Averaging Lemmas and Dispersion Estimates for kinetic equations

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Abstract. Averaging lemmas consist in a regularizing effect on the average of the solution to a linear kinetic equation. Some of the main results are reviewed and their proofs presented in as self contained a way as possible. The use of kinetic formulations for the well posedness of scalar conservation laws is eventually explained as an example of application.
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Introduction

Kinetic equations are a particular case of transport equation in the phase space, *i.e.* on functions $f(x, v)$ of physical *and* velocity variables like

$$\partial_t f + v \cdot \nabla_x f = g, \qquad t \geq 0,\ x, v \ \in\ \mathbb{R}^d.$$

As a solution to a hyperbolic equation, the solution cannot be more regular than the initial data or the right hand-side. However a specific feature of kinetic equations is that the averages in velocity, like

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v)\, \phi(v)\, dv, \quad \phi \in C_c^\infty(\mathbb{R}^d),$$

are indeed more regular. This phenomenon is called velocity averaging.

It was first observed in [24] and then in [23] in a L^2 framework. The final L^p estimate was obtained in [17] (and slighty refined in [3] to get a Sobolev space instead of Besov). The case of a full derivative $g = \nabla_x \cdot h$ was treated in [45] and although it is in many ways a limit case, it is important for some applications as it can replace compensated compactness arguments.

In addition to these works, this course presents and sometimes reformulates some of the results of [6], [17], [22], [31], [32], [36], [37], [45].

There are of course many other interesting contributions investigating averaging lemmas that are not quoted or only briefly mentioned through the text.

1 Kinetic equations: Basic tools

1.1 A short introduction to kinetic equations

For a more complete introduction to kinetic equations and the basic theory, we refer to [6] or [21]. Many proofs are omitted here but are generally well known and not difficult.

1.1.1 The basic equations

During most of this course, we will deal with the simplest equations

$$\partial_t f + \alpha(v) \cdot \nabla_x f = g(t, x, v), \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d,\ v \in \omega, \tag{1.1}$$

where ω is often $\mathbb{R}^d$ (but might only be a subdomain); or with the stationary version

$$\alpha(v) \cdot \nabla_x f = g(x, v), \qquad t \in \mathbb{R}_+,\ x \in O,\ v \in \omega, \tag{1.2}$$

where O is an open, regular subset of $\mathbb{R}^d$ and ω is usually rather the sphere S^{d-1}. The transport coefficient α will always be regular, typically Lipschitz although here bounded would be enough.

Of course (1.1) is really a subcase of (1.2) in dimension $d+1$ and with $\alpha'(v) = (1, \alpha(v))$, $O = \mathbb{R}_+^* \times \mathbb{R}^d$, $\omega = \omega$.

Neither (1.1) nor (1.2) have a unique solution as there are many solutions to

$$\partial_t f + \alpha(v) \cdot \nabla_x f = 0,$$

for instance. Indeed for (1.1) an initial data must be provided

$$f(t=0,x,v) = f^0(x,v), \tag{1.3}$$

and for (1.2) the incoming value of f on the boundary must be specified

$$f(x,v) = f^{in}(x,v), \quad x \in \partial O, \alpha(v)\cdot\nu(x) \le 0, \tag{1.4}$$

where $\nu(x)$ is the outward normal to O at x.

It is then possible to have existence and uniqueness in the space of distributions

Theorem 1.1 *Let $f^0 \in \mathcal{D}'(\mathbb{R}^d \times \omega)$ and $g \in L^1_{loc}(\mathbb{R}_+,\ \mathcal{D}'(\mathbb{R}^d\times\omega))$. Then there is a unique solution in $L^1_{loc}(\mathbb{R}_+,\ \mathcal{D}'(\mathbb{R}^d\times\omega))$ to* (1.1) *with* (1.3) *in the sense of distribution given by*

$$f(t,x,v) = f^0(x-\alpha(v)\,t,v) + \int_0^t g(t-s,x-\alpha(v)\,s,v)\,ds. \tag{1.5}$$

Note that if f solves (1.1) then for any $\phi \in C^\infty_c(\mathbb{R}^d\times\omega)$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\omega} f(t,x,v)\,\phi(x,v) \in L^1_{loc}(\mathbb{R}_+),$$

so f has a trace at $t=0$ in the weak sense and (1.3) perfectly makes sense.
Proof. It is easy to check that (1.5) indeed gives a solution. If f is another solution then define

$$\bar f = f - f^0(x-\alpha(v)\,t,v) - \int_0^t g(t-s,x-\alpha(v)\,s,v)\,ds.$$

Remark that

$$\partial_t \bar f + \alpha(v)\cdot\nabla_x \bar f = 0,$$

and hence $\partial_t(\bar f(t,x+\alpha(v)\,t,v) = 0$ so that $\bar f = 0$. □

An equivalent result may be proved for (1.2) with the condition that the support of the singular part (in x) of the distribution g does not extend to the boundary ∂O.

On the other hand, the modified equation, which we will frequently use,

$$\alpha(v)\cdot\nabla_x f + f = g, \quad x\in\mathbb{R}^d,\ v\in\omega, \tag{1.6}$$

is well posed in the whole $\mathbb{R}^d$ without the need for any boundary condition

Theorem 1.2 *Let $g \in \mathcal{S}'(\mathbb{R}^d \times \omega)$, there exists a unique f in $\mathcal{S}'(\mathbb{R}^d \times \omega)$ solution to* (1.6)*. It is given by*

$$f(x,v) = \int_0^\infty g(x - \alpha(v)\,t, v)\, e^{-t}\, dt. \tag{1.7}$$

1.1.2 Liouville equation

The equation reads

$$\partial_t f + \alpha(v) \cdot \nabla_x f + F(t,x,v) \cdot \nabla_v f = 0, \quad t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \tag{1.8}$$

where F is a given force field. In many applications, like the Vlasov-Maxwell system 1.2, F is in fact computed from the solution f.

Eq. (1.8) describes the dynamics of particles submitted to the force F and as such is connected to the solution of the ODE

$$\begin{aligned} &\frac{d\,X(t,s,x,v)}{dt} = \alpha(V(t,s,x,v)), \quad \frac{d\,V(t,s,x,v)}{dt} = F(t,X,V), \\ &X(s,s,x,v) = x, \quad V(s,s,x,v) = v, \end{aligned} \tag{1.9}$$

which represents the trajectory of a particle starting with position and velocity (x,v) at time $t = s$.

The ODE (1.9) is well posed for instance if

$$\begin{aligned} \alpha(v) \in W^{1,\infty}_{loc}(\mathbb{R}^d), \quad F \in W^{1,\infty}_{loc}(\mathbb{R}_+ \times \mathbb{R}^{2d}), \\ |\alpha| + |F| \leq C(t)\,(1 + |x| + |v|), \end{aligned} \tag{1.10}$$

thanks to Cauchy-Lipschitz Theorem. Weaker assumptions are however enough, $W^{1,1}_{loc}$ and bounded divergence in [16] or even BV_{loc} in [1], but will not be required here.

Under (1.10), (1.8) is also well posed

Theorem 1.3 *Assume* (1.10) *and $\nabla_v \cdot F \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^{2d})$, for any measure valued initial data $f^0 \in M^1(\mathbb{R}^{2d})$, there exists a unique f included in $L^\infty([0,\ T],\ M^1(\mathbb{R}^d))$ solution to* (1.8) *in the sense of distribution and satisfying* (1.3)*. It is given by*

$$f(t,x,v) = f^0(X(0,t,x,v),\ V(0,t,x,v)).$$

If F and α are regular enough (C^∞), the same theorem holds if f^0 is only a distribution.

This theorem implies many properties on f, for example

Proposition 1.1 *(i) $f \geq 0$ if and only if $f^0 \geq 0$.*
(ii) If $f^0 \in L^\infty(\mathbb{R}^{2d})$ then $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^{2d})$ and

$$\|f(t,.,.)\|_{L^\infty(\mathbb{R}^{2d})} \leq \|f^0\|_{L^\infty(\mathbb{R}^{2d})}$$

(iii) If $f^0 \in L^p(\mathbb{R}^{2d})$ then $f \in L^\infty([0,\ T],\ L^p(\mathbb{R}^{2d}))$ and

$$\|f(t,.,.)\|_{L^p(\mathbb{R}^{2d})} \leq \|f^0\|_{L^p(\mathbb{R}^{2d})}\, e^{t\,\|\nabla_v F\|_\infty/p}.$$

From the point of view of averaging lemma, Eq. (1.8) does not have a particularly interesting structure. Indeed most of the time, the acceleration term $F\cdot\nabla_v f$ will be considered as a right hand side with no particular relation to f. Surprisingly enough this is generally optimal.

1.1.3 A simple case: local equilibrium

Let us consider (1.2) in the special case where

$$f(x,v) = \rho(x)\, M(v).$$

This might seem like an over simplification but it will nevertheless provide many examples of optimality later on. For the moment we will be satisfied with a few remarks.

We have

$$M(v)\,\alpha(v)\cdot\nabla_x\rho(x) = g.$$

Let us hence write $g = M(v)\,h(x,v)$.

Assuming that h is a regular function ($L^1 \cap L^\infty$ for example), this provides some regularity for ρ but not necessarily in term of Sobolev spaces.

Notice first that some assumption is needed on α. Indeed if there exists a direction $\xi \in S^{d-1}$ such that $\alpha(v)$ is colinear to ξ or $\alpha \parallel \xi$ for enough v

$$|\{v \in \mathbb{R}^d \mid \alpha(v) \parallel \xi\}| \neq 0,$$

and if M is supported in this set (no matter how regular) then it is only possible to deduce from (1.2) that

$$\xi\cdot\nabla_x\rho \in L^\infty.$$

Nothing can be said a priori about the derivatives in the other directions.

Even if $\alpha(v)$ is not concentrated along some directions like $\alpha(v) = \xi$, some assumption is needed on M. If not, M itself may be concentrated along one direction ξ in which case the same phenomenon occurs.

This shows the two features of all the averaging results that will be proved: Some assumption is needed on $|\{v \in \mathbb{R}^d \mid \alpha(v) \parallel \xi\}|$ and the more regular in velocity f is, the more regular ρ is.

In fact the regularity provided by averaging lemmas (in terms of Sobolev spaces) is in many situations not the optimal way of describing the regularity of solutions to (1.2) (see [10], [12] and [52] for example in the case of scalar conservation laws).

1.2 An application: The Vlasov-Maxwell system

The Vlasov-Maxwell system describes the evolution of charged particles and it reads

$$\partial_t f + v(p) \cdot \nabla_x f + (E(t,x) + v(p) \times B(t,x)) \cdot \nabla_p f = 0, \quad t \geq 0,\ x, p \in \mathbb{R}^d. \quad (1.11)$$

The fields E and B are the electric and magnetic fields and are solutions to Maxwell equations

$$\begin{aligned} &\partial_t E - \operatorname{curl} B = -j, \quad \operatorname{div} E = \rho, \\ &\partial_t B + \operatorname{curl} E = 0, \quad \operatorname{div} B = 0, \end{aligned} \quad (1.12)$$

where ρ and j are the density and current of charged particles and therefore computed from f

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,p)\, dp, \quad j(t,x) = \int_{\mathbb{R}^d} v(p)\, f(t,x,p)\, dp. \quad (1.13)$$

Initial data are required for the system

$$f(t=0,x,p) = f^0(x,p), \quad E(t=0,x) = E^0(x), \quad B(t=0,x) = B^0(x). \quad (1.14)$$

Finally the variable p represents the impulsion of the particles. In the classical case (velocities of the particles much lower than the light speed), it is simply the velocity and

$$v(p) = p.$$

In the relativistic case, the velocity is related to the impulsion through

$$v(p) = \frac{p}{(1+|p|^2)^{1/2}}.$$

For simplicity all physical constants were taken equal to 1.

Globally in time and in dimension 3 and more, only the existence of solutions in the sense of distributions is known (and thus no uniqueness). This was proved in [15] and it is one of the first examples of application of averaging lemmas.

As usual one considers a sequence of classical solutions f_ε, E_ε, B_ε to a regularized system. The form of this system is essentially unimportant as long as it has the same a priori estimates as (1.11)-(1.12). For (1.11) and from the analysis in 1.1.2, one first has

$$\|f_\varepsilon(t,.,.)\|_{L^p(\mathbb{R}^{2d})} \leq \|f^0_\varepsilon\|_{L^p(\mathbb{R}^{2d})}, \quad \forall t\geq 0, \quad \forall p\in[1,\ \infty]. \tag{1.15}$$

The only other available a priori estimate is the conservation of energy

$$\begin{aligned}\int_{\mathbb{R}^{2d}} E(p) f_\varepsilon(t,x,p)\,dx\,dp + \int_{\mathbb{R}^d}(|E_\varepsilon(t,x)|^2+|B_\varepsilon(t,x)|^2)\,dx \leq \\ \int_{\mathbb{R}^{2d}} E(p) f^0_\varepsilon(x,p)\,dx\,dp + \int_{\mathbb{R}^d}(|E^0_\varepsilon(x)|^2+|B^0_\varepsilon(x)|^2)\,dx.\end{aligned} \tag{1.16}$$

This relation is an inequality instead of an equality as the regularized system typically dissipates a bit. The term $E(p)$ is equal to the usual kinetic energy $|p|^2$ in the classical case and to $(1+|p|^2)^{1/2}$ in the relativistic case.

Therefore assuming that

$$f^0\geq 0,\ f^0\in L^1\cap L^\infty(\mathbb{R}^{2d}), \quad \int_{\mathbb{R}^{2d}} E(p)\, f^0\,dx\,dp<\infty, \quad E^0,\ B^0\in L^2(\mathbb{R}^d), \tag{1.17}$$

then the same bounds are uniformly true in ε for $f_\varepsilon(t,.,.)$, $E_\varepsilon(t,.)$ and $B(t,.)$.

On the other hand, we obviously have that

$$\int_{\mathbb{R}^d}\rho_\varepsilon(t,x) = \int_{\mathbb{R}^{2d}} f_\varepsilon(t,x,p) = \int_{\mathbb{R}^{2d}} f^0_\varepsilon. \tag{1.18}$$

In the relativistic case

$$\int_{\mathbb{R}^d}|j_\varepsilon(t,x)|\,dx \leq \int_{\mathbb{R}^{2d}}|f_\varepsilon(t,x,p)| = \int_{\mathbb{R}^{2d}} f^0_\varepsilon, \tag{1.19}$$

while in the classical case, through Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^d} |j_\varepsilon(t,x)|\,dx = \int_{\mathbb{R}^{2d}} |p| f_\varepsilon \le \left(\int_{\mathbb{R}^{2d}} f_\varepsilon\right)^{1/2} \left(\int_{\mathbb{R}^{2d}} |p|^2 f_\varepsilon\right)^{1/2}. \tag{1.20}$$

As a consequence ρ_ε and j_ε are uniformly bounded in L^1.

Moreover a simple interpolation estimate may provide L^p estimates for ρ_ε and j_ε

$$\begin{aligned}\rho_\varepsilon(t,x) &\le \int_{B(0,R)} f_\varepsilon\,dp + \int_{|p|>R} f_\varepsilon\,dp \le R^d \|f_\varepsilon\|_{L^\infty} + \frac{1}{R^\alpha}\int_{\mathbb{R}^d} E(p)\,f_\varepsilon\,dp \\ &\le \|f_\varepsilon\|_{L^\infty}^{\frac{\alpha}{\alpha+d}} \left(\int_{\mathbb{R}^d} E(p)\,f_\varepsilon\,dp\right)^{\frac{d}{d+\alpha}},\end{aligned}$$

through minimization in α; $\alpha = 1$ in the relativistic case and $\alpha = 2$ in the classical case. So

$$\int_{\mathbb{R}^d} (\rho_\varepsilon(t,x))^{\frac{d+\alpha}{d}}\,dx \le \|f_\varepsilon\|_{L^\infty}^{\frac{\alpha}{d}} \int_{\mathbb{R}^{2d}} E(p)\,f_\varepsilon\,dp\,dx.$$

Eventually one may obtain the following uniform bounds

$$\rho_\varepsilon(t,.) \in L^1 \cap L^{(d+\alpha)/d}(\mathbb{R}^d), \quad j_\varepsilon(t,.) \in L^1 \cap L^{(d+\alpha)/(d+\alpha-1)}(\mathbb{R}^d). \tag{1.21}$$

We may thus extract weak-* converging subsequences for f_ε, E_ε, B_ε and ρ_ε, j_ε in the corresponding spaces. One may then try to pass to the limit in (1.11) and (1.12). This works just fine for all terms except

$$(E_\varepsilon(t,x) + v(p)\times B_\varepsilon(t,x))\cdot\nabla_p f_\varepsilon = \nabla_p\cdot(((E_\varepsilon(t,x) + v(p)\times B_\varepsilon(t,x))f_\varepsilon),$$

as it is of course not possible to pass to the limit in a product of only weakly converging sequences.

Unfortunately, it is not possible to prove compactness of f_ε and Maxwell eq. being hyperbolic the compactness of E_ε and B_ε would require it for ρ_ε and j_ε. However for $\phi \in \mathcal{D}(\mathbb{R}^{2d})$

$$\int_{\mathbb{R}^{2d}} E_\varepsilon(t,x)\,f_\varepsilon(t,x,p)\,\phi(x,p)\,dx\,dp = \int_{\mathbb{R}^d} E_\varepsilon(t,x) \int_{\mathbb{R}^d} f_\varepsilon(t,x,p)\,\phi(x,p)\,dp\,dx,$$

and what is only needed is the compactness of moments of f_ε like

$$\int_{\mathbb{R}^d} f_\varepsilon(t,x,p)\,\phi(x,p)\,dp. \tag{1.22}$$

From the estimates proved in the third chapter, one gets that uniformly in ε

$$\int_{\mathbb{R}^d} f_\varepsilon(t,x,p)\,\phi(x,p)\,dp \in H^{1/4}(\mathbb{R}^d),$$

and that all moments are compact. This proves the following

Theorem 1.4 *Assume that* (1.17) *holds then there exists* $f \in L^\infty(\mathbb{R}_+,\ L^1 \cap L^\infty(\mathbb{R}^{2d}))$ *with*

$$\int_{\mathbb{R}^{2d}} E(p)\,f(t,x,p)\,dx\,dp \in L^\infty(\mathbb{R}_+),$$

and $E,\ B \in L^\infty(\mathbb{R}_+,\ L^2(\mathbb{R}^d))$ *solution in the sense of distribution of* (1.11)-(1.12).

Note finally that from the compactness of the moments like $\int_{\mathbb{R}^d} f_\varepsilon(t,x,p)\,\phi(x,p)\,dp$, it would be possible to deduce the compactness of ρ_ε and j_ε and then of E_ε and B_ε. This is not necessary to obtain the existence though.

2 The L^2 estimate

2.1 Presentation

This chapter is entirely devoted to proving the following: If f and g satisfy Eq. (1.2) namely

$$\alpha(v)\cdot\nabla_x f = g, \quad x \in \mathbb{R}^d,\ v \in \omega,$$

with $f,\ g \in L^2(\mathbb{R}^d \times \omega)$ then the moment

$$\rho(x) = \int_\omega f(x,v)\,dv, \tag{2.1}$$

belongs to the Hilbert space $H^k(\mathbb{R}^d)$ with k depending on the assumptions on $\alpha:\ \omega \longrightarrow \mathbb{R}^d$ but at best $k = 1/2$.

Following [6] and [32], (1.2) is rewritten as

$$\alpha(v)\cdot\nabla_x f + f = f + g,$$

and we get

$$\rho(x) = T\,f + T\,g,$$

with

$$T\,f(x) = \int_\omega \int_0^\infty f(x-\alpha(v)t,\ v)\, e^{-t}\, dt\, dv. \tag{2.2}$$

The aim is now to determine the exponent k such that T is continuous from $L^2(\mathbb{R}^d \times \omega)$ to $H^k(\mathbb{R}^d)$. For further use, we will work with

$$T_s\, f(x) = \int_\omega \int_0^\infty f(x-\alpha(v)t,\ v)\, t^{-s}\, e^{-t}\, dt\, dv. \tag{2.3}$$

This estimate on T is the core estimate for averaging lemmas. With the exception of the one with a full derivative in [45], most others estimates can be derived from it, usually through some kind of interpolation procedure. The L^2 regularizing effect presented here was first obtained in [24] and precised in [22], [23].

The operator T and in particular its dual T^* in the case $\alpha(v) = v$

$$T^* h(x,v) = \int_0^\infty h(x+v\,t)\, e^{-t}\, dt$$

are related to the X-ray transform $X:\ \mathbb{R}^d \longrightarrow \mathbb{R}^d \times S^{d-1}$

$$X\, h(x,v) = \int_{-\infty}^\infty h(x+vt)\, dt.$$

Note that T takes a function of two variables x and v and makes it into a function of only x (because of the average), so conversely the dual T^* takes a function h of only the x variable and makes it into a function T^*h of the two variables x and v.

This operator was studied separately in harmonic analysis (see for instance [9], [18], [54]) but with emphasis on mixed type inequalities like the continuity from $L^p(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d,\ L^p(S^{d-1}))$ and not on the gain of differentiability which is our main goal here. These other inequalities are nevertheless very usefull and can be seen as a kind of dispersion estimates for (1.2).

Note that even though this chapter deals uniquely with the stationary case, most of the proofs can easily be adapted to the unstationary case (1.1) (which can anyway be obtained as a subcase of this one) or to more general averages like (1.22).

Finally the Fourier transform in x is denoted $\mathcal{F}$ and we recall that it is an isometry on $L^2(\mathbb{R}^d)$ and that

$$H^k(\mathbb{R}^d) = \left\{\rho \in \mathcal{S}'(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} (1+|\xi|)^{2k}\, |\mathcal{F}\rho(\xi)|^2\, d\xi < \infty \right\}.$$

The homogeneous Sobolev space (used in the next chapter) is simply

$$\dot{H}^k(\mathbb{R}^d) = \left\{ \rho \in \mathcal{S}'(\mathbb{R}^d) \,\Big|\, \int_{\mathbb{R}^d} |\xi|^{2k}\, |\mathcal{F}\,\rho(\xi)|^2\, d\xi < \infty \right\}.$$

2.2 Averaging lemmas through Fourier transform

The proof here is mainly taken and adapted from [6]. Applying Fourier transform to (2.3), one gets

$$\mathcal{F}\,T_s\,f = \int_\omega \mathcal{F}\,f(\xi, v) \int_0^\infty e^{-i\,t\,\alpha(v)\cdot\xi}\, \frac{e^{-t}}{t^s}\, dt\, dv.$$

This is simply equal to

$$\int_\omega \frac{\mathcal{F}\,f(\xi, v)}{1 + i\alpha(v)\cdot\xi}\, dv,$$

if $s = 0$.

Denote

$$\chi(z) = \int_0^\infty e^{-i\,t\,z}\, \frac{e^{-t}}{t^s}\, dt.$$

Notice that of course

$$|\chi(z)| \le \int_0^\infty \frac{e^{-t}}{t^s}\, dt \le C < \infty,$$

provided that $s < 1$. This already gives that

$$|\mathcal{F}\,T_s\,f| \le \int_\omega |\mathcal{F}\,f(\xi, v)|\, dv,$$

and thanks to Cauchy-Schwarz that

$$\int_{\mathbb{R}^d} |T_s\,f(x)|^2\, dx \le |\omega| \int_{\mathbb{R}^d\times\omega} |f(x, v)|^2\, dx\, dv. \tag{2.4}$$

On the other hand, if $|z| \ge 1$, we have in addition

$$\begin{aligned} |\chi(z)| &\le \left| \int_0^K t^{-s}\, dt \right| + \left| \int_K^\infty e^{-i\,t\,z}\, \frac{e^{-t}}{t^s}\, dt \right| \\ &\le C\,K^{1-s} + \left| \frac{1}{z} \int_K^\infty e^{-t}\, |t^{-s} - s\,t^{-s-1}|\, dt \right| \\ &\le C K^{1-s} + \frac{C}{|z|}\, K^{-s} \le \frac{C}{|z|^{1-s}}, \end{aligned}$$

through minimization in K. The combination of both yields

$$|\chi(z)| \leq \frac{C}{1+|z|^{1-s}}.$$

Now by Cauchy-Schwarz, we have that

$$\begin{aligned}|\mathcal{F}\, T_s\, f|^2 &\leq \int_\omega |\mathcal{F}\, f(\xi,v)|^2\, dv \ \int_\omega |\chi(\xi\cdot\alpha(v))|^2\, dv \\ &\leq \int_\omega |\mathcal{F}\, f(\xi,v)|^2\, dv \ \int_\omega \frac{C}{1+|\alpha(v)\cdot\xi|^{2-2s}}\, dv.\end{aligned}$$

We recall that for all $\phi \in C^1(\mathbb{R})$

$$\int_\omega \phi\left(|\alpha(v)\cdot\xi|\right)\, dv = -\int_0^\infty \phi'(y)\ |\{v\in\omega\,;\ |\alpha(v)\cdot\xi| < y\}|\, dy.$$

Let us assume that

$$\forall\zeta\in S^{d-1},\ \forall\varepsilon\in\mathbb{R}_+, \qquad |\{v\in\omega\,;\ |\alpha(v)\cdot\zeta|<\varepsilon\}| \leq \varepsilon^\theta. \tag{2.5}$$

We obtain that

$$\int_\omega \frac{C}{1+|\alpha(v)\cdot\xi|^{2-2s}}\, dv \leq \int_0^\infty \frac{C}{1+|y|^{3-2s}}\ \frac{y^\theta}{|\xi|^\theta}\, dy \leq \frac{C}{|\xi|^\theta},$$

provided that $\theta - 3 + 2s < -1$. Together with (2.4) and assuming that $|\omega| < \infty$, this implies that

$$\int_{\mathbb{R}^d} (1+|\xi|)^\theta\, |\mathcal{F}\, T_s\, f|^2\, d\xi \leq C \int_{\mathbb{R}^d\times\omega} |f(x,v)|^2\, dx\, dv.$$

As a consequence we have proved the

Theorem 2.1 *Assume $|\omega| < \infty$, that* (2.5) *holds and that $\theta + 2s < 2$ then T_s is continuous from $L^2(\mathbb{R}^d\times\omega)$ to $H^{\theta/2}(\mathbb{R}^d)$.*
Consequently if $|\omega| < \infty$, (2.5) *holds, and if $f,\ g \in L^2(\mathbb{R}^d\times\omega)$ satisfy* (1.2) *then ρ defined through* (2.1) *belongs to $H^{\theta/2}(\mathbb{R}^d)$.*

Notice finally that θ is at most 1, in the case $\alpha(v) = v$ and $\omega = S^{d-1}$ for instance. If $\theta = 1$ then s can at most be equal to $1/2$ and the average ρ belongs to $H^{1/2}$.

2.3 Real space method for averaging lemmas

The use of Fourier transform is not strictly necessary for averaging lemmas; it is sometimes useful to proceed otherwise, for discretized problems like in [5] for instance. The proofs however rely on orthogonality properties of the operator T so that a direct proof is difficult. The method presented here uses instead a $T\,T^*$ argument and is taken from [32]. We restrict ourselves to the case

$$\alpha(v) = v, \quad \omega = S^{d-1}, \tag{2.6}$$

to simplify the exposition and since the general case was already dealt with in section 2.2.

The dual of operator T_s is

$$T_s^* h(x,v) = \int_0^\infty h(x+vt)\, t^{-s}\, e^{-t}\, dt. \tag{2.7}$$

It is then equivalent to prove the lemma and to show that T_s^* sends $H^{-1/2}$ in $L^2(\mathbb{R}^d \times S^{d-1})$ or $L^2(\mathbb{R}^d)$ in $L^2(S^{d-1},\ H^{1/2}(\mathbb{R}^d))$ since T_s^* commutes with the derivation in x.

Denote by Δ_x^θ the differentiation operator

$$\Delta_x^\theta h = \mathcal{F}^{-1}\left(|\xi|^{2\theta}\,\mathcal{F}\,h\right),$$

with obviously $\Delta_x^1 = -\Delta$ the laplacian.

Now compute

$$\int_{\mathbb{R}^{2d}} \Delta_x^{1/4} T_s^* h \cdot \Delta_x^{1/4} T_s^* h\, dx\, dv = \int_{\mathbb{R}^d} \Delta_x^{1/2} T_s\, T_s^* h \cdot h(x)\, dx.$$

We then observe that

$$\begin{aligned} T_s\, T_s^* h(x) &= \int_0^\infty \int_0^\infty \int_{S^{d-1}} \frac{1}{(ut)^s} h(x+(t-u)v) e^{-t-u}\, dv\, du\, dt \\ &= 2\int_0^\infty \int_0^t \int_{S^{d-1}} \frac{1}{(ut)^s} h(x+(t-u)v) e^{-t-u}\, dv\, du\, dt. \end{aligned}$$

With two changes of variables from $t-u$ to τ and from the polar coordinates τv to y

$$\begin{aligned} \bar{T}_s\, \bar{T}_s^* h(x) &= \int_0^\infty \int_0^t \int_{S^{d-1}} \frac{1}{t^s (t-\tau)^s} h(x+\tau v) e^{-2t+\tau}\, dv\, d\tau\, dt \\ &= \int_0^\infty \int_{|y|\le t} \frac{1}{t^s} h(x-y)\ \frac{e^{-2t+|y|}}{(t-|y|)^s} \cdot \frac{dy}{|y|^{d-1}}\, dt. \end{aligned}$$

Hence when differentiating $T_s\,T_s^*$, we obtain exactly the structure of a Riesz transform provided still that $s < 1/2$. Therefore the operator $T_s\,T_s^*$ is continuous from $L^2(\mathbb{R}^d)$ to $\dot H^1(\mathbb{R}^d)$ or $\Delta_x^{1/2} T_s\,T_s^*$ is continuous inside $L^2(\mathbb{R}^d)$.

We finally recover Theorem 2.1. This proof is even slighty simpler than the previous one but only in the simple case of (2.6), the general case would be somewhat more complicated.

2.4 A direct proof

We present here a direct method in L^2 for the dual operator T^* from [32]. The proof is much longer than the two previous ones, it is nevertheless interesting because it more clearly exhibits the orthogonality argument at the core of the result.

Precisely we prove the slighty suboptimal

Proposition 2.1 *The operator T_s^* with* (2.6) *is continuous from $L^2(\mathbb{R}^d)$ in $L^2_v(S^{d-1},\ H^\theta(\mathbb{R}^d)$ for $\theta < 1/2$ provided $s < 1/2$.*

A direct proof could be written for T_s by adapting the one for T_s^*, it would even be slighty longer though.

In the spirit of [18], we first prove Proposition 2.1 for characteristic functions of sets and even only for sets which are composed of small hypercubes C_i. The heart of the argument is that for an operator $\tilde T$ derived from T_s^* (it is a derivative of a regularization of T_s^*) then the scalar product

$$\int_{\mathbb{R}^d}\int_{S^{d-1}} \tilde T \mathbb{I}_{C_i}\, \tilde T \mathbb{I}_{C_j}\, dv\, dx$$

is very small provided the two cubes C_i and C_j are far apart.

Hence if $h = \mathbb{I}_E$ and E is composed of N hypercubes then the L^2 norm of $\tilde T h$ behaves only like $\sqrt{N}$ times the L^2 norm for one hypercube $\tilde T \mathbb{I}_C$. For L^1 or L^∞ though, the norm of $\tilde T h$ behaves like N times the norm for one hypercube.

This gain of one $\sqrt{N}$ is typical of such orthogonality argument (or almost orthogonality like here) and it is responsible for the gain of 1/2 derivative.

2.4.1 The case of characteristic functions: Reduction of the problem

The first point to note is that we may work in a domain S_0 in v which is included in $\{v \in S^{d-1},\ 1/4d < v_i < 1/2\ \forall i \le d\}$ instead of working in the whole sphere since the sphere may be decomposed in a finite number of domains of the same form as S_0 and the result is the same on any of them due to the invariance by rotation of the problem.

Next for any $N>0$, we say that a set E belongs to $\mathcal{C}_N$ if it is the union of closed squares (or cubes or hypercubes) of the form $[i_1/N,\ i_1/N+1/N]\times \ldots \times [i_d/N,\ i_d/N+1/N]$ where $i_1,\ldots,i_d$ are integers. Of course we choose this form for $\mathcal{C}_N$ because the "bad" directions which are along the axis of coordinates do not belong to S_0. Then we prove

Lemma 2.1 *For any $N>0$ and any $E\in\mathcal{C}_N$, we have for $\theta<1/2$ and $s<1/2$*

$$\|T_s^*\mathbb{I}_E\|^2_{L^2_v(S_0,\ H^\theta(\mathbb{R}^d))}\leq C|E|.$$

Proof. We compute directly the norm using the well known expression

$$\|T_s^*\mathbb{I}_E\|^2_{L^2_vH^\theta_x}=\int_{x,\,y\in\mathbb{R}^d}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^2|x-y|^{-d-2\theta}\,dv\,dy\,dx.$$

Let us decompose according to the distance between x and y

$$\|T_s^*\mathbb{I}_E\|^2_{L^2_vH^\theta_x}=\int_{|x-y|\geq 1}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^2|x-y|^{-d-2\theta}\,dv\,dy\,dx$$
$$+\sum_{i=1}^\infty\int_{2^{-i}\leq|x-y|<2^{-i+1}}\ldots$$

Of course the first term is dominated by the power 2 of the norm of $T_s^*\mathbb{I}_E$ in $L^2_{x,v}$ which is trivially bounded by the measure of E, as we already noticed that T_s^* is bounded from $L^2(\mathbb{R}^d)$ to $L^2(S_0\times\mathbb{R}^d)$. Since we do not want to get the precised critical case $\theta=1/2$, it is therefore enough to show that for any M and any $\theta<1/2$

$$\int_{1/M\leq|x-y|<2/M}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^2M^{d+2\theta}\,dv\,dy\,dx\leq C|E|.\quad(2.8)$$

Indeed fixing $\theta<1/2$ and choosing $\theta'\in]\theta,\ 1/2[$, one would have from (2.8) with θ' that

$$\|T_s^*\mathbb{I}_E\|^2_{L^2_vH^\theta_x}\leq C|E|+\sum_{i=1}^\infty C\,|E|\times 2^{-i(2\theta'-2\theta)}\leq C'\,|E|.$$

The next point to note, is that we may limit ourselves to the case where E has a fixed bounded diameter K independent of M or i and where we integrate over a ball of the same diameter. Indeed let us fix a ball, then

$$\int_{x\in B(x_0,K)}\int_{1/M\le|x-y|<2/M}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^2M^{d+2\theta}\,dv\,dy\,dx$$
$$\le C\int_{B(x_0,K)}\int_{|x-y|\sim 1/M}\int_v|T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(x,v)-T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(y,v)|^2M^{d+2\theta}$$
$$+Ce^{-K}\int_{B(x_0,K)}\int_{1/M\le|x-y|<2/M}\int_v(|T_s^*\mathbb{I}_E(x,v)|^2+|T_s^*\mathbb{I}_E(y,v)|^2)M^{d+2\theta},$$

because of the e^{-t} term in T_s^* of course. If we are able to prove that for $\theta'>\theta$ but with $\theta'<1/2$

$$\int_{B(x_0,K)}\int_{1/M\le|y-x|<2/M}\int_v|T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(x,v)-T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(y,v)|^2M^{d+2\theta'}$$
$$\le C_K|E\cap B(x_0,2K)|, \tag{2.9}$$

summing on the balls, we get

$$\int_{x\in\mathbb{R}^d}\int_{1/M\le|x-y|<2/M}\int_{v\in S_0}|T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^2M^{d+2\theta}\,dv\,dy\,dx$$
$$\le C_KM^{\theta-\theta'}|E|$$
$$+Ce^{-K}\int_{\mathbb{R}^d}\int_{1/M\le|x-y|<2/M}\int_v(|T_s^*\mathbb{I}_E(x,v)|^2+|T_s^*\mathbb{I}_E(y,v)|^2)M^{d+2\theta}$$
$$\le C_KM^{\theta-\theta'}|E|+Ce^{-K}M^{d+2\theta}|E|.$$

A simple scaling argument shows that, in (2.9), C_K is dominated by a power of K (depending on p). So choosing eventually K in terms of M we may deduce (2.8) from (2.9). Hence from now on, E will have a given finite diameter and the integrals in x or y will be taken inside a ball.

Before finaly turning to proving (2.9), we remark that we may choose $M=N$ (not a great surprise). If $E\in\mathcal{C}_N$ then E belongs to every $\mathcal{C}_{2^iN}$ simply by dividing each hypercube in 2^{di} smaller identical hypercubes: So we may always take $N\ge M$. And if (2.9) is true for $M=N$, it is true for all $M \leq N$ since for instance

$$\int_{2/N\leq|x-y|<4/N}\int_v |T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^2\left(\frac{N}{2}\right)^{d+2\theta} dv\,dy\,dx$$

$$\leq 2\int_{2/N\leq|x-y|<4/N}\int_v |T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}E(x+(y-x)/2,v)|^2\left(\frac{N}{2}\right)^{d+2\theta}$$

$$+\,2\int_{2/N\leq|x-y|<4/N}\int_v |T_s^*\mathbb{I}_E(x+(y-x)/2,v)-T_s^*\mathbb{I}E(y,v)|^2\left(\frac{N}{2}\right)^{d+2\theta}$$

$$\leq \frac{4}{2^{d+2\theta}}N^{2\theta-2\theta'}\int_{1/N\leq|x-y|<2/N}\int_v |T_s^*\mathbb{I}_E(x,v)-T_s^*\mathbb{I}_E(y,v)|^2 N^{d+2\theta'}\,dv\,dy\,dx.$$

Then $4N^{2\theta-2\theta'}$ is less than 1 (unless N is of order one but the proof is trivial then) if $\theta' \geq \theta + C/\ln N$. So (2.9) for $M = N$ implies (2.9) for $M = N/2$ and by repeating the same argument $\ln N/\ln\ln N$ times, for $\ln N \leq M \leq N$ with a final number of derivatives equal to $\theta_f = \theta_0 - C/\ln\ln N$, which is all right. Now of course if $M \leq \ln N$ then the argument is obvious because we may lose at most a $\ln N$ factor which does not matter.

The last reduction of the problem we make is to regularize T_s^*. Indeed by the same kind of argument, we may take T_s^* of the form

$$T_s^*\mathbb{I}_E = \int_0^\infty \mathbb{I}_E(x+vt)\frac{e^{-t}}{(1/N+t)^s}\,dt,$$

and denoting C_i, $1 \leq i \leq n$, the hypercubes which compose E and x_i their center, we approximate $T_s^*\mathbb{I}_E$ by

$$T_N(x,v) = \sum_{i=1}^n l_i(x,v)\phi_i(x),$$

$$l_i(x,v) = \int_0^\infty \mathbb{I}_{C_i}(x+vt)\,dt, \quad \phi_i(x) = \frac{e^{-|x-x_i|}}{(1/N+|x-x_i|)^s}.$$

We may do so because

$$|T_N(x,v) - T_s^*\mathbb{I}_E(x,v)| \leq C\int_0^\infty \mathbb{I}_E(x+vt)N^{s-1}\frac{e^{-t}}{1/N+t}\,dt.$$

Therefore since $s+\theta < 1$, we have

$$\int_{2/N\leq|x-y|<4/N}\int_v |(T_s^*\mathbb{I}_E-T_N)(x,v)|^2 N^{d+2\theta}\,dv\,dy\,dx \leq C\|T_1^*\mathbb{I}_E\|^2_{L^2_{x,v}} \leq C|E|,$$

and in proving (2.9), we may replace $T_s^*\mathbb{I}_E$ by T_N.

Instead of (2.9), we prove

$$\begin{aligned}\sup_{|\xi|\le 1}\int_{B(0,K)}\int_{v\in S_0}&|\nabla_x T_N(x+\xi,v)|^2\,dv\,dx\\ &\le \int_{B(0,2K)}\int_{v\in S_0}|\nabla_x T_N(x,v)|^2\,dv\,dx \le N^{2-2\theta}\,|E|.\end{aligned}\tag{2.10}$$

Estimate (2.10) implies (2.9). Indeed, writing

$$\begin{aligned}|T_N(x,v)-T_N(y,v)| &= \left|\int_0^1 (y-x)\nabla_x T_N(x+s(y-x),v)\,ds\right|\\ &\le |x-y|\times\int_0^1 |\nabla_x T_N(x+s(y-x),v)|\,ds,\end{aligned}$$

and inserting this in the left hand side of (2.9), we find after a simple Hölder estimate in s

$$\begin{aligned}&\int_{B(0,K)}\int_{1/N\le|y-x|<2/N}\int_v |T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(x,v)-T_s^*\mathbb{I}_{E\cap B(x_0,2K)}(y,v)|^2N^{d+2\theta}\\ &\le \int_0^1\int_{B(x_0,K)}\int_{1/N\le|\xi|<2/N}\int_v |\nabla_x T_N(x+s\xi,v)|^2N^{d+2\theta-2}\,dv\,dy\,dx\,ds\\ &\le \int_0^1\int_{|\xi|\le 2/N}\int_{B(x_0,K)}\int_v |\nabla_x T_N(x+s\xi,v)|^2N^{d+2\theta-2}\,dv\,dx\,dy\,ds \le C|E|,\end{aligned}$$

if (2.10) holds. To prove (2.10), we compute the derivative of T_N which may be decomposed into

$$\begin{aligned}|\nabla_x T_N(x,v)| = \left|\sum_{i=1}^n \nabla_x l_i(x,v)\phi_i(x) + l_i(x,v)\nabla_x\phi_i(x)\right| &\le \left|\sum_i \nabla_x l_i(x,v)\phi_i(x)\right|\\ &+ CN^s\sum_i l_i(x)\frac{e^{-|x-x_i|}}{1/N+|x-x_i|}.\end{aligned}$$

The last term is not a problem, it leads to the same computation as for the approximation of $T_s^*\mathbb{I}_E$ by T_N (as $s+\theta<1$) and so we do not repeat it here. We focus on the first term instead.

It is easy to compute $\nabla_x l_i$. It has a non zero component only in the space orthogonal to v. We denote by $L(x,v)$ the line passing through x and of direction v and by $n_i^+(x,v)$ the outward normal of the side of the hypercube C_i through which $L(x,v)$ enters C_i and n_i^- the outward normal of the side of the hypercube through which $L(x,v)$ leaves. Then

$$e\cdot\nabla_x l_i(x,v)=\frac{e\cdot n_i^+}{v\cdot n_i^+}-\frac{e\cdot n_i^-}{v\cdot n_i^-}. \tag{2.11}$$

Consequently this derivative is zero if the two sides are parallel and since $v\in S_0$,

$$\left|\sum_{i=1}^n \nabla_x l_i(x,v)\,\phi_i(x)\right|\leq CKN. \tag{2.12}$$

This estimate would not however provide any gain in derivative.

Since $v\cdot\nabla_x l_i=0$, it is enough to do the proof for the first $d-1$ components $\partial_k l_i$ of $\nabla_x l_i$. We choose $k=1$: the computation for any other $k\leq d-1$ is the same because of the symmetry in S_0.

2.4.2 The orthogonality argument

Define $\mathcal{N}_i$ as the set of j such that C_j intersects one of the half lines centered inside C_i and of direction inside S_0 (because of the definition of S_0, for any x, on a line connecting x, C_i and C_j, C_i is between x and C_j).

Note that, with B_i the set of x such that $L(x,v)$ enters C_i on a given chosen side C_i^k, $k=1\dots 2^d$,

$$\int_{B(0,2K)}\int_{S_0}\left|\sum_{i=1}^n \partial_{x_1}l_i\phi_i(x)\right|^2 dv\,dx=2^d\sum_{i=1}^n\sum_{j\in\mathcal{N}_i}\int_{S_0}\int_{B_i}\partial_{x_1}l_i\,\phi_i\,\partial_{x_1}l_j\,\phi_j\,dx\,dv.$$

Then we perform a change of variable from (x) to (η,t) where $t=|x-x_i|$ and $\eta+x_i$ is the point where $L(x,v)$ crosses the chosen side of C_i (thus $|\eta|\leq 1/N$) to get

$$\int_{B(0,2K)}\int_{S_0}\left|\sum_{i=1}^n \partial_{x_1}l_i\phi_i(x)\right|^2 dv\,dx$$
$$\leq C\sum_{i,j=1}^n\int_{S_0}\int_{t\leq 2K}\int_{\eta\in C_i^k-x_i}(\partial_{x_1}l_i\,\phi_i\,\partial_{x_1}l_j\,\phi_j)(x_i+vt+\eta,v)\,\psi(\eta,t,v)d\eta\,dt\,dv.$$

Define for η a vector with $|\eta| \le 1/N$

$$\Delta_i^\eta(t) = \sum_{j\in\mathcal{N}_i} \int_{S_0} \partial_{x_1} l_i(\eta{+}x_i{+}vt, v)\phi_i(\eta{+}x_i{+}vt)$$
$$\times \partial_{x_1} l_j(\eta{+}x_i{+}vt, v)\phi_j(\eta{+}x_i{+}vt)\,\psi(v)\,dv.$$

The estimate that we are looking for is a consequence of

$$|\Delta_i^\eta(x)| \le Ct^{-2s} \times \log N. \tag{2.13}$$

Indeed since ψ is a perfectly regular function, we may switch the order of integration and apply (2.13) to find

$$\int_{B(0,2K)} \int_{S_0} \left| \sum_{i=1}^n \nabla_x l_i \phi_i(x) \right|^2 dv\,dx \le C\,\log N \sum_{i=1}^n \int_{t\le 2K} \int_{\eta\in C_i^k - x_i} t^{-2s}\,d\eta\,dt$$
$$\le C\,\log N \sum_{i=1}^n N^{1-d} \le CN\,\log N\,|E|,$$

which would finish to prove (2.10) and the lemma. The bound (2.13) is thus the almost orthogonality property that we want.

Fix $j \in \mathcal{N}_i$, a real t and a side of C_i, we denote by S_i the subspace of S_0 so that $L(x_0, v)$ enters C_i on the chosen side and therefore $\partial_{x_1} l_i$ is a constant. Then since $\partial_{x_1} l_j$ is non zero as a function of v, on a space of measure $C(|x_i - x_j|\,N)^{1-d}$,

$$\left| \int_{S_i} \partial_{x_1} l_j(\eta + x_i - vt, v)\,\psi(v)\,dv \right| \le CN^{-d+1} \times |x_i - x_j|^{-d+1}.$$

But using the cancellations and provided ψ is a regular function, we can prove the better inequality

$$\left| \int_{S_1} \partial_{x_1} l_j(\eta + x_i - vt, v)\,\psi(v)\,dv \right| \le CN^{-d} \times |x_i - x_j|^{-d}. \tag{2.14}$$

This additional cancellation is behind (2.13).

Denote by C_j^1 and C_j^2 the sides of C_j whose normal vectors n_j^1 and n_j^2 are parallel to e_1 and $\alpha_j^k(x, v)$ the function with value 1 if $L(x, v)$ intersects C_j^k and 0 otherwise. Note that since $v \in S_0$, there cannot exist $v, v' \in S_0$

such that $L(x,v)$ enters the hypercube on the side C_j^1 but $L(x,v')$ leaves the hypercube on C_j^2 or the converse. Therefore

$$\left|\int_{S_i} \partial_{x_1} l_j(\eta+x_i-vt,v)\,\psi\,dv\right| \leq \left|\int_{S_i} (\alpha_j^1(\eta+x_i-vt,v)-\alpha_j^2(\eta+x_i-vt,v))\,\frac{\psi}{v_1}dv\right|.$$

We know that $\alpha_j^2(x,v)=\alpha_j^1(x,R_{ij}v)$ with R_{ij} such that $|R_{ij}v-v|\leq C/N\,|x_i-x_j|$. Since the functions ψ and $1/v_1$ are C^∞ over S_0, we immediately get (2.14) from the fact that α_j^k is the indicatrix of a subset of S_i of diameter at most $C/(N\,|x_i-x_j|)$.

Now note that in $\Delta_i(t)$, in fact $\phi_i(\eta+x_i-vt)$ and $\phi_j(\eta+x_i-tv)$ are almost constant since $|\eta+x_i-vt|$ is equal to $t\pm 1/N$ and $|\eta+x_j-x_i+tv|$ to $|x_j-x_i|+t\pm 1/N$ (the points x_i-tv, x_i and x_j are almost on the same line if ∇l_j is not zero). So up to an approximation of the kind we already performed, we may take it constant and we then have thanks to (2.14)

$$\begin{aligned}|\Delta_i^\eta(t)| &\leq CN^{-d}t^{-s}\sum_{j\in\mathcal{N}_i}(|x_i-x_j|+t)^{-s}\,|x_i-x_j|^{-d}\\ &\leq CN^{-d}t^{-s}\sum_{k=1}^{N}(k/N+t)^{-s}\,(k/N)^{-d}\times k^{d-1},\end{aligned}$$

summing first on all $j\in\mathcal{N}_i'$ which are at the same distance of x_i. Eventually we find (2.13).

2.4.3 The general case and the proof of Prop. 2.1

The proof uses Lemma 2.1 and a standard approximation procedure.

Let us consider any nonnegative function f with compact support and which is constant on any hypercubes of the form $[i_1/N,\ i_1/N+1/N]\times\ldots\times[i_d/N,\ i_d/N+1/N]$ for a given integer N. Therefore f takes only a finite number of positive values $0<\alpha_1<\ldots<\alpha_n$. Denoting by E_i the set of points x where f is equal to α_i, we know that $E_i\in\mathcal{C}_N$ from the assumption on f. Hence for any $\theta<1/2$ by Lemma 2.1

$$\|T_s^*f\|_{L_v^2H_x^\theta}\leq\sum_{i=1}^n\alpha_i\|T_s^*\mathbb{I}_{E_i}\|_{L_v^2H_x^\theta}\leq C\sum_{i=1}^n\alpha_i|E|^{1/2}.$$

Denote by $f^*(t)$ the decreasing rearrangement corresponding to f (see [2]). Then $f^*(t)$ has value α_i on the interval $[\beta_{i+1},\ \beta_i]$ with $\beta_i = \sum_{j=i}^n |E_j|$. Consequently the Lorentz norm of f satisfies

$$\|f\|_{L^{2,1}} = \int_0^\infty t^{1/2} f^*(t) \frac{dt}{t} = \sum_{i=1}^n \alpha_i (\beta_i^{1/2} - \beta_{i+1}^{1/2}) \geq C \sum_{i=1}^n \alpha_i |E_i|^{1/2}.$$

So eventually we showed that for any $\theta < 1/2$

$$\|T_s^* f\|_{L^2_v H^\theta_x} \leq C \|f\|_{L^{2,1}}.$$

Since $L^{2,1}$ is embedded in H^{-k} for any $k > 0$ and since we do not care about the critical case, this implies that for any $\theta < 1/2$ and any function f as described at the beginning

$$\|T_s^* f\|_{L^2_v W^{\theta,2}_x} \leq C \|f\|_{L^2}.$$

Now it is enough to note that functions with compact support and whose level sets belong to $\mathcal{C}_N$ for a given N, are dense in L^2 which concludes the proof of Prop. 2.1.

3 The L^p estimates

3.1 Presentation

Consider a solution to

$$\alpha(v) \cdot \nabla_x f = \Delta_x^{a/2} g, \quad x \in \mathbb{R}^d,\ v \in M, \quad a < 1, \tag{3.1}$$

and with the average for some $\Phi \in C_c^\infty(M)$ and M a regular hypersurface of $\mathbb{R}^d$

$$\rho_\Phi(x) = \int_M f(x,v)\, \Phi(v)\, dv. \tag{3.2}$$

Let us assume that

$$\exists C,\ \forall \xi \in S^{d-1},\ \forall \varepsilon \qquad |\{v \in M \text{ s.t. } |\alpha(v) \cdot \xi| \leq \varepsilon\}| \leq C\, \varepsilon^k. \tag{3.3}$$

Note that in the usual unstationary case $\alpha(v) = (1, a(v))$, $M = \mathbb{R}^{d-1}$ and the previous condition simply becomes

$$\exists C,\ \forall \xi \in R^{d-1},\ \forall \tau\ \forall \varepsilon \qquad |\{v \in \mathbb{R}^{d-1} \text{ s.t. } |a(v) \cdot \xi - \tau| \leq \varepsilon\}| \leq C\, \varepsilon^k. \tag{3.4}$$

Then the following holds

Theorem 3.1 *Let f and g satisfy* (3.1) *and*

$$\begin{aligned} f \;&\in \dot W_v^{\beta,\,p_1}(M,\ L_x^{p_2}(\mathbb{R}^d)), \qquad && \beta \geq 0,\\ g \;&\in \dot W_v^{\gamma,\,q_1}(M,\ L^{q_2}(\mathbb{R}^d)), \qquad && -\infty < \gamma < 1 - k/2, \end{aligned} \tag{3.5}$$

with $1 < p_2, q_2 < \infty$, $1 \leq p_1 \leq \min(p_2,\ p_2^*)$ *and* $1 \leq q_1 \leq \min(q_2,\ q_2^*)$ *where for a general* p, p^* *is the dual exponent of* p*, and assume moreover that* $\gamma - 1/q_1 < 0$. *Then,*

$$\|\rho\|_{\dot B^{s,r}_{\infty,\infty}} \leq C\|f\|^{1-\theta}_{W_v^{\beta,p_1}(L_x^{p_2})} \times \|g\|^{\theta}_{W_v^{\gamma,q_1}(L_x^{q_2})},$$

with

$$\begin{aligned} &\frac{1}{r} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}, \quad s = (1-a)\,k\,\theta,\\ &\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}. \end{aligned} \tag{3.6}$$

For simplicity in this chapter we consider only the simplified setting: The equation reads

$$v\cdot\nabla_x f = \Delta_x^a\, g, \quad x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \quad a<1, \tag{3.7}$$

and the average is

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v)\,\phi(v)\,dv. \tag{3.8}$$

The aim is to prove and investigate the optimality of

Theorem 3.2 *Let f and g satisfy* (3.7) *and*

$$\begin{aligned} f \;&\in \dot W_v^{\beta,\,p_1}(\mathbb{R}^d,\ L_x^{p_2}(\mathbb{R}^d)), \qquad && \beta \geq 0,\\ g \;&\in \dot W_v^{\gamma,\,q_1}(\mathbb{R}^d,\ L_x^{q_2}(\mathbb{R}^d)), \qquad && -\infty < \gamma < 1, \end{aligned} \tag{3.9}$$

with $1 < p_2, q_2 < \infty$, $1 \leq p_1 \leq \min(p_2,\ p_2^*)$ *and* $1 \leq q_1 \leq \min(q_2,\ q_2^*)$ *where for a general* p, p^* *is the dual exponent of* p*, and assume moreover that* $\gamma - 1/q_1 < 0$. *Then,*

$$\|\rho\|_{\dot B^{s,r}_{\infty,\infty}} \leq C\|f\|^{1-\theta}_{W_v^{\beta,p_1}(L_x^{p_2})} \times \|g\|^{\theta}_{W_v^{\gamma,q_1}(L_x^{q_2})},$$

with

$$\begin{aligned} &\frac{1}{r} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}, \quad s = (1-a)\theta, \\ &\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}. \end{aligned} \tag{3.10}$$

This result essentially uses the L^2 regularizing effect given by Th. 2.1 and a lot of interpolation. The definition of the spaces $\dot{W}^{s,p}$ and $\dot{B}^{s,p}_{\infty,\infty}$ are recalled later on.

Notice that as predicted by the simple example in the first chapter the regularity of the average ρ depends only on the regularity in velocity of f and g. The more general case of (1.2), (2.1) with the condition (2.5) would just give the same result provided $\beta, \gamma \leq 0$ (the regularity would in fact depend on the exponent in (2.5) with the one given in Th. 3.2 if it is 1). However dealing with $\beta > 0$ or $\gamma > 0$ would likely require a more stringent assumption; At least it is not known how to do it with only (2.5).

For a large part (the case $p_1 = p_2$, $q_1 = q_2$, $\beta = 0$ and $\gamma \leq 0$), Theorem 3.2 was proved in [17] using dyadic decomposition in the Fourier space to interpolate and obtaining the average ρ in the Besov space $B^{s,r}_\infty$. This was improved in [3] using product Hardy spaces for the interpolation with an average in the Sobolev space $W^{s,r}$. We also refer to [6], [13], [46].

The case of positive derivatives in v (but still with $p_1 = p_2$, $q_1 = q_2$) was obtained in [31] with a simpler but less effective interpolation method that we use here also.

3.2 Sobolev, Besov spaces and real interpolation

This section only aims at recalling or introducing the basic tools that we will need. No proof is included and the reader should refer to [2] for instance for more details and information.

Definition 3.1 *Let E and F be two Banach spaces. An interpolated space at order θ between E and F is a space G included in $E+F$ such that for all operators T continuous in E and in F then T is continuous in G and*

$$\|T\|_G \leq \|T\|_E^{1-\theta} \, \|T\|_F^\theta.$$

Note that there is no reason why the interpolate should be unique (and in most cases it is not). The definition in fact works also if T is an operator between two Banach spaces

Proposition 3.1 *Let T be a continuous operator from E_1 to E_2 and from F_1 to F_2. Let G_i be an interpolated space at order θ between E_i and F_i. Then T is continuous from G_1 to G_2 and*

$$\|T\|_{G_1\to G_2} \leq \|T\|_{E_1\to E_2}^{1-\theta}\, \|T\|_{F_1\to F_2}^{\theta}.$$

It is for example well known that an interpolate at order θ between the spaces $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ is the space $L^r(\mathbb{R}^d)$ with

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

Let us recall the definition of Sobolev spaces

$$W^{1,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) \mid \nabla f \in L^p(\mathbb{R}^d)\},$$
$$W^{-1,p}(\mathbb{R}^d) = \{f = g + \nabla \cdot h \mid g \in L^p(\mathbb{R}^d),\ h \in (L^p(\mathbb{R}^d))^d\},$$

and homogeneous Sobolev spaces

$$\dot W^{1,p}(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d) \mid \nabla f \in L^p(\mathbb{R}^d)\},$$
$$\dot W^{-1,p}(\mathbb{R}^d) = \{f = \nabla \cdot h \mid h \in (L^p(\mathbb{R}^d))^d\},$$

with the obvious extensions for $W^{k,p}$ where $k \in \mathbb{Z}$. Then the spaces $W^{s,p}(\mathbb{R}^d)$ with $s \in \mathbb{R}$ can be obtained by interpolation: For instance if $s \in [0,\ 1]$ then $\dot W^{s,p}(\mathbb{R}^d)$ is an interpolate at order s between $L^p(\mathbb{R}^d)$ and $\dot W^{1,p}(\mathbb{R}^d)$. If $1 < p < \infty$ then an equivalent definition is that $f \in \dot W^{s,p}(\mathbb{R}^d)$ iff $\Delta^{s/2} f \in L^p(\mathbb{R}^d)$.

Different approaches exist to interpolation theory namely the complex methods and the real methods which are a bit more complicated but more constructive. We describe here one real method: The so-called K-theory from [35].

For E and F two Banach spaces and $\rho \in E + F$ we define

$$K_\rho(t) = \inf_{\rho=\rho_1+\rho_2} (\|\rho_1\|_E + t\|\rho_2\|_F). \tag{3.11}$$

We define $(E,\ F)_{\theta,k}$ as the space of functions ρ such that

$$\left(\int_0^\infty (K_\rho(t)\, t^{-\theta})^k\, \frac{dt}{t}\right)^{1/k} < \infty,$$

and in the particular case $k=\infty$

$$\sup_t K_\rho(t)\, t^{-\theta} < \infty.$$

All spaces $(E,\ F)_{\theta,k}$ for any $\theta\in]0,\ 1[$, $k\in[1,\ \infty]$ are interpolated spaces at order θ. The method generates all Besov spaces (and Lorentz spaces for the interpolation between L^p and L^q). We will use it only for $k=\infty$ and describe the main interpolated spaces.

The space $(W^{s_1,p}(\mathbb{R}^d),\ W^{s_2,p}(\mathbb{R}^d))_{\theta,\infty}$ is the Besov space $B^{s,p}_\infty(\mathbb{R}^d)$ with

$$s=(1-\theta)\, s_1+\theta\, s_2.$$

This space is very close from the Sobolev space and in particular

$$W^{s,p}(\mathbb{R}^d)\subset B^{s,p}_\infty(\mathbb{R}^d)\subset W^{s',p}(\mathbb{R}^d)\quad \forall s'<s.$$

For the homogeneous spaces $(\dot W^{s_1,p}(\mathbb{R}^d),\ \dot W^{s_2,p}(\mathbb{R}^d))_{\theta,\infty}$, we obtain the homogeneous Besov space $\dot B^{s,p}_\infty(\mathbb{R}^d)$ with on a compact support Ω

$$\dot W^{s,p}(\Omega)\subset \dot B^{s,p}_\infty(\Omega)\subset \dot W^{s',p}(\Omega)\quad \forall s'<s.$$

Unfortunately the space $(W^{s_1,p}(\mathbb{R}^d),\ W^{s_2,q}(\mathbb{R}^d))_{\theta,\infty}$ is not a Besov space if $p\neq q$, we denote it $B^{s,r}_{\infty,\infty}$ but it also satisfies

$$W^{s,p}(\mathbb{R}^d)\subset B^{s,p}_{\infty,\infty}(\mathbb{R}^d)\subset W^{s',p}(\mathbb{R}^d)\quad \forall s'<s.$$

3.3 Proof of the Theorem

We regularize the operator $v\cdot\nabla_x$ by adding λf (λ is a parameter of interpolation which will be chosen later in terms of f and g)

$$(\lambda+v\cdot\nabla_x)\, f(x,v)=\Delta_x^{\alpha/2} g(x,v)+\lambda f(x,v).$$

We denote by T_λ the operator

$$T_\lambda f(x)=\int_0^\infty\int_{\mathbb{R}^d} f(x-vt,\ v)\, e^{-\lambda t}\,\phi(v)\, dv\, dt. \tag{3.12}$$

Consequently

$$\rho(x)=\int_{\mathbb{R}^d} f(x,v)\,\phi(v)\, dv=\lambda T_\lambda f+\Delta_x^{a/2} T_\lambda g. \tag{3.13}$$

We study this operator T_λ in the next subsection and conclude the proof of Theorem 3.2 in the last one.

3.3.1 Estimates for T_λ

We prove

Proposition 3.2 *For any* $1 \le p_1 \le \min(p_2,\, p_2^*)$ *with* $1 < p_2 < \infty$, *for any* s *with* $s \le 1/p_1$, *we have*

$$T_\lambda :\ \dot W^{s,p_1}_{loc,v}(R^d,\ L^{p_2}_x(\mathbb{R}^d)) \longrightarrow \dot W^{1+s-1/p_1,p_2}(\mathbb{R}^d),\ \textit{with norm } C\lambda^{s-1/p_1}.$$

Notice first that with a simple change of variable

$$T_\lambda f(x) = \frac{1}{\lambda}\int_0^\infty \int_{\mathbb{R}^d} f(x - vt/\lambda,\, v)\, e^{-t}\, \phi(v)\, dv\, dt = \frac{1}{\lambda}\, T f_\lambda(\lambda x),$$

with $f_\lambda(x) = f(x/\lambda, v)$. Therefore it is enough to show Prop. 3.2 for $\lambda = 1$, *i.e.* for the operator T.

We begin with the simple case where we only have L^1 regularity in velocity. In this case T can at best exchange derivability in v for derivability in x, more precisely we have

Lemma 3.1 $\forall\ 0 \le s < 1,\ T :\ \dot W^{s,1}_{v,loc}(\mathbb{R}^d,\ L^p_x(\mathbb{R}^d)) \longrightarrow \dot W^{s,p}(\mathbb{R}^d)$, *for every* $1 \le p \le \infty$.

Proof. It is a direct computation, once one has noticed that

$$\partial_{x_i} f(x - vt, v) = -\frac{1}{t}\partial_{v_i}(f(x - vt,\, v))) + \frac{1}{t}(\partial_{v_i} f)(x - vt,\, v).$$

First of all, simply by commuting the integrals, it is obvious that

$$\left\| \int_{\mathbb{R}^d} f(x - vt,\, v)\, \phi(v)\, dv \right\|_{L^p} \le C \|f\|_{L^1_v L^p_x},$$

where C does not depend on t. Then we also obtain from our remark that

$$\left\| \partial_{x_i} \int_{\mathbb{R}^d} f(x - vt,\, v)\, \phi(v)\, dv \right\|_{L^p} \le \frac{C}{t} \|f\|_{W^{1,1}_v L^p_x}.$$

By interpolation, we conclude that for any $s < 1$

$$\left\| \int_{\mathbb{R}^d} f(x - vt,\, v)\, \phi(v)\, dv \right\|_{\dot W^{s,p}} \le \frac{C}{t^s} \|f\|_{W^{s,1}_v L^p_x},$$

and by integrating in t against e^{-t} we get the desired result. $\square$

With exactly the same idea, one obtains for negative derivatives,

Lemma 3.2 $\forall\, s \le 0,\ T:\ \dot W_v^{s,1}(\mathbb{R}^d,\ L_x^p(\mathbb{R}^d)) \longrightarrow \dot W^{s,p}(\mathbb{R}^d)$.

It remains to combine this with the L^2 case provided by Theorem 2.1. In fact one has for any $s \in \mathbb{R}$

$$\Delta_x^{s/2} h(x+vt) = \Delta_v^{s/2} h(x+vt)\, t^{-s},$$

which implies for the dual operator T^* and if $s<1$

$$\Delta_x^{s/2} T^* h = \Delta_v^{s/2} \int_0^\infty h(x+vt) \frac{e^{-t}}{t^s}\, dt = \Delta_v^{s/2} T_s^* h,$$

according to the definition of T_s (2.3). As we precisely proved Th. 2.1 for T_s and therefore T_s^* one obtains

Lemma 3.3 $\forall s < 1/2,\ T:\ \dot H_v^s(L_x^2) \longrightarrow \dot H^{s+1/2}$.

To obtain the behaviour of T on any space of the form $\dot W_v^{s,p_1}(L_x^{p_2})$, we only have to interpolate between Lemma 3.1 and Lemma 3.3. A slight problem arises because the operator $\Delta_x^{s/2}$ does not operate nicely on L^1.

For any $1 < p_2 < 2$, we first point out that the proof of Lemma 3.1 also shows that T sends $\dot W_v^{s,1}(\mathcal{H}_x^1)$ in $\Delta_x^{-s/2}\mathcal{H}^1$ with $\mathcal{H}^1$ the Hardy space; This would also be true with any Banach space whose norm is invariant by translation (*i.e.* the norm of $f(x+h)$ is equal to the norm of f).

Then we can interpolate without any problem between $\dot W_v^{s,1}(\mathcal{H}_x^1)$ and $\dot H_v^s L_x^2$ to obtain $\dot W_v^{s,p_2} L_x^{p_2}$ whose image by T is in the interpolation of $\Delta_x^{-s/2}\mathcal{H}^1$ and $\dot H^{s+1/2}$, that is $\dot W^{1-1/p_2,p_2}$. Finally we interpolate between $\dot W_v^{s,1}(L_x^{p_2})$ and $\dot W_v^{s,p_2} L_x^{p_2}$, which is the space $\dot W_v^{s,p_1} L_x^{p_2}$ with its image in the interpolate between $\dot W^{s,p_2}$ and $\dot W^{1-1/p_2,p_2}$. That precisely gives Prop. 3.2.

3.3.2 Conclusion of the proof of Theorem 3.2

We are ready to prove Theorem 3.2. We first do it with the additional assumption that $\beta < 1/p_1$. Indeed with that we may apply Proposition 3.2 to both f and g.

We have

$$\rho = \rho^1 + \rho^2 = \lambda T_\lambda f + \Delta_x^{a/2}\, T_\lambda g,$$

with by Proposition 3.2

$$\|\rho^1\|_{\dot W^{1+\beta-1/p_1,\,p_2}} \le C\,\lambda \times \lambda^{\beta-1/p_1} \times \|f\|_{\dot W_v^{\beta,p_1} L_x^{p_2}},$$
$$\|\rho^2\|_{\dot W^{1+\gamma-1/q_1-a,\,q_2}} \le C\lambda^{\gamma-1/q_1} \times \|g\|_{\dot W_v^{\gamma,q_1} L_x^{q_2}}.$$

We then minimize in λ according to the K-method of real interpolation which was earlier described. We take

$$\lambda = t^{1/(1+\beta-1/p_1-\gamma+1/q_1)},$$

and we indeed find

$$K(t) \leq t^\theta \times \|f\|^{1-\theta}_{\dot W^{\beta,p_1}_v L^{p_2}_x} \times \|g\|^{\theta}_{\dot W^{\gamma,q_1}_v L^{q_2}_x},$$

with

$$\theta = \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1},$$

as given by Theorem 3.2. Consequently ρ belongs to the space $\dot B^{s,r}_{\infty,\infty}$ as the interpolation of order (θ,∞) of the two spaces $\dot W^{1+\beta-1/p_1,p_2}$ and $\dot W^{1+\gamma-1/q_1-a,q_2}$.

It only remains to indicate how we prove Theorem 3.2 for $\beta \geq 1/p_1$. Clearly if Proposition 3.2 were true for these values, we would be done since there would not be any difficulty with the previous argument of real interpolation.

If one tries to prove any of the lemmas in the previous subsection for $\beta \geq 1/p_1$, the problem is that we do not have enough integrability in t. More precisely, we would have to integrate a term in t^{-k} with $k \geq 1$ which is not possible. However

$$\begin{aligned}
T_\lambda f &= \int_0^\infty \int_{\mathbb{R}^d} \partial_t(t) f(x-vt,\, v) e^{-\lambda t}\, \phi(v)\, dv\, dt \\
&= \int_0^\infty \int_{\mathbb{R}^d} f(x-vt,\, v)\, \lambda t e^{-\lambda t}\, \phi(v) \\
&\qquad + \int_0^\infty \int_{\mathbb{R}^d} v\cdot\nabla_x f(x-vt,\ v) t e^{-\lambda t}\, \phi(v) \\
&= \int_0^\infty \int_{\mathbb{R}^d} f(x-vt,\, v)\, \lambda t e^{-\lambda t}\, \phi(v) \\
&\qquad + \frac{1}{\lambda} \int_0^\infty \int_{\mathbb{R}^d} \Delta_x^{a/2} g(x-vt,\, v) \lambda t e^{-\lambda t}\, \phi(v).
\end{aligned}$$

The first term has the same homogeneity as $T_\lambda f$ but with more integrability around the origin in t. The second term, once it is multiplied by λ behaves exactly like the usual $T_\lambda g$.

Therefore, repeating this simple trick as many times as necessary, we avoid any problem of integrability in t for $T_\lambda f$ and we may consider β as large as we want.

Notice finally that this would not work for $T_\lambda g$ because we have used that $v \cdot \nabla_x f = \Delta_x^{a/2} g$ and we do not have anything like that for g. This is only natural as one cannot expect to gain more than one derivative from averaging lemmas.

3.4 Optimality

This is the exact analogue in a slighty more general situation of the two notes [36] and [37], which show that the usual averaging lemmas (with $p_1 = p_2$, $q_1 = q_2$ and $\beta = 0$) are optimal.

They are given in dimension two for simplicity. We do it in two steps. For the first one consider two C_c^∞ functions a and b and take

$$\begin{aligned} f_N(x, v) &= N^{\delta(1/p_1-\beta)} \times a(N\,x_1, x_2/N)\; b(N^\delta v_1), \\ g_N(x, v) &= N^{1-\delta+\delta/p_1-\delta\beta} \times \partial_1 a(N\,x_1, x_2/N)\; N^\delta v_1\, b(N^\delta v_1). \end{aligned} \tag{3.14}$$

We then simply choose δ such that g_N belongs to the space $W_v^{\gamma,q_1}(L_x^{q_2})$ uniformly in N for every q_2, so

$$\delta = \frac{1}{1 - 1/p_1 + \beta + 1/q_1 - \gamma}.$$

Notice that if $\gamma < 0$, we also have to require that $wb(w)$ be the γ derivative of some function. Moreover, we have

$$v \cdot \nabla_x f_N = g_N + h_N,$$

with for any r

$$\|h_N\|_{L_v^1(W_x^{1,r})} \leq CN^{-2\delta}.$$

Therefore the contribution from h_N to the regularity of the average is one full derivative and (from the point of view of counterexample) we may neglect this term.

To finish with this counterexample, it is enough to notice that for any $1 \leq r \leq \infty$

$$\|\rho_N\|_{\dot{W}^{s,r}} \geq N^{s-\delta(1-1/p_1+\beta)}.$$

Hence for this norm to be bounded uniformly in N, we need that

$$s \leq \delta(1 - 1/p_1 + \beta) = \frac{1 - 1/p_1 + \beta}{1 - 1/p_1 + \beta + 1/q_1 - \gamma},$$

which is precisely the value given by Theorem 3.2. This counterexample also shows that, provided $p_1 \leq p_2$ and $q_1 \leq q_2$, the regularity gained by averaging does not depend on the integrability in x of either f or g.

Now we prove that the exponent r given by Theorem 3.2 is optimal. To do so we consider

$$\begin{aligned} f_N(x, v) &= N^{1/p_2 + \delta(1/p_1 - \beta)} \times a(N\,x_1, x_2)\; b(N^\delta v_1), \\ g_N(x, v) &= N^{1+1/p_2 - \delta + \delta/p_1 - \delta\beta} \times \partial_1 a(N\,x_1, x_2)\; N^\delta v_1\, b(N^\delta v_1). \end{aligned} \tag{3.15}$$

To bound uniformly g_N in the space given by (4.3) (f_N was correctly normalized), we need to take

$$\delta = \frac{1 + 1/p_2 - 1/q_2}{1 - 1/p_1 + \beta + 1/q_1 - \gamma}$$

We again have

$$v \cdot \nabla_x f_N = g_N + h_N,$$

with h_N more regular than g_N and so negligible for our purpose. Finally

$$\|\rho_N\|_{W^{s,r}} \geq N^{s + 1/p_2 - 1/r - \delta(1 - 1/p_1 + \beta)}.$$

Since we already know that s is at most the value given by Theorem 3.2, we take that one and deduce that for ρ_N to be uniformly bounded, we need that

$$\frac{1}{r} = \frac{1}{p_2} - \frac{s}{p_2} + \frac{s}{q_2},$$

which is the value given by theorem 3.2. If we care only about local regularity then any $1/r$ larger than this will do of course.

4 Limit Cases

Some limitations of Theorem 3.2 are investigated here. The first two are the case of a full derivative ($a = 1$ in (3.7)) and the case of only L^1 regularity in velocity for f; in both cases, only compactness can be expected from averaging lemmas and of course no gain of derivatives. These two situations however have important uses: The first one as it replaces compensated compactness arguments in some cases (see for example [40] and [50] for compensated compactness) and the second one for Boltzmann equation or other collisional models. The last part of the chapter is devoted to the limitation $p_1 \leq \min(p_2,\ p_2^*)$ (or $q_1 \leq \min(q_2,\ q_2^*)$) and it illustrates the complexity of averaging lemmas with mixed norm.

4.1 The case of a full derivative

The main result here was obtained in [45]. We deal with Eq. (3.1) with $a = 1$ or

$$v \cdot \nabla_x f = \mathrm{div}_x g, \quad x \in \mathbb{R}^d,\ v \in S^{d-1}. \tag{4.1}$$

Very little can be expected in this case as indeed all functions f satisfy (4.1) with a right hand side just as regular as themselves. Nevertheless (4.1) is enough to ensure some compactness for the average

$$\rho(x) = \int_{S^{d-1}} f(x,v)\, dv. \tag{4.2}$$

In fact one may first prove the

Theorem 4.1 *Let f and g satisfy* (3.7) *and*

$$\begin{aligned} f \ &\in \dot W_v^{\beta,\,p_1}(S^{d-1},\ L_x^{p_2}(\mathbb{R}^d)), \qquad \beta \geq 0,\\ g \ &\in \dot W_v^{\gamma,\,q_1}(S^{d-1},\ L^{q_2}(\mathbb{R}^d)), \qquad -\infty < \gamma < 1, \end{aligned} \tag{4.3}$$

with $1 < p_2, q_2 < \infty$, $1 \leq p_1 \leq \min(p_2,\ p_2^*)$ and $1 \leq q_1 \leq \min(q_2,\ q_2^*)$ where for a general p, p^* is the dual exponent of p, and assume moreover that $\gamma - 1/q_1 < 0$. Then,

$$\|\rho\|_{B^{0,r}_{\infty,\infty}} \leq C \|f\|^{1-\theta}_{W_v^{\beta,p_1}(L_x^{p_2})} \times \|g\|^{\theta}_{W_v^{\gamma,q_1}(L_x^{q_2})},$$

with

$$\begin{aligned} \frac{1}{r} &= \frac{1-\theta}{p_2} + \frac{\theta}{q_2},\\ \theta &= \frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}. \end{aligned} \tag{4.4}$$

The space $B^{0,r}_{\infty,\infty}$ is again obtained by interpolation but here as ρ trivially belongs to $L^{p_2}(\mathbb{R}^d)$ we have that ρ belongs to all $L^{r'}$ with $r' \in [p_2,\ r[$ or $]r,\ p_2]$. Moreover it is possible to deduce from Theorem 4.1

Corollary 4.1 *Consider two sequences f_n and g_n solutions to* (4.1). *Assume moreover that f_n is uniformly bounded in $\dot W^{\beta,p_1}_v(S^{d-1},\ L^{p_2}(\mathbb{R}^d))$ with*

$$\beta \geq 0,\ 1 < p_2 < \infty,\ 1 \leq p_1 \leq \min(p_2,\ p_2^*),$$

and that g_n is uniformly bounded and **compact** *in $\dot W^{\gamma,q_1}_v(S^{d-1},\ L^{q_2}(\mathbb{R}^d))$ with*

$$-\infty < \gamma < 1,\ 1 < q_2 < \infty,\ 1 \leq q_1 \leq \min(q_2,\ q_2^*).$$

Then the sequence ρ_n is compact in any $L^{r'}$ with $r' \in]p_2,\ r[$ or $]r,\ p_2[$ and r given by (4.4).

These two results were obtained in [45] (with a different decomposition of the operator $v \cdot \nabla_x$ and thus with ρ in a true Besov space). They are quite useful for kinetic formulations as the next chapter illustrates.
Proof of Cor. (4.1)**.** It is an almost straightforward consequence of Theorem 4.1. As f_n is uniformly bounded, it converges weak$-*$ to some limit f (at least after extraction of a sub sequence). On the other hand, still after extraction, g_n converges strongly to some limit g and thus

$$v \cdot \nabla_x f = \mathrm{div}_x\, g,$$

or

$$v \cdot \nabla_x (f_n - f) = \mathrm{div}_x\, (g_n - g).$$

Applying now Th. 4.1 to $f_n - f$ and $g_n - g$, we find that

$$\|\rho - \rho_n\|_{B^{0,r}_{\infty,\infty}} \leq C\|f - f_n\|^{1-\theta}_{W^{\beta,p_1}_v(L^{p_2}_x)} \times \|g - g_n\|^{\theta}_{W^{\gamma,q_1}_v(L^{q_2}_x)}.$$

As $g_n - g$ strongly converges toward 0 and f_n is uniformly bounded, we deduce that $\rho_n - \rho$ converges strongly toward 0 in $B^{0,r}_{\infty,\infty}$. Therefore it also does in all $L^{r'}$ with $r' \in]p_2,\ r[$ or $]r,\ p_2[$ since $\rho - \rho_n$ is uniformly bounded in L^{p_2}.

Proof of Theorem 4.1. We follow the steps of the proof of Th. 3.2 and decompose

$$\rho = \rho_1 + \rho_2 = \lambda T_\lambda f + \mathrm{div}_x\, T_\lambda g.$$

From Prop. 3.2 we get that

$$\|\rho^1\|_{\dot W^{1+\beta-1/p_1,\,p_2}} \le C\,\lambda\times\lambda^{\beta-1/p_1}\times\|f\|_{\dot W^{\beta,p_1}_v L^{p_2}_x},$$
$$\|\rho^2\|_{\dot W^{\gamma-1/q_1,\,q_2}} \le C\lambda^{\gamma-1/q_1}\times\|g\|_{\dot W^{\gamma,q_1}_v L^{q_2}_x}.$$

So again minimizing in λ in the functional $K(t)$ we take

$$\lambda = t^{1/(1+\beta-1/p_1-\gamma+1/q_1)},$$

and we indeed find

$$K(t)\le t^\theta\times\|f\|^{1-\theta}_{W^{\beta,p_1}_v L^{p_2}_x}\times\|g\|^{\theta}_{W^{\gamma,q_1}_v L^{q_2}_x},$$

with

$$\theta=\frac{1+\beta-1/p_1}{1+\beta-1/p_1-\gamma+1/q_1}.$$

Therefore ρ belongs to $B^{s,r}_{\infty,\infty}$ and it only remains to notice that

$$s=(1-\theta)\,(1+\beta-1/p_1)+\theta\,(\gamma-1/q_1)=0,$$

which finishes the proof. □

4.2 L^1 integrability only for f

Theorem 3.2 does not give any regularity for the average if $1/p_1-\beta=1$. A case of notable interest is

$$v\cdot\nabla_x f=g, \tag{4.5}$$

where f is only in $L^1(\mathbb{R}^d\times S^{d-1})$. It is notably crucial for collisional models: See [15] for the existence of renormalized solutions to Boltzmann equation, and [26], [27], [48] for the famous derivation of hydrodynamic limits.

In that situation the average ρ is not generally in any Sobolev spaces. Even though it was shown in [25] that some compactness property still holds namely

Theorem 4.2 *Let f_n and g_n be two sequences uniformly bounded in the space $L^1(\mathbb{R}^d\times S^{d-1})$ and solutions to (4.5). Assume moreover that the sequence f_n is uniformly equi-integrable in v. Then the sequences of averages ρ_n is compact in $L^1_{loc}(\mathbb{R}^d)$.*

The proof relies first on the fact that if f_n is equi-integrable in velocity then it is in both variables:

Proposition 4.1 *Let f_n and g_n be two sequences uniformly bounded in $L^1(\mathbb{R}^d \times S^{d-1})$ and solutions to (4.5). If the sequence f_n is uniformly equi-integrable in $v \in S^{d-1}$ then it is uniformly equi-integrable in $(x, v) \in \mathbb{R}^d \times S^{d-1}$.*

It is then possible to get

Theorem 4.3 *Let f_n and g_n be two sequences uniformly bounded in the space $L^1(\mathbb{R}^d \times S^{d-1})$ and solutions to (4.5). Assume moreover that the sequence f_n is uniformly equi-integrable in $(x, v) \in \mathbb{R}^d \times S^{d-1}$. Then the sequence of averages ρ_n is compact in $L^1_{loc}(\mathbb{R}^d)$.*

With the additional assumption that g_n is equi-integrable, this last result was already noticed in [23].

We only give here the proof of 4.3 with a slight variant of the method used in [25].

If f and g satisfy (4.5), and if there is an increasing function $\Phi \in C(\mathbb{R}_+)$ with $\phi(\xi)/\xi$ increasing and $\Phi(\xi)/\xi \longrightarrow \infty$ as $\xi \to \infty$ and such that

$$I(f) = \int_{\mathbb{R}^d \times S^{d-1}} \Phi(|f(x, v)|)\, dx\, dv < \infty,$$

then there exists a function $\varepsilon(h)$ depending only on Φ with $\lim \varepsilon(h) = 0$ as $h \to 0$ and such that for any $\phi \in C^1_c(\mathbb{R}^d,\ \mathbb{R}_+)$

$$\int_{\mathbb{R}^d} |\rho(x+h) - \rho(x)|\, \phi(x)\, dx \le C_\phi\, \varepsilon(h)\ (\|f\|_{L^1} + \|g\|_{L^1} + I(f)). \qquad (4.6)$$

Of course (4.6) would imply Theorem 4.3.

Notice that

$$v \cdot \nabla_x(\phi f) = g\,\phi + f\, v \cdot \nabla\phi.$$

Now decompose

$$(\lambda + v \cdot \nabla_x)\,(\phi f) = \bar{g} + \lambda f_1^M + \lambda f_2^M,$$

with

$$f_1^M = \phi\, f\, \mathbb{I}_{|f| \le M}, \quad f_2^M = \phi\, f\, \mathbb{I}_{|f| > M}, \quad \bar{g} = g\,\phi + f\, v \cdot \nabla_x \phi.$$

Then

$$\phi\,\rho = T_\lambda\,\bar g + \lambda T_\lambda f_1^M + \lambda T_\lambda\, f_2^M.$$

Obviously

$$\int_{\mathbb{R}^d} |\rho(x+h)-\rho(x)|\,\phi(x)\,dx \le \int_{\mathbb{R}^d} |\phi(x+h)\rho(x+h)-\phi(x)\rho(x)|$$
$$+\, h\|\nabla\phi\|_{L^\infty}\,\|\rho\|_{L^1}$$
$$\le \int_{\mathbb{R}^d} |T_\lambda \bar g(x+h)-T_\lambda\,\bar g|\,dx + \int_{\mathbb{R}^d} |\lambda T_\lambda f_1^M(x+h)-\lambda T_\lambda f_1^M|\,dx$$
$$+ \int_{\mathbb{R}^d} |\lambda T_\lambda\, f_2^M(x+h)-\lambda T_\lambda\, f_2^M|\,dx + C_\phi\, h\|f\|_{L^1}$$
$$\le 2\|T_\lambda \bar g\|_{L^1} + 2\lambda\,\|T_\lambda f_2^M\|_{L^1} + \int_{\mathbb{R}^d} |\lambda T_\lambda f_1^M(x+h)-\lambda T_\lambda f_1^M|\,dx + C_\phi\, h\|f\|_{L^1}.$$

From Prop. 3.2, we have

$$\|T_\lambda \bar g\|_{L^1} \le \frac{C}{\lambda}\,\|\bar g\|_{L^1} \le \frac{C}{\lambda}\,(\|g\|_{L^1} + C_\phi\|f\|_{L^1}),$$

and

$$\|T_\lambda f_2^M\|_{L^1} \le \frac{C}{\lambda}\,\|f_2^M\|_{L^1} \le \frac{C}{\lambda}\,\frac{M}{\Phi(M)}\,I(f),$$

as (remember that $\phi(\xi)/\xi$ is increasing)

$$\int_{\mathbb{R}^d\times S^{d-1}} |f(x,v)|\,\mathbb{I}_{|f|>M}\,dx\,dv = \int_{\mathbb{R}^d\times S^{d-1}} \Phi(|f(x,v)|)\,\mathbb{I}_{|f|>M}\,\frac{|f|}{\Phi|f|}\,dx\,dv$$
$$\le \sup_{\xi>M}\frac{\xi}{\Phi(\xi)}\int_{\mathbb{R}^d\times S^{d-1}} \Phi(|f(x,v)|)\,dx\,dv.$$

For the last term $T_\lambda f_1^M$, notice first that is is compactly supported in the support of ϕ so

$$\|T_\lambda f_1^M\|_{W^{1/2,1}(\mathbb{R}^d)} \le C_\phi \|T_\lambda f_1^M\|_{H^{1/2}(\mathbb{R}^d)}.$$

Furthermore as f_1^M belongs to $L^2(\mathbb{R}^d\times S^{d-1})$ then

$$\|T_\lambda f_1^M\|_{H^{1/2}(\mathbb{R}^d)} \le C\,\lambda^{-1/2}\,\|f_1^M\|_{L^2(\mathbb{R}^d\times S^{d-1})} \le C\,\lambda^{-1/2}\,M^{1/2}\,\|f_1^M\|_{L^1}^{1/2}.$$

Consequently

$$\int_{\mathbb{R}^d} |\lambda T_\lambda f_1^M(x+h) - \lambda T_\lambda f_1^M|\,dx \le h^{1/2}\,\|T_\lambda f_1^M\|_{W^{1/2,1}(\mathbb{R}^d)}$$
$$\le C_\phi\, h^{1/2}\,\lambda^{1/2}\,M^{1/2}\,\|f_1^M\|_{L^1}^{1/2}.$$

Combining all estimates, one obtains

$$\int_{\mathbb{R}^d} |\rho(x+h)-\rho(x)|\,\phi(x)\,dx \le \frac{C}{\lambda}\,(\|g\|_{L^1} + C_\phi\|f\|_{L^1}) + C\,\frac{M}{\Phi(M)}\,I(f)$$
$$+ C_\phi\,\lambda^{1/2}\,h^{1/2}\,M^{1/2}\,\|f_1^M\|_{L^1}^{1/2} + C_\phi\,h\,\|f\|_{L^1}.$$

For any h, it only remains to minimize in λ and M to obtain (4.6).

Notice finally that in most applications, Φ is equal to $\xi\ \log\xi$ (from entropy bounds). In that case, the function $\varepsilon(h)$ is

$$\varepsilon(h) = \frac{1}{\log 1/h}.$$

4.3 Mixed norm inequalities

A disappointing condition in Theorem 3.2 is that $p_1 \le \min(p_2,\ p_2^*)$ (and the same for q_1 and q_2). First of all it tells that the best case would be when f or g belongs to $L^p(\mathbb{R}^d \times S^{d-1})$ and any additional integrability in velocity is "lost". This somehow contradicts the idea that the regularity of the average depends only on the regularity in velocity; This idea though is supported by some heuristics arguments and results like [53] (where the average is however obtained in a weak space).

It turns out that this question is probably quite difficult. We give a proof (but only in dimension 2) which improves Theorem 3.2 but also an example showing that this cannot be carried out too far (namely if f is only L^1 in x then nothing may be gained even it is L^∞ in v).

4.3.1 An improvement on the condition $p_1 \le p_2$

We can prove the following

Proposition 4.2 *Let f, g in $L^{4/3}(\mathbb{R}^2,\ L^2(S^1))$ satisfy* (4.5). *Assume moreover that f and g are even in v then the average $\rho = \int_{S^1} f(x,v)\,dv$ belongs to $W^{s,4/3}(\mathbb{R}^2)$ for all $s < 1/2$.*

Therefore we gain 1/2 derivative on the average even though we work only in $L^{4/3}$ in x. This result is proved only in dimension 2, an equivalent in higher dimension is unknown (and would probably involve another critical space than $L^{4/3}$). In dimension 2, $L^{4/3}$ is probably critical in the sense that any L^p with $p < 4/3$ would give less than 1/2 derivative.

If f or g is in $L^2(S^1,\ L^{4/3}(\mathbb{R}^2))$, then nothing is known. Notice that, of course, by Hölder estimates, $L^{4/3}(\mathbb{R}^2,\ L^2(S^1))$ is stronger (included in) than $L^2(S^1,\ L^{4/3}(\mathbb{R}^2))$.

The assumption f even in v is not necessary, and Prop. 4.2 can of course be combined with the other estimates in chapter 3 by interpolation to give more elaborate estimates (we refer to [32] for more details).

Proof of Prop. 4.2. We again decompose

$$\rho = T\,f + T\,g.$$

Since f and g are even

$$T\,f = \int_{S^{d-1}} \int_0^\infty f(x - vt, v)\, e^{-t}\, dt = \int_{S^{d-1}} \int_{-\infty}^\infty f(x - vt, v)\, e^{-t}\, dt = T_0\, f.$$

Now we have to prove that T_0 is continuous from $L^{4/3}(\mathbb{R}^2,\ L^2(S^1))$ to the space $W^{s,4/3}(\mathbb{R}^2)$ for any $s < 1/2$. By duality this is equivalent to the continuity of the dual T_0^* from $L^4(\mathbb{R}^2)$ to $W^{s,4}(\mathbb{R}^2,\ L^2(S^1))$. Finally with a decomposition similar to the one performed in subsection 2.4.3, it is enough to show

Lemma 4.1 *For any set E and any $0 \le \theta < 1/2$,*

$$\|\Delta_x^{\theta/2} T_0^* \mathbb{I}_E\|^4_{L^4_x(\mathbb{R}^2,\, L^2_v(S^1))} \le C\,|E|. \tag{4.7}$$

Proof of Lemma 4.1. First of all, we decompose the sphere S^1 into subdomains S_k with $k = 1,\, 2$ such that $|v_k| > 1/2$ in S_k. Of course it is enough to prove (4.7) with S_k instead of S^1 and by symmetry we do it only for S_1. Now we are going to make two reductions.

Step 1: Reduction to the compactly supported case.

We explain why it is enough to prove for any $K > 0$ and any set $E \in B(0, K)$, the inequality

$$\|\Delta_x^{\theta/2} T_0^* \mathbb{I}_E\|^4_{L^4_x(B(0,K),\, L^2_v(S_1))} \le C(K)\,|E|. \tag{4.8}$$

Take any set $E\subset\mathbb{R}^2$ with finite measure and any $K>0$. We decompose E into $\cup_i E_i$ with $E_i\subset B(x_i,K)$ and $|x_i-x_j|>K/2$ and $E_i\cap E_j=\emptyset$, $\forall i\neq j$. Then

$$\mathbb{I}_E(y)=\sum_i \mathbb{I}_{E_i}(y),$$

and consequently

$$T_0^*\mathbb{I}_E(x,v)=\sum_i T_0^*\mathbb{I}_{E_i}(x,v)\,\mathbb{I}_{B(x_i,2K)}(x)+\sum_i T_0^*\mathbb{I}_E(x,v)\,\mathbb{I}_{|x-x_i|>2K}=I+II.$$

Now, of course because of the condition $|x_i-x_j|>K/2$

$$\begin{aligned}\int_{\mathbb{R}^2}\left(\int_{S_1}|\Delta_x^{\theta/2}\,I|^2\,dv\right)^2dx&=C\sum_i\int_{B(x_i,2K)}\left(\int_{S_1}|\Delta_x^{\theta/2}\,T_0^*\mathbb{I}_{E_i}(x,v)|^2\,dv\right)^2dx\\&\le C(2K)\sum_i|E_i|\le C(2K)\,|E|,\end{aligned}$$

since (4.8) is obviously invariant by translation and hence true as well if we replace $B(0,K)$ by $B(y,K)$ for any y.

As for the second term, we remark that, as $E_i\subset B(x_i,K)$

$$T_0^*\mathbb{I}_{E_i}(x,v)\,\mathbb{I}_{|x-x_i|>2K}\le e^{-|x-x_i|/2-K/2},$$

and that furthermore (that inequality is proved in [18]), for any x

$$\int_{S_1}|T_0^*\mathbb{I}_{E_i}(x,v)|^2\,dv\le C\,|E_i|.$$

Eventually we simply bound in L^4

$$\begin{aligned}\int_{\mathbb{R}^2}\left(\int_{S_1}|II|^2\,dv\right)^2dx&\le C\,e^{-K}\sum_{i,j}|E_i|^{1/2}\,|E_j|^{1/2}\int_{\mathbb{R}^2}e^{-|x-x_i|/2-|x-x_j|/2}\,dx\\&\le C\,e^{-K}\,|E|.\end{aligned}$$

We have decomposed $T_0^*\mathbb{I}_E$ into two terms for any K. The first one belongs to $W_x^{\theta,4}(L_v^2)$ with norm $(C(2K)\,|E|)^{1/4}$ (which is obviously at most polynomial in K) and the second one in L^4 with norm $e^{-K/4}\,|E|^{1/4}$. By real interpolation, we deduce that $T_0^*\mathbb{I}_E$ belongs to $W_x^{\theta',4}(L_v^2)$ with norm $C|E|^{1/4}$ for any $\theta'<\theta$, which is exactly what we want.

Step 2: Reduction to the X-ray transform.

The aim here is to get back the case where $T_0^*\mathbb{I}_E(x,v)$ is invariant along any line with direction v like the X-ray transform. So first of all, we write

$$
\begin{aligned}
|\Delta_x^{\theta/2} T_0^* \mathbb{I}_E(x,v)| &= |\Delta_x^{\theta/2} \int_{-\infty}^{0} v\cdot\nabla_x T_0^*\mathbb{I}_E(x+tv,v)\,dt| \\
&\leq \int_{-\infty}^{+\infty} |\Delta_x^{\theta/2}\, v\cdot\nabla_x T_0^*\mathbb{I}_E(x+tv,v)|\,dt.
\end{aligned}
$$

All these expressions make sense because now $E\subset B(0,K)$

$$
\begin{aligned}
v\cdot\nabla_x T_0^*\mathbb{I}_E(x+tv,v) &= \int_0^\infty v\cdot\nabla_x \mathbb{I}_E(x+tv+rv)\,e^{-r}\,dr \\
&= \int_0^\infty \frac{\partial}{\partial r}\big(\mathbb{I}_E(x+tv+rv)\big)\,e^{-r}\,dr \\
&= -\int_0^\infty \mathbb{I}_E(x+tv+rv)\ e^{-r}\,dr,
\end{aligned}
\tag{4.9}
$$

by integration by parts in r. Then T_0^* is the integral on the whole line by (2.3) and so

$$
v\cdot\nabla_x T_0^*\mathbb{I}_E(x+tv,v) = \int_{-\infty}^{\infty} \mathbb{I}_E(x+tv+rv)\,e^{-|r|}\times\frac{r}{|r|}\,dr. \tag{4.10}
$$

Now we denote

$$
T\mathbb{I}_E(x,v) = \int_{-\infty}^{+\infty} |\Delta_x^{\theta/2}\, v\cdot\nabla_x T_s^*\mathbb{I}_E(x+tv,v)|\,dt.
$$

Thanks to (4.9) and (4.10), we know the following properties on T, for some $\theta'>0$ (in fact $\theta'=1/2-\theta$)

$$
v\cdot\nabla_x T\,\mathbb{I}_E(x,v)=0,\quad \|\Delta_x^{\theta'/2}\,T\,\mathbb{I}_E\|_{L^2_{B(0,K)\times S_1}}\leq C\,|E|^{1/2}. \tag{4.11}
$$

We want to deduce from (4.11)

$$
\|T\mathbb{I}_E\|^4_{L^4_x(B(0,K),\,L^2_v(S_1))}\leq C(K)\,|E|. \tag{4.12}
$$

Step 3: Deduction of (4.12) *from* (4.11).

We begin with

$$\|T\mathbb{I}_E\|^4{}_{L^4_x(B(0,K),\,L^2_v(S_1))} = \int_{B(0,K)} \left(\int_{v\in S_1} |T\mathbb{I}_E(x,v)|^2\,dv\right)^2 dx$$
$$= \int_{B(0,K)}\int_{v,w\in S_1} |T\mathbb{I}_E(x,v)|^2 \times |T\mathbb{I}_E(x,w)|^2\,dv\,dw\,dx$$
$$= \int_{v\in S_1}\int_{x\in B(0,K)}\int_{w\in S_1} |T\mathbb{I}_E(x,v)|^2\; |T\mathbb{I}_E(x,w)|^2\,dw\,dx\,dv.$$

We change variables in x decomposing x in $y+lv$ with y in the plane H_1 of equation $x_1=0$. Since $|v_1|>1/2$, the jacobian of the transformation is bounded and as all the terms in the integral are non negative, we may simply bound

$$\|T\mathbb{I}_E\|^4_{L^4_x(B(0,K),\,L^2_v(S^1))} \le \int_{v\in S_1}\int_{y\in H_1}\int_{l=-K}^{K}\int_{w\in S_1} |T\,\mathbb{I}_E(y+lv,v)|^2 \times |T\mathbb{I}_E(y+lv,w)|^2\,dw\,dl\,dy\,dv$$
$$\le \int_{v\in S_1}\int_{y\in H_1} |T\,\mathbb{I}_E(y,v)|^2 \times \left(\int_{l=-K}^{K}\int_{w\in S_1} |T\,\mathbb{I}_E(y+lv,w)|^2 dw\,dl\right) dy\,dv,$$

because $Tf(x,v)$ is constant on any line with direction v and therefore $T\,\mathbb{I}_E(y+lv,v)$ does not depend on l. We denote

$$I(y,v) = \int_{l=-K}^{K}\int_{w\in S_1} |T\mathbb{I}_E(y+lv,w)|^2\,dw\,dl,$$

and we want to show that I belongs to L^∞. So we fix y and v and we first decompose S_1 into the union of S_1^i with $S_1^i=\{w\in S^1,\ 2^{-i-1}<|v-w|<2^{-i}\}$ and so

$$I(l,v) = \sum_{i=0}^{\infty} I_i(l,v) = \sum_{i=0}^{\infty} \int_{l=-K}^{K}\int_{w\in S_1^i} |T\,\mathbb{I}_E(y+lv,w)|^2\,dw\,dl.$$

Of course $T\,\mathbb{I}_E(y+lv,w)$ is constant along any line with direction w so we may bound

$$I_i \le \frac{1}{2K}\int_{w\in S_1^i}\int_{l=-K}^{K}\int_{s=-K}^{K} |T\,\mathbb{I}_E(y+sw+lv,w)|^2\,ds\,dl\,dw.$$

We change again variables from l and s to $z = y + sw + lv$. We denote by $C_{y,v,w}$ the set $\{y + sw + lv,\ |s| \leq K,\ |l| \leq K\}$ and by $|(v,w)|$ the sinus of the angle between v and w. Then

$$\begin{aligned} I_i &\leq \frac{1}{2K} \int_{w\in S_1^i} \int_{z\in C_{y,v,w}} |T\,\mathbb{I}_E(z,w)|^2 \frac{dz\,dw}{|(v,w)|} \\ &\leq \frac{2^{i+1}}{2K} \int_{w\in S_1^i} \int_{z\in C_{y,v,w}} |T\,\mathbb{I}_E(z,w)|^2\,dz\,dw. \end{aligned}$$

Denote $C_{y,v} = \bigcup_{w\in S_1^i} C_{y,v,w}$ and $\tilde{E} = E \cap C_{y,v}$. Clearly, as all the terms are non negative

$$I_i \leq \frac{2^{i+1}}{2K} \int_{w\in S_1^i} \int_{z\in C_{y,v}} |T\,\mathbb{I}_{\tilde{E}}(z,w)|^2\,dz\,dw.$$

Using a Hölder estimate, we find for any $p > 2$,

$$\begin{aligned} I_i &\leq \frac{2^{i+1}}{2K} \times |C_{y,v}|^{1-2/p} \times \int_{w\in S_1^i} \left(\int_{z\in C_{y,v}} |T\,\mathbb{I}_{\tilde{E}}(z,w)|^p\,dz \right)^{2/p} dw \\ &\leq C(K)\,2^{i+1} \times 2^{-i(1-2/p)} \times \int_{w\in S^1} \left(\int_{z\in B(0,2K)} |T\,\mathbb{I}_{\tilde{E}}(z,w)|^p\,dz \right)^{2/p} dw, \end{aligned}$$

because the measure of $C_{y,v}$ is bounded by a constant depending on K times 2^{-i}. Now by Sobolev embedding, for $1/2 - \theta'/2 \leq 1/p < 1/2$, the last integral is dominated by the $L^2_w H^{\theta'}_z$ norm of $T\mathbb{I}_{\tilde{E}}$. Therefore, taking $1/p = 1/2 - \theta'/2$, we get by (4.11)

$$\begin{aligned} I_i &\leq C(K)\,2^{i+1} \times 2^{-i\theta'} \times \int_{w\in S^1} \int_{z\in B(0,2K)} |\Delta_x^{\theta'/2} T\,\mathbb{I}_{\tilde{E}}(z,w)|^2\,dz\,dw \\ &\leq C(K)\,2^{i+1} \times 2^{-i\theta'} \times C\,|\tilde{E}| \leq C(K) \times 2^{-i\theta'}, \end{aligned}$$

because the measure of $\tilde{E}$ is less than the measure of $C_{y,v}$. Eventually we may sum up the series and get

$$I = \sum_{i=0}^{\infty} I_i \leq C(K).$$

This has as immediate consequence that

$$\begin{aligned}\|\Delta_x^{s/2} T \mathbb{I}_E\|^4_{L^4_x(B(0,K),\, L^2_v(S_1))} &\leq C(K) \int_{v\in S_1}\int_{y\in H_1} |\Delta_x^{s/2} T\, \mathbb{I}_E(y,v)|^2\, dy\, dw\\ &\leq C(K)\times |E|,\end{aligned}$$

using again the known L^2 estimate (4.11) on T. $\square$

Note that it is relatively simple to find a set E for which the lemma would be false if $p > 4$ in dimension two. Indeed, one may take for example a set composed of the N sets E_i of equations in polar coordinates r, θ, $\theta \in [i/N,\ i/N + i/2N]$ and $r \leq 1$. Then $|E| \geq 1$ and for any x in the square of size $1/N$ centered at the origin $\int_v |\Delta_x^{1/4} \mathbb{I}_E(x,v)|^2\, dv = N$ and so to have

$$N^{-2}\times N^p \leq \int_{B(0,2K)} \left(\int_v |\Delta_x^{1/4}\, \mathbb{I}_E(x,v)|^2\, dv\right)^{p/2} dx \leq C N^{p/2},$$

one must have $p \leq 4$.

4.3.2 A example in L^1

The example that we give below shows that in $L^1(\mathbb{R}^d)$ (in x) no derivative may be gained.

Consider the following function g_N

$$g_N(x,v) = \sum_{i=1}^N \sum_{j=1}^N (-1)^i\, \mathbb{I}_{|x_1 - i/N|\leq 1/N^2} \times \delta(x_2 = j/N)\times \Phi_N(v).$$

Instead of true dirac masses, we should take approximations of them in L^1 so that g_N belong to L^1_x. However to keep things as simple as possible, we will do just as if Dirac masses belong to L^1. Then, we obviously have

$$\|g_N\|_{L^1_x L^\infty_v} = N\times N\times N^{-2}\times \|\Phi_N\|_{L^\infty} \leq 1.$$

The function Φ_N will be determined later on but with an L^∞ norm less than one.

Next we define f_N by means of g_N

$$f_N(x,v) = a(x)\times \int_0^\infty g_N(x - vt, v)\, dt,$$

with $a(x)$ a regular function with compact support and value 1 in the ball of radius 2. Therefore we have

$$v \cdot \nabla_x f_N = g_N + h_N,$$

with

$$h_N = (v \cdot \nabla_x a) \times \int_0^\infty g_N(x - vt, v)\, dt.$$

It is obvious that h_N is at least as regular as g_N and so

$$\|v \cdot \nabla_x f_N\|_{L^1_x L^\infty_v} \leq C. \tag{4.13}$$

Now let us compute the $L^1_x L^\infty_v$ norm of f_N. Given x and v the value of f_N depends on the number of times the line issued from x, and with direction v, crosses one of the small segments of which g_N is composed. This almost never happens. For instance, if Nx_2 is an integer and if v is along the x_1-axis, then f_N is the average of Dirac masses. This case is avoided by assuming that $\Phi((a, 0)) = 0$, for any a and it ensures that f_N does not exhibit any Dirac mass itself.

However, it remains the other cases where for example $x_1 = i/N \pm 1/N^2$ for some i. Then if $|v_1| \leq 1/N^2$, $f(x, v)$ is of order N. Finally the norm of f_N may be estimated as

$$\|f_N\|_{L^1_x L^\infty_v} \leq C\,(1 + N \times N \times N^{-2}) \leq C. \tag{4.14}$$

For ρ_N those points of concentration of f_N do not have any importance. Indeed ρ_N is the average of f_N in v and if f_N is of order N at some points, it is only for values of v in an angular sector of size N^{-2}. Consequently, ρ_N is at most of order one. Then consider a segment with relative coordinates (a, b) (relative with respect to x), this segment is seen from x with an angular variation of

$$\max\left(\frac{1}{N^2 b}\,,\ \frac{b}{N^2\, a^2}\right).$$

Hence for a given x which is typically at a distance $1/2N$ of the closest line $x_2 = j/N$, the measure of the set of velocities v, such that the corresponding line crosses at least one segment, is

$$\sum_{j=1}^{N} \left(j \times \frac{1}{N\, j} + \sum_{i=j}^{N} \frac{j/N}{N^2\, i^2/N^2} \right) \sim 1.$$

Note that this also justifies that a given line almost never intersects more than one segment.

Now of course there is the question of the alterning signs in g_N which could produce cancellations in ρ_N. This is where the definition of Φ_N, and the fact that it is L^∞ but not in any Sobolev space, plays a crucial role. Indeed let us choose a Φ_N such that ρ_N is indeed of order 1 at the point $(1/2,\,1/2)$ for instance. This is possible but only because we do not need any derivability on Φ_N.

Then notice that ρ is almost periodic of period $2/N$. If the segments in g_N where equidistributed in the whole space, it would be exactly periodic but as it is, some small perturbation has to be expected from the compact support in g_N. Because the derivative of ρ_N is obviously at most of order N, this means that ρ_N is of order one on a domain a measure of order one also.

To conlcude this counterexample, we remark that ρ_N changes sign if we add $1/N$ to x_1 due to the alterning signs in g_N. Therefore, the derivative of ρ_N is exactly of order N and

$$\|\rho_N\|_{W^{s,1}_{loc}} \sim N^s. \tag{4.15}$$

The combination of (4.13), (4.14) and (4.15) shows that, although f_N and g_N are uniformly bounded in $L^1_x L^\infty_v$, ρ_N is not uniformly bounded in any $W^{s,1}_{loc}$, $s > 0$.

We turn to the case of exponents $p \geq 2$. We use polar coordinates in x and v, hence $x = re^{i\theta}$ $v = e^{i\phi}$. We take

$$g_N(x,v) = e^{iN\theta}\mathbb{I}_{r\leq N} \times e^{-iN\phi},$$

such that

$$\|g_N\|_{L^q_x L^\infty_v} = N^{2/q}.$$

As in the previous case, we define f_N as

$$f_N(x,v) = \left(\int_0^\infty g(x - vt, v)\,dt\right) \times a(r/N),$$

for a a C^∞_c function. We obtain

$$\|v \cdot \nabla_x f_N\|_{L^q_x L^\infty_v} \sim N^{2/q}. \tag{4.16}$$

Given any $x = re^{i\theta}$, if we choose $v = e^{i(\theta+\pi)}$, then $f_N(x, v)$ is equal to N, so that

$$\|f_N\|_{L^p_x L^\infty_v} \sim N^{1+2/p}. \tag{4.17}$$

Now given x and assuming that v is not parallel to x, then there are cancellations in the integral defining f_N. As a matter of fact, the order of f_N is the typical length on which there cannot be any cancellation. It is easy to see that this length is N/r or N if $r \leq 1$. Therefore, given the oscillation in ρ_N coming from the $e^{iN\theta}$ in g_N

$$\|\rho_N\|_{W^{s,1}_{loc}} \sim N^{1+s}. \tag{4.18}$$

As previously, this norm has to be bounded by the norm of g_N to the power s times the norm of f_N to the power $1-s$. Estimates (4.16), (4.17) and (4.18) have as a consequence that s has to satisfy

$$1 + s \leq \frac{2s}{q} + 1 - s + \frac{2}{p} - \frac{2s}{p},$$

or

$$s \leq \frac{1/p}{1 - 1/q + 1/p}.$$

This again corresponds to the result predicted by Theorem 3.2.

Before ending this subsection, we would like to point out that these examples do not rigorously allow us to conlude that the conditions $p_1 \leq \min(p_2, p_2^*)$, or the same for q_i, are absolutely necessary. At least a counterexample with an exponent $p_2 < 2$ for f and an exponent $q_2 > 2$ for g (or the converse) is missing.

5 Application to Scalar Conservation Laws

The purpose of this chapter is to present a (relatively short and self-contained) application of the previous results namely for kinetic formulations of scalar conservation laws. Its scope is much too limited to give an overview of conservation laws, or even scalar conservation laws, kinetic formulations or regularity results for these equations. Hence many major contributions to the field are not described. We refer the interested reader to [42] for a more complete description of kinetic formulations, and to [11], [28], or [47] for example for an introduction to the theory of conservation laws.

Scalar conservation laws are hyperbolic equations on a scalar $u(t,x) \in \mathbb{R}$

$$\begin{aligned} &\partial_t u + \nabla_x \cdot (A(u(t,x))) = 0, \quad t \geq 0,\ x \in \mathbb{R}^d, \\ &u(t=0,x) = u^0(x), \end{aligned} \tag{5.1}$$

where the flux A will always be regular here, namely $A \in C^2(\mathbb{R},\ \mathbb{R}^d)$.

The characteristics for Eq. (5.1) are lines. More precisely if u is a regular (C^1) solution then

$$u(t, x + ta(u^0(x))) = u^0(x),$$

where $a(\xi) = A'(\xi)$. Of course this also shows that regular solutions cannot exist in general for all times : if $x = x_1 + ta(u^0(x_1)) = x_2 + ta(u^0(x_2))$, then $u(t,x)$ would have to be equal to both $u^0(x_1)$ and $u^0(x_2)$.

This lack of regular solutions in large times requires the use of weak solutions for which, unfortunately, there is no uniqueness; hence the introduction of entropy to discriminate.

We present here the theory of entropy solutions through kinetic formulations, as this is the simplest way to apply averaging lemmas. Other approaches are of course as valid; the more tradional being the use of vanishing viscosity and Kruzkov entropy.

5.1 Kinetic Formulation

5.1.1 The definition

The kinetic formulation for scalar laws was first introduced in [38] (and at the same time in [39] for isentropic gas dynamic).

Assume that u is a classical solution to (5.1). Define then

$$f(t,x,v) = \begin{cases} 1 & \text{if } 0 \leq v < u(t,x), \\ -1 & \text{if } u(t,x) < v \leq 0, \\ 0 & \text{in the other cases.} \end{cases} \tag{5.2}$$

Compute (in the sense of distribution)

$$\begin{aligned} \partial_t f = \partial_t u\, \delta(u(t,x) - v) &= -a(u(t,x)) \cdot \nabla_x u(t,x)\, \delta(u(t,x) - v) \\ &= -a(v) \cdot \nabla_x u(t,x)\, \delta(u(t,x) - v) = -a(v) \cdot \nabla_x f, \end{aligned}$$

and so f solves the free transport equation. When u is no more C^1 this computation cannot be done. Instead one may define

Definition: A function $u \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ is an entropy solution to (5.1) if and only if there exists a non negative measure $m \in M^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^{2d})$, such that the function f defined through (5.2) satisfies

$$\partial_t f + a(v) \cdot \nabla_x f = \partial_v m. \tag{5.3}$$

Note that if f satisfies (5.3) then f is of bounded variation in time with value in a negative Sobolev space $(BV_{loc}(\mathbb{R}_+,\ W^{-1-0,1}(\mathbb{R}^{d+1})))$. Therefore the trace of f at $t = 0$ ($t = 0+$ more precisely) is well defined and since u can be recovered through

$$u(t, x) = \int_{\mathbb{R}} f(t, x, v)\, dv \tag{5.4}$$

the value of u at $t = 0$ makes perfect sense and one may add an initial condition to the definition.

Through the rest of this chapter, we will assume for simplicity the equivalent of (2.5)

$$\exists C,\ \forall \xi \in \mathbb{R}^d,\ \forall \tau,\ \forall \varepsilon \in \mathbb{R}_+, \qquad |\{v \in \mathbb{R};\ |a(v) \cdot \xi - \tau| \leq \varepsilon\}| \leq \varepsilon. \tag{5.5}$$

This condition enables us to use the equivalent of Theorem 3.2. However most of the theory would remain valid under the weaker assumption

$$\exists C,\ \forall \xi \in \mathbb{R}^d,\ \forall \tau,\ \forall \varepsilon \in \mathbb{R}_+, \qquad |\{v \in \mathbb{R};\ |a(v) \cdot \xi - \tau| \leq \varepsilon\}| \leq \varepsilon^\theta, \tag{5.6}$$

for some $\theta > 0$.

The main aim of this chapter is to prove the following

Theorem 5.1 *Assume* (5.5). *For any $u^0 \in L^1(\mathbb{R}^d)$, there exists a unique function $u \in L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^d))$, entropy solution to* (5.1) *with $u(t = 0) = u^0$. Moreover if $u^0 \in L^\infty$ the solution satisfies $u \in W^{s,3/2}_{loc}(\mathbb{R}^*_+ \times \mathbb{R}^d)$ for any $s < 1/3$.*

5.1.2 Propagation of L^p norms

Let us begin by showing the easiest property of entropy solutions, namely

Proposition 5.1 *Take any $\phi \in C^2(\mathbb{R})$, convex and assume that*

$$\int_{\mathbb{R}^d} \phi(u^0(x))\, dx < \infty,$$

then for any $t > 0$ and any entropy solution u with initial data u^0

$$\int_{\mathbb{R}^d} \phi(u(t,x))\, dx \le \int_{\mathbb{R}^d} \phi(u^0(x))\, dx.$$

In particular if $u^0 \in L^p$ then $u \in L^\infty(\mathbb{R}_+,\ L^p(\mathbb{R}^d))$.

Proof. Define ϕ_n a sequence converging toward ϕ with $\phi_n'' \in C_c(\mathbb{R})$. Notice that because of (5.2)

$$\int_{\mathbb{R}^d} \phi_n(u(t,x))\, dx = \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\, f(t,x,v)\, dx\, dv.$$

Now multiplying (5.3) by $\phi_n'(v)$, integrating in space and velocity

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\, f(t,x,v)\, dx\, dv &= \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\, \partial_v m\, dx\, dv \\ &= -\int_{\mathbb{R}^d\times\mathbb{R}} \phi_n''(v)\, m\, dx\, dv \le 0,\end{aligned}$$

because $\phi_n'' \ge 0$ and $m \ge 0$. Consequently

$$\begin{aligned}\int_{\mathbb{R}^d} \phi_n(u(t,x))\, dx &= \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\, f(t,x,v)\, dx\, dv \\ &\le \int_{\mathbb{R}^d\times\mathbb{R}} \phi_n'(v)\, f(0,x,v)\, dx\, dv \\ &= \int_{\mathbb{R}^d} \phi_n(u^0(x))\, dx,\end{aligned}$$

and passing to the limit in n, one obtains the proposition. Notice that the sign of the measure is crucial to get the estimate. □

5.2 Existence by transport-collapse approximation

As usual the existence of a solution is obtained by an approximation procedure. The classical one is the vanishing viscosity method, it however requires the use of compensated compactness arguments or velocity averaging lemmas with a full derivative in x (see 4.1). Here we instead use the transport and collapse introduced by Y. Brenier (see for example [7], [51] for the convergence of the method or [29] for other relaxation schemes).

5.2.1 Presentation of the method

For any n we define the function f_n recursively on the intervals $]i/n,\ (i{+}1)/n]$. The approximate solution u_n is then always given by

$$u_n(t,x) = \int_{\mathbb{R}} f_n(t,x,v)\,dv. \tag{5.7}$$

Step 0: Initialization We start with

$$f_n(0,x,v) = \begin{cases} 1 & \text{if } 0 \le v < u^0(t,x), \\ -1 & \text{if } u^0(t,x) < v \le 0, \\ 0 & \text{in the other cases.} \end{cases}$$

Step 1: Transport. Given $f_n(i/n,x,v)$, f_n on $]i/n,\ (i{+}1)/n[$ is the solution to

$$\partial_t f_n + a(v)\cdot\nabla_x f_n = 0,$$

with the corresponding initial data at $t = i/n$. Finally u_n on $]i/n,\ (i+1)/n[$ is given by (5.7). This explicitly gives

$$f_n(t,x,v) = f_n(i/n, x - a(v)\,(t - i/n), v).$$

Notice however that on this interval, one does not necessarily have the constraint (5.2) as there is nothing in the free transport equation to ensure it.

Step 2: Collapse We introduce the non linear collapse operator L on the functions of the variable v by

$$Lf(v) = \begin{cases} 1 & \text{if } 0 \le v < \displaystyle\int_{\mathbb{R}} f(v)\,dv, \\ -1 & \text{if } \displaystyle\int_{\mathbb{R}} f(v)\,dv < v \le 0, \\ 0 & \text{in the other cases.} \end{cases} \tag{5.8}$$

Then one defines

$$f_n((i+1)/n,x,v) = L\,(f_n(i/n, x - a(v)/n, v)) = L\,f_n((i+1)/n-, x, v),$$

where $f_n((i+1)/n-, x, v)$ is the limit of $f_n(t,x,v)$ for $t \to (i+1)/n$ with $t < (i+1)/n$.

Therefore one recovers for all i

$$f_n(i/n,x,v) = \begin{cases} 1 & \text{if } 0 \le v < u_n(i/n,x), \\ -1 & \text{if } u_n(i/n,x) < v \le 0, \\ 0 & \text{in the other cases.} \end{cases} \tag{5.9}$$

Finally let us point out the main property of the collapse operator. For any f with $\sup|f| \le 1$ and any regular function $\phi(v)$ with $\phi'(v) \ge 0$

$$\int_{\mathbb{R}} \phi(v) L\, f(v)\, dv \le \int_{\mathbb{R}} \phi(v)\, f(v)\, dv. \tag{5.10}$$

The proof of this estimate is not given here (see [7]).

5.2.2 Convergence to an entropy solution

In the sense of distribution f_n satisfies

$$\partial_t f_n + a(v) \cdot \nabla_x f_n = g_n, \tag{5.11}$$

with

$$g_n = \sum_{i=1}^{\infty} \delta(t - i/n)\, (f_n(i/n,x,v) - f_n(i/n-,x,v)).$$

Moreover

$$\sup|f_n(0,x,v)| = 1, \qquad \int_{\mathbb{R}^{d+1}} |f_n(0,x,v)\, dx\, dv = \int_{\mathbb{R}^d} u^0(x)\, dx < \infty,$$

and by induction on the intervals $[i/n,\ (i+1)/n]$, for any $t > 0$

$$\|f_n(t,.,.)\|_{L^1(\mathbb{R}^{d+1})} = \|u_n(t,.)\|_{L^1(\mathbb{R}^d)} = \|u^0\|_{L^1}, \qquad \sup_{x,v} |f_n(t,x,v)| = 1.$$

Hence we may extract a converging subsequence, still denoted f_n,

$$f_n \longrightarrow f, \quad w - *L^\infty.$$

And in addition we may use (5.10) to deduce that for any test function $\Phi(x,v)$ with $\partial_v \Phi \ge 0$

$$\int_{\mathbb{R}d+1} \Phi(x,v)\, (f_n(i/n,x,v) - f_n(i/n-,x,v))\, dx\, dv \le 0.$$

Hence there exists a non negative measure $M_{i,n}(x,v)$ such that

$$(f_n(i/n,x,v) - f_n(i/n-,x,v)) = \partial_v M_{i,n}(x,v).$$

Obviously this implies that

$$g_n = \partial_v m_n, \quad m_n \geq 0, \tag{5.12}$$

with

$$m_n(t,x,v) = \sum_{i=1}^{n} \delta(t-i/n)\, M_{i,n}(x,v).$$

Now define Φ_M

$$\begin{aligned} &\Phi_M(v) = v \quad \text{for } |v| \leq M, \\ &\Phi_M(v) = M \quad \text{for } v \geq M, \\ &\Phi_M(v) = -M \quad \text{for } v \leq -M, \end{aligned}$$

Multiplying eq. (5.11) by Φ_M and integrating on $[0,\ T] \times \mathbb{R}^{d+1}$, one gets

$$\int_{\mathbb{R}^{d+1}} \Phi_M(f_n(T,x,v) - f_n(0,x,v))\, dx\, dv = -\int_0^T \int_{\mathbb{R}^{d+1}} \partial_v \Phi_M\, dm_n(t,x,v).$$

So from the L^1 estimate on f_n

$$\int_0^T \int_{-M}^{M} \int_{\mathbb{R}^d} dm_n(t,x,v) \leq 2M \|f_n(t,.,.)\|_{L^1} \leq 2M \|u^0\|_{L^1(\mathbb{R}^d)}.$$

Therefore still extracting a subsequence, we obtain

$$m_n \longrightarrow m, \quad w - *M^1_{loc}$$

with m a non negative measure in $M^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1})$. The limit f then satisfies

$$\partial_t f + a(v) \cdot \nabla_x f = \partial_v m.$$

It remains to show that the constraint (5.2) holds. Assuming that u_n is compact in L^1 then this follows from (5.9) and we are done.

5.2.3 Compactness of u_n

Take a function $\Phi \in C^\infty(\mathbb{R})$ satisfying

$$\Phi(v) = 1 \quad \text{if } |v| \le 1, \quad \Phi(v) = 0 \quad \text{if } |v| \ge 2, \quad 0 \le \Phi(v) \le 1 \quad \forall v.$$

Then define

$$u_n^R = \int_{\mathbb{R}} f_n(t,x,v)\, \Phi(v/R)\, dv.$$

This u_n^R is an average of f_n as defined by Eq. (3.2). Moreover we have

$$\partial_t f_n + a(v) \cdot \nabla_x f_n = \partial_v m_n,$$

with m_n bounded in any $W^{-r,p}([0,\ T] \times \mathbb{R}^d \times [-R,\ R])$ for $r > 0$ and $p < (1 - r/d)^{-1}$ as

$$\|m_n\|_{W^{-r,1}([0,\ T]\times\mathbb{R}^d\times[-R,\ R]} \le C_r \int_{[0,\ T]\times\mathbb{R}^d\times[-R,\ R]} dm_n \le C_r\, R\, \|u^0\|_{L^1}.$$

Next the supremum of f_n is less than 1 so f_n is locally in any L^p and in particular

$$\|f_n\|_{L^2([0,\ T]\times B(0,K)\times[-R,\ R])} \le C\,\sqrt{TKR}.$$

Using Theorem 3.1, one gets that u_n^R belongs to $W^{s,5/3}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ for any $s < 1/5$ with

$$\|u_n^R\|_{W^{s,5/3}([0,\ T]\times B(0,K))} \le C(s,T,K,R), \tag{5.13}$$

and therefore u_n^R is locally compact so that

$$u_n^R \longrightarrow u^R = \int_{\mathbb{R}} f(t,x,v)\, \Phi(v/R)\, dv. \qquad \text{in } L^{5/3}_{loc}. \tag{5.14}$$

Now as $u^0 \in L^1$ there exists an even convex function $\chi \in C^2(\mathbb{R})$ with $\chi(0) = 0$, $\chi(\xi)/|\xi| \longrightarrow +\infty$ as $|\xi| \to +\infty$ and such that

$$\int_{\mathbb{R}^d} \chi(u^0(x))\, dx < \infty.$$

Note that from the definition of f_n this implies that

$$\int_{\mathbb{R}^d\times\mathbb{R}} \chi'(v)\, f_n(t=0,x,v)\, dv\, dx = \int_{\mathbb{R}^d} \chi(u^0(x))\, dx < \infty.$$

Indeed assume that $u^0(x) \geq 0$ (the negative case being the same as χ is even) then

$$\int_{\mathbb{R}} \chi'(v)\, f_n(t=0,x,v)\, dv = \int_0^{u^0(x)} \chi'(v)\, dv = \chi(u^0(x)).$$

Multiplying Eq. (5.11) by χ' and integrating, one gets from (5.12)

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} |\chi'(v)|\, |f_n(t,x,v)|\, dv\, dx = \frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} \chi'(v)\, f_n(t,x,v)\, dv\, dx$$
$$= \int_{\mathbb{R}^d\times\mathbb{R}} g_n\, \chi'\, dx\, dv = -\int_{\mathbb{R}^d\times\mathbb{R}} m_n\, \chi''(v)\, dv\, dx \leq 0.$$

This shows that

$$\int_{\mathbb{R}^d} |u_n - u_n^R|\, dx \leq \int_{\mathbb{R}^d}\int_{|v|\geq R} |f_n(t,x,v)|\, dv$$
$$\leq \frac{1}{|\chi'(R)|}\int_{\mathbb{R}^d\times\mathbb{R}} \chi'\, f_n\, dx\, dv \leq \frac{1}{|\chi'(R)|}\int_{\mathbb{R}^d} \chi(u^0(x))\, dx,$$

and so $u_n - u_n^R$ in L^1 goes to 0 as R tends to infinity, **uniformly** in n. From the compactness of u_n^R (5.14), we deduce the compactness of u_n in L^1_{loc} and we are done.

The heart of the argument here is the compactness provided by averaging lemma. Non optimal averaging lemmas would be enough though.

5.3 Uniqueness and propagation of BV bound

Uniqueness for scalar conservation was first obtained in [34]. We give here a formal argument corresponding to the proof in [43] which uses directly the kinetic formulation.

5.3.1 Uniqueness

Consider two entropy solutions u_1 and u_2 to the scalar law, then

Proposition 5.2 *(L^1 contractivity) We have for any $t > 0$*

$$\|u_1(t,.) - u_2(t,.)\|_{L^1(\mathbb{R}^d)} \leq \|u_1^0 - u_2^0\|_{L^1(\mathbb{R}^d)}.$$

This of course implies the uniqueness of the solution but it does even more than that (see the next subsection).

Denote f_1 and f_2 the two functions defined from u_1 and u_2 by (5.2) and m_1, m_2 the measures in (5.3). For simplicity assume that u_1 and u_2 are non negative and hence so are f_1 and f_2.

First note that as a consequence $f_i^2 = f_i$. The function f_i^2 solves the same equation but multiplying (5.3) by $2f_i$ we also get

$$\partial_t f_i^2 + a(v) \cdot \nabla_x f_i^2 = 2f_i\, \partial_v m_i.$$

Thus

$$2f_i\, \partial_v m_i = \partial_v m_i,$$

and

$$\int_{\mathbb{R}} f_i\, \partial_v m_i\, dv = 0. \tag{5.15}$$

Of course this is only formal. The rigourous argument requires the use of convolution as in [43].

Now use (5.3) for f_1 and f_2 and compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} |f_1 - f_2|^2\, dx\, dv &= \int_{\mathbb{R}^d\times\mathbb{R}} (f_1 - f_2)(\partial_v m_1 - \partial_v m_2)\\ &= -\int_{\mathbb{R}^d\times\mathbb{R}} (f_1\, \partial_v m_2 + f_2\, \partial_v m_1),\end{aligned}$$

by (5.15). As f_i is non increasing

$$\int_{\mathbb{R}^d\times\mathbb{R}} f_1\, \partial_v m_2\, dx\, dv = -\int_{\mathbb{R}^d\times\mathbb{R}} \partial_v f_1 m_2\, dx\, dv \geq 0,$$

and the same is true for the other term. Finally

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}} |f_1 - f_2|^2\, dx\, dv \leq 0.$$

To conclude note that $|f_1 - f_2|$ is equal to 0 if $0 \leq v \leq u_1$ and $0 \leq v \leq u_2$ or if $v > u_1$ and $v > u_2$; it is equal to 1 if $u_1 < v < u_2$ or $u_2 < v < u_1$. Therefore

$$\int_{\mathbb{R}^d\times\mathbb{R}} |f_1 - f_2|^2\, dx\, dv = \int_{\mathbb{R}^d} |u_1 - u_2|\, dx,$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u_1 - u_2|\, dx \leq 0.$$

Again this computation is only formal.

5.3.2 Propagation of BV norm

Take any h in $\mathbb{R}^d$ and apply Prop. 5.2 for a solution $u(t,x)$ and the solution $u(t,x+h)$ which corresponds to the initial data $u^0(x+h)$, it shows that

$$\int_{\mathbb{R}^d} |u(t,x+h)-u(t,x)|\,dx \le \int_{\mathbb{R}^d} |u^0(x+h)-u^0(x)|\,dx,$$

and so

$$\int_{\mathbb{R}^d} \frac{|u(t,x+h)-u(t,x)|}{|h|}\,dx \le \int_{\mathbb{R}^d} |\nabla_x u^0(x)|\,dx. \tag{5.16}$$

Hence as a corollary

Corollary 5.1 *Let u be an entropy solution to* (5.1) *and assume that $u^0 \in BV(\mathbb{R}^d)$ then $u(t,.) \in BV(\mathbb{R}^d)$ and*

$$\|u(t,.)\|_{BV} \le \|u^0\|_{BV}.$$

Note that typically, there is no equality (the inequality is strict). Indeed the equality holds only as long as there is a strong solution.

There are many ways to prove this result. The regularity will typically be shown to hold uniformly for a sequence of approximating solutions. If one considers the sequence f_n obtained through transport-collapse, it is easy to check that

$$\|f_n(t,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))} = \|f_n(i/n+,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))}, \quad \forall t \in [i/n,\ (i+1)/n[,$$

and that the collapse operator contracts the BV norm or

$$\|f_n(i/n+,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))} \le \|f_n(i/n-,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))}.$$

One then gets that

$$\|f_n(t,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))} \le \|f_n(0,.,.)\|_{BV(\mathbb{R}^d,\ M^1(\mathbb{R}))} = \|u^0\|_{BV(\mathbb{R}^d)},$$

and from that an estimate on the BV norm of u_n.

Finally let us point out that the proof through uniqueness given here is interesting because if one goes back to the estimate on f, it bounds

$$\int_{\mathbb{R}^d\times\mathbb{R}} \frac{|f(t,x+h,v)-f(t,x,v)|^2}{|h|}\,dx\,dv,$$

which is not the BV norm of f as it would be natural but looks in fact like the $H^{1/2}$ norm on f. Of course as u is the average of f

$$\|u(t,.)\|_{BV} = \|f(t,.,.)\|_{BV_x(M^1_v)},$$

and this in turn dominates any $H^s_x(L^2_v)$ norm of f with $s < 1/2$ (by interpolation as f is also BV in velocity by its definition). However it is only the very specific form of f which provides the bound **the other way around**. In fact the argument in section 5.3.1 could be used to directly bound

$$\|f\|^2_{H^s_x(L^2_v)} = \int_{\mathbb{R}^{2d}\times\mathbb{R}} \frac{|f(t,x,v) - f(t,y,v)|^2}{|x-y|^{2s+d}}\, dx\, dy\, dv.$$

5.4 Regularization

The first section proves a regularization of the solution in L^p. Then averaging lemmas may be directly applied to get the regularization of Theorem 5.1. We finish by showing Oleĭnik BV regularizing property and with some comments on other (non Sobolev) regularity properties.

5.4.1 Dispersion estimates and L^p regularity

As all other regularizing effects shown here, this one relies only on the properties of free transport. The use of dispersion estimates on the kinetic equation to improve the L^p norm of u was already performed in [38] (see also [42] where a simple illustration is given and [44] or [19] for a more complete treatment but purely for kinetic equations).

Let us first take the simple example

$$\partial_t f + v\cdot\nabla_x f = 0, \quad f(t=0,x,v) = \frac{1}{1+|v|^k}\,\frac{1}{1+|x|^l}.$$

Then the solution is simply

$$f(t,x,v) = \frac{1}{1+|v|^n}\,\frac{1}{1+|x-v\,t|^l}.$$

Therefore although $\int (1+|v|^k)\, f(t=0)\, dv \in L^1_{loc}(\mathbb{R}^d)$ only for $k < n-d$, we have that $\int (1+|v|^k)\, f(t)\, dv \in L^1_{loc}(\mathbb{R}^d)$ for any $t > 0$ and for all $k < n+l-d$.

This shows that the solution enjoys additional decay in velocity, depending on its decay in space.

The same feature is true for the solution f to (5.3) with the additional remark that moments in velocity imply L^p norm for u. In order to simplify the exposition, let us assume that

$$|v|\,|a'(v)| \leq C|a(v)|, \quad C\,|a(v)| \geq |v|^l, \quad \text{for } |v| \text{ large enough.} \tag{5.17}$$

Note that from (5.6), one would expect $l = \theta$ but of course (5.6) does not imply any estimate like (5.17) (which is not strictly necessary in addition). Now we have

Proposition 5.3 *Assume that for some $p \geq 1$, $u^0 \in L^p(\mathbb{R}^d)$, then for any $t > 0$, $u(t,.) \in L^{p+l}_{loc}(\mathbb{R}^d)$ and for any $k > 1$*

$$\int_0^T \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}}{(1+|x|)^k}\,|f(t,x,v)|\,dx\,dv\,dt \leq C \int_{\mathbb{R}^d} |u^0(x)|^p\,dx. \tag{5.18}$$

Proof. By Eq. (5.3) we have

$$f(t,x,v) = f(0, x - a(v)t, v) + \int_0^t \partial_v m(t-s, x - a(v)\,s, v)\,ds$$

Therefore for $t > 0$

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-2}}{(1+|x|)^k} v f(t,x,v)\,dx\,dv\,dt &= \int_0^T \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}}{(1+|x|)^k}\,|f(t,x,v)| \\
= \int_0^T \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}}{(1+|x|)^k} &|f(0, x - a(v)t, v)|\,dx\,dv\,dt \\
+ \int_0^T \int_0^t \int_{\mathbb{R}^{d+1}} &\frac{|v|^{p+l-2}}{(1+|x|)^k}\, v\,\partial_v m(t-s, x-a(v)s, v)\,dx\,dv\,ds\,dt.
\end{aligned}$$

On the one hand, for $k > 1$, by (5.17)

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^{d+1}} &\frac{|v|^{p+l-1}}{(1+|x|)^k} |f(0, x-a(v)t, v)|\,dx\,dv\,dt \\
&= \int_0^T \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}}{(1+|x+a(v)\,t|)^k}\,|f(0,x,v)|\,dx\,dv\,dt \\
&\leq C \int_{\mathbb{R}^d+1} |v|^{p-1} f(0,x,v) \int_0^T \frac{|a(v)|}{(1+|x+a(v)\,t|)^k}\,dt\,dx\,dv \\
&\leq C \int_{\mathbb{R}^d+1} |v|^{p-1} f(0,x,v)\,dx\,dv = C \int_{\mathbb{R}^d} |u^0(x)|^p\,dx.
\end{aligned}$$

On the other hand with a change of variable, and an integration by parts

$$\int_0^T \int_0^t \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-2}}{(1+|x|)^k}\, v\,\partial_v m(t-s, x-a(v)s, v)\,dx\,dv\,ds\,dt$$

$$= \int_0^T \int_0^t \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-2}}{(1+|x+a(v)\,s|)^k}\, v\,\partial_v m(t-s, x, v)\,dx\,dv\,ds\,dt$$

$$= -(p+l)\int_0^T \int_0^t \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}}{(1+|x+a(v)\,s|)^k} m(t-s,x,v)\,dx\,dv\,ds$$

$$-\,k\int_0^T \int_0^t \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}\;a'(v)\cdot(x+a(v)s)}{(1+|x+a(v)\,s|)^{k+1}\times|x+a(v)\,s|}\;m(t-s,x,v)\,dx\,dv\,ds\,dt$$

$$\leq C\,k\int_0^T \int_0^{T-s} \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-2}}{(1+|x+a(v)\,s|)^{k+1}}\;m(r,x,v)\,dx\,dv\,dr\,ds$$

as m is non negative and by (5.17). With the same argument as for $f(0)$ one concludes that

$$\int_0^T \int_0^t \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-2}}{(1+|x|)^k}\, v\,\partial_v m(t-s, x-a(v)s, v)\,dx\,dv\,ds\,dt$$

$$\leq C\,k\int_0^T \int_{\mathbb{R}^{d+1}} |v|^{p-2}\,m(r,x,v)\int_0^T \frac{|a(v)|}{(1+|x+a(v)\,s|)^{k+1}}\,ds\,dx\,dv\,dr$$

$$\leq C\,k\int_0^T \int_{\mathbb{R}^{d+1}} |v|^{p-2}\,m(r,x,v)\,dx\,dv\,dr.$$

Notice that from the estimate on the propagation of the L^p bound for u

$$\int_{\mathbb{R}^d} |u(t,x)|^p\,dx + \int_0^t \int_{\mathbb{R}^{d+1}} |v|^{p-2}\,m(t,x,v)\,dx\,dv\,dt \leq \int_{\mathbb{R}^d} |u^0(x)|^p\,dx,$$

so finally

$$\int_0^T \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}}{(1+|x|)^k}\,|f(t,x,v)|\,dx\,dv\,dt \leq C\int_{\mathbb{R}^d} |u^0(x)|^p\,dx.$$

□

It is in fact conjectured that with suitable assumptions on a, then the solution u becomes immediately bounded. This is still unproved however. If one could prove an estimate like

$$\int_0^T \int_{\mathbb{R}^{d+1}} \frac{|v|^{p+l-1}}{(1+|x|)^k}\,|f(t,x,v)|\,dx\,dv\,dt \leq C\int_{\mathbb{R}^d} \frac{|u^0(x)|^p}{(1+|x|)^{k'}}\,dx,$$

then by boostraping it, it would show that the solution belongs to L^p for any $p < \infty$. But notice that the argument of Prop. 5.3 may precisely only be used once because it requires the full L^p of u^0 and not only a weighted L^p.

5.4.2 Regularization by averaging lemma

Define as before for a regular Φ

$$u^R = \int_{\mathbb{R}} f(t,x,v)\, \Phi(v/R)\, dv.$$

Note that from the definition of f (5.2)

$$\int_{\mathbb{R}} |\partial_v f(t,x,v)|\, dv = 2.$$

Indeed (at least for a continuous u), assuming for simplicity that $u(t,x) \geq 0$

$$\partial_v f(t,x,v) = \delta(v - u(t,x)) - \delta(v),$$

integrating this gives the claimed estimate, which is then easy to extend by density for any $u \in L^1_{loc}$.

So that

$$\|f\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d, BV_{loc}(\mathbb{R}))} \leq C.$$

As on the other hand $\|f\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^{d+1})} = 1$, by interpolation for any $s < 1/2$

$$\|f\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d,\ H^s(\mathbb{R}))} \leq C.$$

Because $\|f\|_{L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^{d+1}))} = \|u\|_{L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^d))}$, with a last interpolation for any $T > 0$

$$\|f\|_{L^2([0,\ T] \times \mathbb{R}^d,\ H^s(\mathbb{R}))} \leq C\,(\|u\|_{L^\infty(\mathbb{R}_+,\ L^1(\mathbb{R}^d)}).$$

Since m is a locally bounded measure, it belongs to $W^{\theta,1}_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1})$ for any $\theta < 0$. Thanks to (5.5), we may apply Theorem 3.1 and get

$$u^R \in W^{s,3/2}_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1}), \quad \forall\, s < 1/3.$$

Now if $u \in L^\infty$ then for $R > \|u\|_{L^\infty}$, $u^R = u$ and

$$u \in W^{s,3/2}_{loc}(\mathbb{R}_+ \times \mathbb{R}^{d+1}), \quad \forall\, s < 1/3. \tag{5.19}$$

This is the regularity given in Theorem 5.1

Proposition 5.4 *Assume that* $u^0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ *and that* u *is the entropy solution to* (5.1). *Then if* (5.5) *holds,* $u \in W^{s,3/2}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ *for any* $s < 1/3$.

If u is only in L^p, then the argument would be more complicated and we only give a sketch. It is necessary to do another interpolation using R as a parameter, namely

$$u = u^R + v^R, \quad v^R = \int_{\mathbb{R}} (1 - \Phi(v/R))\, f\, dv.$$

The control on v^R is simple, for any $T > 0$, $K > 0$,

$$\|v^R\|_{L^1([0,\ T]\times B(0,K))} \leq \int_0^T \int_{B(0,K)} \int_{|v|\geq R} f\, dv\, dx\, dt$$
$$\leq \frac{1}{R^{p-1}} \int_0^T \int_{B(0,K)} \int_{\mathbb{R}} |v|^{p-1} f\, dv\, dx\, dt \leq \frac{1}{R^{p-1}} \|u\|^p_{L^p([0,\ T]\times B(0,K))}.$$

So it would remain to bound the behaviour of $\|u^R\|_{W^{s,3/2}}$ in terms of R. That is more delicate because it depends on the behaviour of $a(v)$ for large v so the final result also depends on this behaviour.

Let us only give the example where $d = 1$ and $v = a(v)$. Define then $f_R(t,x,v) = R\, f(t/R, x, R\, v)$ and $m_R(t,x,v) = R^{-1}\, m(t/R, x, R\, v)$

$$\partial_t f_R + v\, \nabla_x f_R = \partial_t f + Rv\, \nabla_x f = \partial_v m_R,$$

and

$$u^R(t/R, x) = \int_{\mathbb{R}} \Phi(v/R)\, f(t/R, x, v)\, dv = \int_{\mathbb{R}} \Phi(v)\, f_R\, dv.$$

Moreover

$$\|u^R\|^{3/2}_{W^{s,3/2}([0,\ T]\times B(0,K))} \leq \frac{(1+R^{-1})^{3/2}}{R} \|u^R(t/R, x)\|^{3/2}_{W^{s,3/2}([0,RT]\times B(0,K))}$$
$$\leq \sum_{k=1}^{R} \frac{(1+R^{-1})^{3/2}}{R} \|u^R(t/R, x)\|^{3/2}_{W^{s,3/2}([T(k-1),Tk]\times B(0,K))}.$$

By Theorem 3.2, as $[T(k-1),\ Tk]$ is of length T, for $r < 1/2$

$$\|u^R(t/R, x)\|_{W^{s,3/2}([T(k-1),Tk]\times B(0,K))} \leq \|f_R\|^{1-s}_{L^2([T(k-1),Tk]\times B(0,K),\ H^r(\mathbb{R}))}$$
$$\times \|m_R\|^s_{M^1([T(k-1),Tk]\times B(0,K)\times[-2,\ 2])}.$$

Note that as $r < 1/2$

$$\|f_R\|^2_{L^2([T(k-1),\,Tk]\times B(0,K),\ H^r(\mathbb{R}))} \leq R^2 \int_{T(k-1)/R}^{Tk/R} \int_{B(0,K)} \int_{\mathbb{R}} |\partial_v^r f(t,x,v)|^2$$
$$\leq R\,K\,T\,\|f\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R},\ H^r(\mathbb{R}))} \leq C\,R.$$

With the same computation

$$\int_{T(k-1)}^{Tk} \int_{B(0,K)} \int_{-2}^{2} m_R(t,x,v)\,dv\,dx\,dt$$
$$\leq R^{-1} \int_{T(k-1)/R}^{Tk/R} \int_{B(0,K)} \int_{|v|\leq 2R} m(t,x,v)\,dv\,dx\,dt.$$

And from the computation on the propagation of the L^p norm of u (see the corresponding subsection), one deduces that

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |v|^{p-2} m(t,x,v)\,dv\,dx\,dt \leq \|u^0\|.$$

So finally with $\tilde{p} = \min(2,p)$

$$\int_{T(k-1)}^{Tk} \int_{B(0,K)} \int_{-2}^{2} m_R(t,x,v)\,dv\,dx\,dt \leq C\,R^{2-\tilde{p}-1},$$

and

$$\|u^R(t/R,x)\|_{W^{s,3/2}([T(k-1),\,Tk]\times B(0,K))} \leq C\,R^{1/2-s/2+2s-s\tilde{p}-s} = C\,R^{1/2+s/2-s\tilde{p}}.$$

Concluding

$$\|u^R\|_{W^{s,3/2}([0,\ T]\times B(0,K))} \leq C\,R^{1/2-s/2+2s-s\tilde{p}-s} = C\,R^{1/2+s/2-s\tilde{p}},$$

so that using the real method of interpolation described earlier, we get that $u \in W^{r,q}_{loc}$ with $r < (p-1)/(2(p-s\tilde{p}-1/2+s/2))$ and $1/q = (1-2r)+4r/3$.

5.4.3 Oleĭnik BV regularization

It is possible to show that the solution immediately becomes BV in the particular case of a strictly convex flux in dimension 1: $\inf a'(v) > 0$.

The original argument was given in [41] for the vanishing viscosity approximation, with first proving a semi-Lipschitz bound on u. Here we instead use the transport collapse scheme described in subsection 5.2.1.

To simplify assume that

$$a(v) = v, \quad u^0 \geq 0, \quad u^0 \in L^\infty(\mathbb{R}). \tag{5.20}$$

This is not a huge hypothesis as anyway the computation can only be done for strictly increasing $a(v)$.

The following uniform bound holds for the sequences f_n and u_n defined in 5.2.1

Proposition 5.5 *For any $t > 0$, any $R > 0$*

$$\|t\,\partial_x u_n(t,.) - 1\|_{M^1([-R,\ R])} \leq 2R\|u^0\|_{L^\infty} + 2t\,\|u^0\|_{L^\infty}^2.$$

Proof. We argue by induction on every interval $]i/n,\ (i+1)/n]$. Let us start with the first $]0,\ 1/n]$. For $t < 1/n$, f_n is simply the solution to the free transport and hence

$$f_n(t,x,v) = f(0, x - vt, v).$$

So

$$\begin{aligned}
\partial_x u_n(t,x) &= \int_{\mathbb{R}} \partial_x f_n(0, x - vt, v)\,dv \\
&= \int_{\mathbb{R}} (-\frac{1}{t}\partial_v(f_n(0, x - vt, v)) + \frac{1}{t}(\partial_v f_n)(0, x - vt, v))\,dv \\
&= \frac{1}{t}\int_{\mathbb{R}} (\partial_v f_n)(0, x - vt, v)\,dv.
\end{aligned}$$

As such for $0 < t < 1/n$, since $f(0)$ satisfies (5.2)

$$t\,\partial_x u_n(t,x) - 1 = \int_{\mathbb{R}} (\delta(v) - \delta(v - u^0(x - vt)))\,dv - 1 = -\int_{\mathbb{R}} \delta(v - u^0(x - vt))\,dv.$$

Therefore

$$\begin{aligned}
\int_{-R}^{R} |\partial_x u_n(t,x) - 1|\,dx &= \int_{\mathbb{R}} \int_{-R+vt}^{R+vt} \delta(v - u^0(x))\,dx\,dv \\
&\leq \int_{-R-\|u^0\|_{L^\infty}\,t}^{R+\|u^0\|_{L^\infty}\,t} \int_{\mathbb{R}} \delta(v - u^0(x))\,dx\,dv \leq 2\,R\,\|u^0\|_{L^\infty}.
\end{aligned}$$

As u_n is continuous in time at $t = i/n$ (the collapse operator only modifies f_n), the same estimate is true at $t = 1/n$.

Next, assume that the estimate is true at time $t = i/n$. Define

$$g_n(i, x, v) = f_n(i/n+, x + v\, i/n, v),$$

and notice that

$$\partial_v g_n = (\partial_v f_n)(i/n+, x + v\, i/n, v) + \frac{i}{n} \partial_x f_n(i/n+, x + v\, i/n, v).$$

On the other hand for $t \in]i/n,\ (i+1)/n]$

$$u_n(t, x) = \int_{\mathbb{R}} f_n(t, x, v)\, dv = \int_{\mathbb{R}} g_n(i, x - vt, v)\, dv,$$

so with the same argument as before

$$\begin{aligned} \partial_x u_n =& \frac{1}{t} \int_{\mathbb{R}} (\partial_v g_n)(i, x - vt.v)\, dv = \frac{1}{t} \int_{\mathbb{R}} (\partial_v f_n)(i/n+, x + v\,(i/n - t), v) \\ &+ \frac{1}{t}\, \frac{i}{n} \int_{\mathbb{R}} \partial_x f_n(i/n+, x + v\,(i/n - t), v)\, dv. \end{aligned}$$

By the definition of $f_n(i/n+)$ (5.8), one gets the induction relation

$$\begin{aligned} t\, \partial_x u_n - 1 &= \int_{\mathbb{R}} (\delta(v) - \delta(v - u_n(i/n, x + v\,(i/n - t))))\, dv - 1 \\ &+ \frac{i}{n} \int_{\mathbb{R}} \partial_x u_n(i/n, x + v\,(i/n - t), v)\, \delta(v - u_n(i/n, x + v\,(i/n - t)))\, dv \\ &= \int_{\mathbb{R}} (\frac{i}{n} \partial_x u_n(i/n, x + v\,(i/n - t)) - 1)\, \delta(v - u_n(i/n, x + v\,(i/n - t)))\, dv. \end{aligned} \tag{5.21}$$

Consequently for $i/n < t < (i+1)/n$

$$\begin{aligned} \int_{-R}^{R} |t\, \partial_x u_n - 1|\, dx \leq& \int_{-R-(t-i/n)\, \|u^0\|_{L^\infty}}^{R+(t-i/n)\, \|u^0\|_{L^\infty}} \int_{\mathbb{R}} |i/n \partial_x u_n(i/n, x) - 1| \\ & \delta(v - u_n(i/n, x))\, dv\, dx \\ &\leq \int_{-R-(t-i/n)\, \|u^0\|_{L^\infty}}^{R+(t-i/n)\, \|u^0\|_{L^\infty}} |i/n \partial_x u_n(i/n, x) - 1|\, dx \\ &\leq 2(R + (t - i/n)\, \|u^0\|_{L^\infty})\, \|u^0\|_{L^\infty} + \frac{2\,i}{n}\, \|u^0\|_{L^\infty}^2 \\ &\leq 2R\|u^0\|_{L^\infty} + 2\,t\|u^0\|_{L^\infty}, \end{aligned}$$

because we have assumed that $u(i/n, x)$ satisfies the estimate. $\square$

5.4.4 Comments on the regularization effect

In dimension one there is a wide gap between the previous BV regularity and the $1/3$ derivative provided by averaging lemmas. So of course it is natural to wonder whether in higher dimensions (or for non convex fluxes) one could not improve the result of averaging lemmas and possibly reach BV, which for many reasons would be a crucial step.

The answer is not known for entropy solutions. However there is a counterexample if one considers the larger class of solutions with bounded entropy production. Those are u such that the function f defined through (5.2) solves (5.3) with m a bounded measure (but not necessarily non negative). Obviously there is no uniqueness in this case. For those solutions an example detailed in [12] shows that the $1/3$ derivative is optimal.

What is missing is a precise and careful use of the sign of m (this is true as well for the L^p regularization). The kind of techniques that are used for averaging lemmas do not make that easy however, and again it is not sure at all that the regularity can be improved anyway. We refer to [8] for more on the regularity of scalar conservation laws (not necessarily with averaging lemmas).

Regularity in Sobolev spaces is not the only interesting property of solutions. For instance BV regularity is interesting over $W^{1/3,3/2}$ because, in particular, it provides the existence of strong traces of the solution. Instead of trying to get BV bound, one may directly study the traces though. This kind of approach is better able to take advantage of the structure of Eq. (5.3). So for example, strong traces are proved to exist for the solution in [52] (even for solutions with only bounded entropy). More recently it was shown that the solutions enjoy a "BV like" structure (see [10]).

Finally let us mention that kinetic formulations and the corresponding averaging results are not limited to scalar conservation laws: See [39] or [30] for other examples, [49] for a class of hyperbolic equations with possibly degenerate second order terms, or again [42].

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