with

$$I_{x,v} = \{t,\ |\alpha(V(t,x,v))-\xi_n| \geq a_n\,\eta/2\},$$
$$\sup_{[t_i^n,\ s_i^n]} |\alpha(V)-\xi_n| = a_n\,\eta,\ \ \inf_{[t_i^n,\ s_i^n]} |\alpha(V)-\xi_n| \leq a_n\,\eta/2.$$

Similarly one defines $I_{x,v}^\delta$ and the intervals $[t_i^{n,\delta},\ s_i^{n,\delta}]$ for $i = 0\ldots k_n^\delta$. Note that k_n and k_n^δ depend on (x,v).

Define now

$$l_n = \sum_{i=0}^{k_n} (s_i^n - t_i^n), \quad l_n^\delta = \sum_{i=0}^{k_n} (s_i^{n,\delta} - t_i^{n,\delta}),$$

and

$$\omega(\eta, K) = \{(x,v) \in \Omega \ s.t. \ \sum_n l_n \geq K\,\eta\},$$
$$\omega^\delta(\eta, K) = \{(x,v) \in \Omega \ s.t. \ \sum_n l_n^\delta \geq K\,\eta\}.$$

Similarly to Lemma 4.1 one can deduce

Lemma 4.2 *Assume that $\sum_n a_n < \infty$, then there exists a constant C depending on $\sum_n a_n$ s.t.*

$$|\omega(\eta, K)| \leq \frac{C}{K}, \quad |\omega^\delta(\eta, K)| \leq \frac{C}{K}.$$

Proof. Simply note that

$$\omega(\eta, K) \subset \big\{(x,v), \quad |\{t, \ V(t,x,v) \in \bigcup_n O_n\}| \geq K\,\eta\big\}.$$

Then similarly to the proof of Lemma 4.1

$$\begin{aligned} K\,\eta\,|\omega(\eta,K)| &\leq \int_{\omega(\eta,K)} \int_0^T \mathbb{I}_{V(t,x,v)\in\bigcup_n O_n}\,dt\,dx\,dv \\ &\leq \int_0^T \int_\Omega \mathbb{I}_{V(t,x,v)\in\bigcup_n O_n}\,dx\,dv\,dt = \int_0^T \int_{\tilde\Omega} \mathbb{I}_{v\in\bigcup_n O_n}\,dx\,dv \\ &\leq T \sum_n |O_n| \leq T\,\eta \sum a_n, \end{aligned}$$

which shows the result. □
We are now ready to define $\bar\delta(t,x,v,|\delta|)$. We put

$$\bar\delta(0,x,v,|\delta|) = |\delta|, \quad \partial_t \bar\delta(0,x,v,|\delta|) = C\,\|F\|_\infty \sum_n (l_n + l_n^\delta). \tag{4.5}$$

Note that from Lemma 4.2 and the fact that η tends to 0 as $|\delta| \to 0$, $\bar\delta$ indeed converges to 0 for *a.e.* (x,v).

From the computation for Q_δ in the proof of Proposition 2.1, one sees that

$$\begin{aligned}\frac{d}{dt}R_\delta(T) \leq & 2\int_0^T \frac{V(t)-V^\delta(t)}{A_\delta(t,x,v)} \\ & \int_0^t (F(s,X(s,x,v)) - F(s,X^\delta(s,x,v)))\,ds\,dt \\ & - 2\int_0^T \frac{\partial_t\bar\delta(t,x,v,|\delta|)}{A_\delta(t,x,v)}\,dt,\end{aligned}$$

with

$$\begin{aligned}A_\delta(t,x,v) =& \bar\delta^2 + \sup_{0\leq s\leq t} |X(s,x,v)-X^\delta(s,x,v)|^2 \\ & + \int_0^t |V(s,x,v)-V^\delta(s,x,v)|^2\,ds.\end{aligned}$$

Compute

$$\begin{aligned}&\int_0^t (F(s,X(s)) - F(s,X^\delta(s)))\,ds \\ &\quad = \sum_n \int_0^t (F^0(X(s)-\xi_n s) - F^0(X^\delta(s)-\xi_n s))\,\mu_n\,ds \\ &\leq \sum_n \Bigg(C\,\|F\|_\infty\,\mu_n \sum_{i=0}^{k_n} |s_i^n - t_i^n| + C\,\|F\|_\infty\,\mu_n \sum_{i=0}^{k_n^\delta} |s_i^{n,\delta} - t_i^{n,\delta}| \\ &\qquad + \int_{s\notin\cup_i([t_i^n,\ s_i^n]\cup[t_i^{n,\delta},\ s_i^{n,\delta}])} (F^0(X(s)-\xi_n s) - F^0(X^\delta(s)-\xi_n s))\,\mu_n\,ds\Bigg).\end{aligned}$$

Now $[0,\ T]\setminus\cup_i[t_i^n,\ s_i^n]\cup[t_i^{n,\delta},\ s_i^{n,\delta}]$ is included in an union of intervals of the form $[s_i^n,\ t_{i+1}^n]$, $[s_i^{n,\delta},\ t_{i+1}^{n,\delta}]$, $[s_i^n, t_j^{n\delta}]$ or $[s_i^{n,\delta},\ t_j^n]$. This depends only on whether $V(s)$ or $V^\delta(s)$ is the first or the last to be such that $|\alpha(V)-\xi_n| = a_n\,\eta$. Assume for instance that the corresponding interval is $[s_i^n,\ t_{i+1}^n]$.
Define the transforms

$$\Phi_n(s) = X(s) - \xi_n\,s, \quad \Phi_n^\delta(s) = X^\delta(s) - \xi_n\,s$$

and note that by definition Φ_n and Φ_n^δ are invertible on the corresponding interval $[s_i^n,\ t_{i+1}^n]$. Hence denote $S_n(u)$ and $S_n^\delta(u)$ the reciprocal functions.

To bound the next integral, we will make a change of variable from s to $u = \Phi_n(s)$ and then go back to the original variable:

$$\int_{[s_i^n,\ t_{i+1}^n]} (F^0(X(s) - \xi_n s) - F^0(X^\delta(s) - \xi_n s))\, ds$$
$$\leq \int_{\Phi_n(s_i^n)}^{\Phi_n(t_{i+1}^n)} F(u) \left(\frac{1}{\alpha(V(S_n(u))) - \xi_n\, S_n(u)} - \frac{1}{\alpha(V^\delta(S_n^\delta(u))) - \xi_n\, S_n^\delta(u)} \right) du$$
$$+ \|F\|_\infty \frac{\sup_s |X(s) - X^\delta(s)|}{a_n\, \eta},$$

or

$$\int_{[s_i^n,\ t_{i+1}^n]} (F^0(X(s) - \xi_n s) - F^0(X^\delta(s) - \xi_n s))\, ds$$
$$\leq C\, \frac{\|F\|_\infty}{a_n\, \eta} \int_{s_i^n}^{t_{i+1}^n} (|V(s) - V^\delta(s)| + |s - S_n^\delta(\Phi_n(s))|)\, ds$$
$$+ \|F\|_\infty \frac{\sup_s |X(s) - X^\delta(s)|}{a_n\, \eta},$$

simply by using the Lipschitz regularity of α in v and of V, V^δ in time. Next notice that

$$|s - S_n^\delta(\Phi_n(s))| = |S_n^\delta(\Phi_n^\delta(s) - S_n^\delta(\Phi_n(s))| \leq \|S_n^\delta\|_{W^{1,\infty}}\, |\Phi_n^\delta(s) - \Phi_n(s)|$$
$$\leq C\, \frac{\sup_s |X(s) - X^\delta(s)|}{a_n\, \eta}.$$

And hence deduce that

$$\int_{[s_i^n,\ t_{i+1}^n]} (F^0(X(s) - \xi_n s) - F^0(X^\delta(s) - \xi_n s))\, ds \leq C\, \frac{\|F\|_\infty}{a_n\, \eta}$$
$$\left(\int_{s_i^n}^{t_{i+1}^n} |V(s) - V^\delta(s)|\, ds + (1 + |t_{i+1}^n - s_i^n|/(a_n\, \eta)) \sup_s |X(s) - X^\delta(s)| \right). \tag{4.6}$$

Introducing this estimate, one obtains

$$\frac{d}{dt}R_\delta(T) \leq C\,\|F\|_\infty \sum_n \mu_n \int_0^t \frac{|V(t)-V^\delta(t)|}{A_\delta(t,x,v)} \Bigg(l_n + l_n^\delta + \frac{1}{a_n\eta}\int_0^t |V(s)-V^\delta(s)|\,ds + \frac{k_n+k_n^\delta+t/(a_n\eta)}{a_n\,\eta}\sup_{s\leq t}|X(s)-X^\delta(s)| \Bigg) - 2\int_0^T \frac{\partial_t\bar\delta(t,x,v,|\delta|)}{A_\delta(t,x,v)}\,dt.$$

Therefore

$$\frac{d}{dt}R_\delta(T) \leq C\,\|F\|_\infty \sqrt{\log 1/|\delta|}\sum_n \frac{\mu_n}{a_n\,\eta}(1+k_n+k_n^\delta+1/(a_n\eta)) + \int_0^T \frac{|V(t)-V^\delta(t)|\sum_n \mu_n(l_n+l_n^\delta)-\partial_t\bar\delta}{A_\delta(t,x,v)}\,dt,$$

and from the definition (4.5) of $\bar\delta$ and the obvious bound $|V-V^\delta|\leq|\delta|+T\|F\|_\infty$, one simply gets

$$\frac{d}{dt}R_\delta(T) \leq C\,\|F\|_\infty \sqrt{\log 1/|\delta|}\sum_n \frac{\mu_n}{a_n\,\eta}(1+k_n+k_n^\delta+1/(a_n\eta)).$$

By the definition of the intervals and the fact that V is Lipschitz in time

$$|s_i^n - t_i^n| \geq a_n\,\eta/C.$$

Hence

$$l_n \geq k_n\,a_n\,\eta/C,\ \text{ or } k_n \leq \frac{C\,T}{a_n\,eta}.$$

So finally

$$R_\delta(T) \leq C\,T\,\|F\|_\infty \frac{\sqrt{\log 1/|\delta|}}{\eta^2}\sum_n \frac{\mu_n}{a_n^2}. \tag{4.7}$$

Taking for instance $\eta = (\log 1/|\delta|)^{-1/8}$, one indeed concludes the proof of Prop. 4.1, provided that it is possible to choose the a_n s.t. $\sum_n \mu_n/a_n^2$ is finite.
The only other constraint to satisfy on the a_n is that $\sum_n a_n < \infty$. By the bound (1.12), one may simply choose $a_n = n^{-\gamma/2}$. $\square$

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