Identification of the dilute regime in particle sedimentation

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Abstract

We investigate the dynamics of rigid, spherical particles of radius R sinking in a viscous fluid. Both the inertia of the particles and the fluid are neglected. We are interested in a large number N of particles with average distance $d \gg R$. We investigate in which regime (in terms of N and R/d) the particles do not significantly interact and approximately sink like single particles. We rigorously establish the lower bound $N_{crit} \geq C\,(\frac{d}{R})^{3/2}$ for the critical number N_{crit} of particles. This lower bound agrees with the heuristically expected N_{crit} in terms of its scaling in R/d. The main difficulty lies in showing that the particles cannot get significantly closer over a relevant time scale. We use the method of reflection for the Stokes operator to bound the strength of the particle interaction.

1 Introduction

1.1 Motivation of the result

We consider the sedimentation of rigid spherical particles of the same radius in a fluid. The particles interact through the fluid: When one particle moves, it generates a fluid flow which acts on all the other particles.

We neglect the inertia of both particles and fluid. In particular, the fluid flow is quasi stationary and described by the incompressible Stokes system with no–slip boundary condition at the particles' surface. Hence the dynamics are driven by ρ, the difference in weight density between the particles and the fluid, and e, the gravity field. They are limited by the viscosity μ of the fluid. The quasi stationarity assumption on the fluid flow means that the fluid immediately adapts itself to the situation created by the particles' positions and their velocities: It does not "remember" anything about the past of the dynamics. We also neglect the rotation of the particles.

We always assume that the Stokes fluid is at rest at infinity, meaning that the fluid velocity u satisfies $u \to 0$ for $|x| \to \infty$. In such an environment, a single particle of radius R sinks with a velocity

$$V_{single} \ := \ \frac{\rho}{6\,\pi\,\mu}\,R^2\,|e|. \tag{1.1}$$

A small number N of distant particles will only show little interaction; they will sink like a single particle. Hence the velocity V_{cloud} of such a cloud of particles is approximately equal to V_{single}:

$$V_{cloud} \ \approx \ V_{single}.$$

We call this the "non–interacting scenario". But if the number N of particles is large, their interaction may no longer be a small perturbation. In such a regime, the fact that the fluid at rest at infinity (that is, $u \to 0$ for $|x| \to \infty$) may be "screened" from the particles in the interior of the cloud. Hence it is plausible that a macroscopic fluid flow is generated, which makes the cloud sink faster. We call this the "screening scenario". Our goal is to identify the cross–over between these two regimes. The critical number of particles N_{crit} will depend on their radius R and their average distance d, more precisely, on the non-dimensional ratio R/d. In this paper, we investigate the scaling of N_{crit} in R/d. The main result is a rigorous lower bound for N_{crit} as a function of R/d, see Theorem 1.1. We now will give a heuristic argument that this bound is optimal in terms of scaling.

Let us perform a little Gedankenexperiment: The screening scenario is mimicked by treating the cloud of particles as a single "meta–particle". This meta–particle would have diameter $\tilde{R} \sim N^{1/3}\,d$ and a difference $\tilde{\rho} \sim \rho\,(\frac{R}{d})^3$ in density with respect to the fluid. Hence it would sink with a velocity which scales as

$$|\tilde{V}_{cloud}| \;\sim\; \frac{\tilde{\rho}}{\mu}\,\tilde{R}^2|e| \;\sim\; \frac{\rho}{\mu}\,N^{2/3}\,\frac{R^3}{d}|e|. \tag{1.2}$$

Folklore suggests that the screening scenario should prevail over the non–interacting scenario, if and only if the velocity $\tilde{V}_{cloud}$ is larger than the velocity V_{cloud}. Hence one obtains from (1.1) and (1.2) the following guess for the cross–over number N_{crit}

$$N_{crit} \;:=\; \left(\frac{d}{R}\right)^{3/2}. \tag{1.3}$$

Unfortunately, by this logic, we would have to choose yet another scenario, the "clustering scenario", where the particles form (possibly transient) clusters within the cloud. The extreme case of the clustering scenario is mimicked by a single "mega–particle". It would have diameter $R_* \sim N^{1/3}\,R$ and hence sink with a velocity which scales as

$$|V_*| \;\sim\; \frac{\rho}{\mu}\,{R_*}^2|e| \;\sim\; \frac{\rho}{\mu}\,N^{2/3}\,R^2|e|,$$

beating the velocity of the non–interacting scenario V_{cloud}. Nonetheless, we will show in this paper that as long as

$$N \;\ll\; N_{crit},$$

with N_{crit} defined as in (1.3), the particles do not interact and all sink with approximately the velocity V_{single} of a single particle.

In view of the clustering scenario, it is not surprising that the main issue is to rule out that particles get too close to each other. Let us make this more precise and denote by $d_{min}(t)$ the minimal distance between the particles at time t. We will assume that initially, all particles are well–separated in the sense of

$$R \;\ll\; d_{min}(0).$$

The main effort consists in establishing that no clustering occurs over the relevant time scale

$$R \;\ll\; d_{min}(t) \quad \text{for } t = O(\tau).$$

What is the relevant time scale? It is the time τ it takes for a single particle to sink a distance $N^{1/3}\,d$ of the order of the cloud diameter. More precisely, we set

$$\tau \;:=\; \frac{N^{1/3}\,d_{min}(0)}{|V_{single}|}. \tag{1.4}$$

We also denote by Λ the critical parameter

$$\Lambda(t) \ := \ \frac{R\,N^{2/3}}{d_{min}(t)}. \tag{1.5}$$

The main result of this paper is

Theorem 1.1 *There exists a universal constant $C < \infty$ such that if*

$$C\,R \ \le \ d_{min}(0), \quad N \ \ge \ C, \quad \textit{and} \quad C\,\Lambda(0) \le 1$$

we have, where V_k denotes the velocity of particle k,

$$\begin{aligned} d_{min}(t) \ &\ge \ \frac{1}{2}\,d_{min}(0), \\ \sup_k |V_k(t) - V_{single}| \ &\le \ C\,\Lambda(0)\,|V_{single}| \end{aligned}$$

for all t with

$$0 \ \le \ t \ \le \ C^{-1}\,\Lambda^{-2}(0)\,\tau.$$

Theorem 1.1 shows that the non-interacting scenario prevails for $\Lambda \ll 1$ as heuristically expected.

1.2 The underlying dynamics

Let us now give the precise equations of the dynamics. Let $i = 1, \cdots, N$ be an enumeration of the particles. Each particle is assumed to be a spherical ball of radius R and center X_i, moving with the velocity V_i; we denote the ball by B_i.
At a given time t, the fluid flow is described by its pressure field $p \in \mathbb{R}$, its velocity field $u \in \mathbb{R}^3$ and its stress field $\sigma \in \mathrm{Sym}(\mathbb{R}^3)$ related by

$$\sigma_{\alpha,\beta} \ = \ -p\,\delta_{\alpha,\beta} + \mu\,(\partial_\alpha u_\beta + \partial_\beta u_\alpha). \tag{1.6}$$

The flow is assumed to be incompressible, to be at rest at infinity:

$$\nabla \cdot u \ = \ 0 \quad \text{in } \mathbb{R}^3 - \bigcup_i B_i \quad \text{and} \quad u \ \to \ 0 \quad \text{for } |x| \uparrow \infty, \tag{1.7}$$

and to satisfy the no–slip boundary condition at the particle interfaces

$$u \ = \ V_i \quad \text{on } \partial B_i \quad 1 \le i \le N. \tag{1.8}$$

Finally, we postulate quasi stationary balance between viscous and gravitational forces in the bulk and at the interface

$$\nabla \cdot \sigma \;=\; 0 \quad \text{in } \mathbb{R}^3 - \bigcup_i B_i \quad \text{and} \quad \int_{\partial B_i} \sigma \cdot n \, dS + \rho R^3 e \;=\; 0, \; 1 \le i \le N, \tag{1.9}$$

where n denotes the outer normal of B_i. Notice that $\nabla \cdot \sigma$ turns into the Stokes equation for u and p.

The set of assumptions (1.6), (1.7), (1.9) & (1.8) are supposed to be a good model of the real dynamics if the particles are small enough so that the Reynolds number of the fluid is much less than 1 but not too small so as to neglect Brownian motion, and if the density of the particles is comparable to the density of the fluid (so the inertia of the particles is of the same order as the one of the fluid and hence negligible too). This model is used for the computation of sedimentation velocity (see [1], [6] or [9] for instance).

These dynamics are mathematically well–behaved as long as particles do not get too close. In fact, B. Desjardins and M. Esteban have shown that the dynamics — even with Navier-Stokes replacing quasi stationary Stokes — have a solution until the time of first collision between particles (see [5] and [14] for numerical simulations in this case). Notice however that it is not known whether collisions actually occur in this model. Even worse, the assumptions on which the model is based are no longer physically valid in a spatial and temporal neighborhood of a collision. In [11], it is shown that in two space dimensions, a particle cannot "bump" into a boundary. This indicates that in two space dimensions, collisions must occur at zero velocity. This corresponds to the result of [18] where a weak solution of the dynamics is defined if there is only one particle. But of course avoiding collisions is not enough for the asymptotics in N.

For a large number of particles, solving (1.6), (1.7), (1.9) & (1.8) becomes numerically prohibitive. It is therefore crucial to investigate whether certain macroscopic aspects of the particle distribution can be predicted without solving the underlying microscopic dynamics. For instance, one could aim at deriving an evolution of the macroscopic number density of the particles (in the physical space when the inertia of the particles is neglected and in the phase space when it is not). J. Rubinstein and J.B. Keller have thus studied a limit toward a macroscopic system for particles without inertia (see [16] and also [4] for a study of a limit system close to this one but with Brownian motion). If the inertia of the particles is not neglected, it is possible to derive kinetic equations (see [13]). A modeling by kinetic equations has also been studied in the case of a perfect fluid with a potential flow (see [10] and [17]).

Unfortunately, all this limits are formal. In general, we know how to pass from the microscopic dynamics to a macroscopic evolution only when the interaction has no singularity. For kinetic equations, the Boltzmann-Grad limit has been proved by R. Illner and M. Pulvirenti in [12] (see also [3]) but only for a short time. Concerning Vlasov equations, the convergence is proved only for regularized forces (see the book by H. Spohn [19] for instance). This question is also connected with the convergence of particle methods (for the Vlasov-Poisson equation, more details can be found in [2], [20] and [21]).

However for macroscopic systems (when the velocity of one particle depends only on its position), the question appears to be easier; for instance, the limit of point vortex method to the Euler equations has been proved in dimension 2 in [7]. We see Theorem 1.1 as a first step towards rigorously deriving an evolution of the macroscopic number density. For instance, Theorem 1.1 identifies the meaning of “macroscopic”: Sedimentation is self–averaging on length-scales up to $N_{crit}^{1/3}\, d = \frac{d^{3/2}}{R^{1/2}}$; hence the macroscopic number density should be defined with respect to this critical length-scale. Moreover, Theorem 1.1 gives us hope that particle collisions can be ruled out even in the limit of infinite interacting particles over the relevant time scales.

1.3 The structure of the dynamics

The system formed of (1.6), (1.7), (1.9) & (1.8) defines an evolution of the particle centers $\{X_i \in \mathbb{R}^3\}_i$. The dynamics can be written as

$$\dot{X}_i = V_i(X_1, \ldots, X_N), \quad 1 \le i \le N,$$

where the velocity V_i of each particle is only determined by the configuration at the same time, $i.e$ all the positions X_j.

The equations (1.6), (1.7), (1.9) & (1.8) are linear in the velocities V_i. Therefore, it is convenient to introduce the following tensor $\{g_{ij}(X_1, \ldots, X_N) \in \mathrm{Sym}(\mathbb{R}^3)\}_{i,j}$, $1 \le i, j \le N$; For any $U \in \mathbb{R}^3$, $g_{ij}\, U \in \mathbb{R}^3$ is defined through

$$g_{ij}\, U \; = - \int_{\partial B_i} \sigma\, n,$$

where (p, u, σ) denotes the solution of (1.6), (1.7) and

$$\nabla \cdot \sigma \;=\; 0 \quad \text{in } \mathbb{R}^3 - \textstyle\bigcup_k B_k$$
$$u \;=\; U \quad \text{on } \partial B_j \quad \text{and} \quad u \;=\; 0 \quad \text{on } \partial B_k \quad \text{for all } k \neq j.$$

We observe that the tensor $\{g_{ij} \in \mathrm{Sym}(\mathbb{R}^3)\}_{i,j}$ is symmetric and positive definite on $\mathbb{R}^{3\,N}$. Indeed, for $\{U_i^1\}_i$ and $\{U_i^2\}_i$ we have

$$\sum_{1 \le i,j \le N} U_i^1 \cdot g_{ij}\, U_j^2 \;=\; 2\,\mu \int_{\mathbb{R}^3 - \bigcup_i B_i} \sum_{1 \le \alpha,\beta \le 3} \partial_\alpha u_\beta^1\, \partial_\alpha u_\beta^2\, dx,$$

where u^1 and u^2 are the solutions of (1.6), (1.7), (1.9) & (1.8) with $\{V_i\}_i$ replaced by $\{U_i^1\}_i$ resp. $\{U_i^2\}_i$.
With this notation, the velocities V_i are given by the solution of the linear system

$$\sum_{j=1}^{N} g_{ij}\, V_j \;=\; F_i, \quad 1 \le i \le N,$$

where the right hand side vector $\{F_i\}_i$ is the constant gravity force on each particle

$$F_i \;:=\; \rho\, R^3\, e, \quad 1 \le i \le N.$$

Notice that the dynamics have the structure of a gradient flow with respect to the metric tensor $\{g_{ij}\}_{i,j}$ and the gravity potential. It is clear that one has to control the metric tensor $\{g_{ij}\}_{i,j}$ as a function of the particle positions $\{X_i\}_i$ in order to control the dynamics. The control required is summarized in Proposition 1.1. We recall the notation

$$d_{ij} \;:=\; |X_i - X_j|, \quad d_{min} \;=\; \min_{i \neq j} d_{ij} \quad \text{and} \quad \lambda \;=\; \frac{R N^{1/3}}{d_{min}}.$$

In order to avoid writing generic universal constants, we from now on use the notation $\lesssim$ and $\ll$.

Proposition 1.1 *Provided*

$$R \;\ll\; d_{min} \quad \textit{and} \quad \Lambda \;\ll\; 1$$

we have

$$|g_{11} - 6\pi\mu R\,\mathrm{id}| \;\lesssim\; \mu R \frac{R^2 N^{1/3}}{d_{min}^2}, \tag{1.10}$$

$$|g_{12}| \;\lesssim\; \mu R \frac{R}{d_{12}}, \tag{1.11}$$

$$|g_{13} - g_{23}| \;\lesssim\; \mu R d_{12} \left((\frac{R}{d_{13}^2} + \frac{R}{d_{23}^2}) + (\frac{1}{d_{13}} + \frac{1}{d_{23}}) \frac{R^2 N^{1/3}}{d_{min}^2} \right). \tag{1.12}$$

2 From Proposition 1.1 to Theorem 1.1

We first give three lemmas which enable us to control the velocity of each particle and then we prove Theorem 1.1.

2.1 Control on the particles' velocities

We begin with a completely technical lemma which is also used in the last section of the paper

Lemma 2.1 *For any exponent $k \in [0,\ 2]$, we have*

$$\sum_{j \neq i} \frac{1}{d_{ij}^k} \;\lesssim\; \frac{N^{1-k/3}}{d_{min}^k}. \tag{2.1}$$

Lemma 2.2 *Provided $R \ll d_{min}$ and $\Lambda \ll 1$ we have*

$$\sup_k |V_k - V_{single}| \;\lesssim\; \Lambda\, |V_{single}|. \tag{2.2}$$

Lemma 2.3 *Provided $R \ll d_{min}$ and $\Lambda \ll 1$ we have for all $i \neq j$*

$$|V_i - V_j| \;\lesssim\; \frac{R\, N^{1/3}}{d_{min}^2} \sup_k |V_k|\, d_{ij}. \tag{2.3}$$

Proof of Lemma 2.1. Since the property we want to prove is invariant by translation, we may suppose that $i = 1$ and $X_1 = 0$ without any loss of generality. After that, we rearrange the indices such that the sequence $|X_1|, ..., |X_N|$ is non decreasing.

Since, for any $i \neq j$, $|X_i - X_j| \geq d$, the balls $B(X_i, d_{min}/2)$ and $B(X_j, d_{min}/2)$ do not intersect. Moreover for any i between 2 and n, we know that

$$\bigcup_{j=1}^{i} B(X_j, d_{min}/2) \subset B(0, d_{min}/2 + \sup_{j=1,..,i} |X_j|) \subset B(0, 2|X_i|).$$

Comparing the two volumes, we find that

$$i \left(\frac{d_{min}}{2} \right)^3 \leq (2|X_i|)^3,$$

or

$$|X_i| \geq \frac{1}{4} d_{min} i^{1/3}.$$

We finish the proof with an easy comparison of the sum with an integral

$$\begin{aligned} \sum_{i=2}^{N} \frac{1}{|X_1 - X_i|^k} &\leq \frac{4^k}{d_{min}^k} \sum_{i=2}^{N} i^{-k/3} \leq \frac{4^k}{d_{min}^k} \int_0^N x^{-k/3} dx \\ &\leq \frac{4^k}{d_{min}^k} \left[\frac{X^{1-k/3}}{1-k/3} \right]_0^N \leq \frac{4^k}{1-k/3} \times \frac{N^{1-k/3}}{d_{min}^k} \leq 48 \frac{N^{1-k/3}}{d_{min}^k}. \end{aligned}$$

$\square$

Proof of Lemma 2.2. We start by observing that for any i, we have on one hand

$$\sum_k g_{ik} V_k - 6\pi\mu R V_{single} = F_i - \rho R^3 e = 0,$$

and on the other hand

$$\begin{aligned} \sum_k g_{ik} V_k - 6\pi\mu R V_{single} &= (6\pi\mu R)(V_i - V_{single}) \\ &+ (g_{ii} - 6\pi\mu R) V_i \\ &+ \sum_{k \neq i} g_{ik} V_k. \end{aligned}$$

Therefore,

$$6\pi\mu R |V_i - V_{single}| \leq |g_{ii} - 6\pi\mu R \,\mathrm{id}| \times \sup_k |V_k| + \sum_{k \neq i} |g_{ik}| \times \sup_k |V_k|. \quad (2.4)$$

According to Prop. 1.1 we have

$$\begin{aligned} |g_{ii} - 6\,\pi\,\mu\,R\,\mathrm{id}| &\lesssim \mu\,R\,\frac{R^2\,N^{1/3}}{d_{min}^2}, \\ \sum_{k\neq i} |g_{ik}| &\lesssim \mu\,R^2 \sum_{k\neq i} \frac{1}{d_{ik}}, \end{aligned}$$

which yields with Lemma 2.1

$$\sum_{k\neq i} |g_{ik}| \lesssim \mu R \frac{RN^{2/3}}{d_{min}} \lesssim \mu R \Lambda.$$

We may consequently deduce from (2.4)

$$6\pi\mu R\,|V_i - V_{single}| \lesssim \mu R \left(\frac{R^2\,N^{1/3}}{d_{min}^2} + \Lambda \right) \sup_k |V_k|.$$

Note also that

$$\frac{R^2 N^{1/3}}{d_{min}^2} = \frac{RN^{-1/3}}{d_{min}} \times \frac{RN^{2/3}}{d_{min}} \leq \frac{R}{d_{min}} \Lambda \leq \Lambda,$$

so eventually,

$$6\pi\mu R\,|V_i - V_{single}| \lesssim \mu R\,\Lambda \sup_k |V_k| \lesssim \mu R\,\Lambda \sup_k |V_k - V_{single}| + \mu R\,\Lambda\, V_{single}.$$

Taking the supremum in i, we obtain

$$\sup_i |V_i - V_{single}| \lesssim \Lambda \sup_i |V_i - V_{single}| + \Lambda |V_{single}|.$$

This yields Lemma 2.2 since $\Lambda \ll 1$.

□

Proof of Lemma 2.3. The argument is almost the same; For any $i \neq j$, we write

$$\sum_k g_{ik}\,V_k \;-\; \sum_k g_{jk}\,V_k \;=\; F_i - F_j \;=\; 0,$$

whereas we also have

$$\begin{aligned}
\sum_k g_{ik}\,V_k - \sum_k g_{jk}\,V_k &= (6\,\pi\,\mu\,R)\,(V_i - V_j) \\
&+ (g_{ii} - 6\,\pi\,\mu\,R)\,V_i - (g_{jj} - 6\,\pi\,\mu\,R)\,V_j \\
&+ g_{ij}\,V_j - g_{ji}\,V_i \\
&+ \sum_{k\neq i, k\neq j} (g_{ik} - g_{jk})\,V_k.
\end{aligned}$$

Thus, as in the previous lemma

$$\begin{aligned}
6\,\pi\,\mu\,R\,|V_i - V_j| &\leq \Big\{2\,\sup_k |g_{kk} - 6\,\pi\,\mu\,R\,\mathrm{id}| + 2\,|g_{ij}| \\
&+ \sum_{k\neq i, k\neq j} |g_{ik} - g_{jk}|\Big\}\,\sup_k |V_k|. \qquad (2.5)
\end{aligned}$$

We apply Proposition 1.1:

$$\begin{aligned}
\sup_k |g_{kk} - 6\,\pi\,\mu\,R\,\mathrm{id}| &\lesssim \mu\,R\,\frac{R^2\,N^{1/3}}{d_{min}^2} \leq \mu\,R\,\frac{R}{d_{min}}\,\frac{R\,N^{2/3}}{d_{min}} = \mu\,R\,\frac{R}{d_{min}}\,\Lambda, \\
|g_{ij}| &\lesssim \mu\,R\,\frac{R}{d_{ij}} \lesssim \mu\,R\,\frac{R}{d_{min}}, \\
\sum_{k\neq i, k\neq j} |g_{ik} - g_{jk}| &\lesssim \mu\,R\,\Big\{d_{ij}\,R\,(\sum_{k\neq i}\frac{1}{d_{ik}^2} + \sum_{k\neq j}\frac{1}{d_{jk}^2}) \\
&\quad + d_{ij}\,\frac{R^2\,N^{1/3}}{d_{min}^2}\,(\sum_{k\neq i}\frac{1}{d_{ik}} + \sum_{k\neq j}\frac{1}{d_{jk}})\Big\}.
\end{aligned}$$

According to Lemma 2.1, the last estimate turns into

$$\begin{aligned}
\sum_{k\neq i, k\neq j} |g_{ik} - g_{jk}| &\lesssim \mu\,R\,d_{ij}\,\left\{\frac{R\,N^{1/3}}{d_{min}^2} + \frac{R^2\,N}{d_{min}^3}\right\} \\
&= \mu\,R\,d_{ij}\,\left\{\frac{R\,N^{1/3}}{d_{min}^2} + \frac{R\,N^{1/3}}{d_{min}^2} \times \frac{R\,N^{2/3}}{d_{min}}\right\} \\
&= \mu\,R\,d_{ij}\,\frac{R\,N^{1/3}}{d_{min}^2}\,(1 + \Lambda).
\end{aligned}$$

Inserting these three estimates into (2.5) yields

$$|V_i - V_j| \lesssim \left\{ \frac{R}{d_{min}} (1+\Lambda) + d_{ij} \frac{R\, N^{1/3}}{d_{min}^2} (1+\Lambda) \right\} \sup_k |V_k|,$$

and we conclude by observing that $\Lambda \ll 1$ and

$$d_{ij} \frac{R\, N^{1/3}}{d_{min}^2} \geq \frac{R\, N^{1/3}}{d_{min}} \geq \frac{R}{d_{min}}.$$

□

2.2 Proof of Theorem 1.1

Let $C < \infty$ denote the maximum generic constant in the estimates of Lemmas 2.2 and 2.3. Let t_* be such that

$$d_{min}(t) \geq \frac{1}{2} d_{min}(0) \quad \text{for } 0 \leq t \leq t_*.$$

Then we have $\Lambda(t) \leq 2\Lambda(0)$ for $o \leq t \leq t_*$ and therefore, provided $\Lambda(0)$ was small enough, we may apply all previous results up to time t_*.
According to Lemma 2.2, we have

$$\sup_k |V_k(t) - V_{single}| \lesssim \Lambda(0)\, |V_{single}|,$$

and hence in particular, using $\Lambda(0) \ll 1$,

$$\sup_k |V_k(t)| \lesssim |V_{single}|,$$

Therefore, we obtain from Lemma 2.3

$$|V_i(t) - V_j(t)| \lesssim \frac{R^2\, N}{d_{min}(0)^3} |V_{single}|\, d_{ij}(t),$$

Let C denote the universal constant implicit in the above estimate, i. e.

$$|V_i(t) - V_j(t)| \leq C \frac{R^2\, N}{d_{min}(0)^3} |V_{single}|\, d_{ij}(t),$$

Since

$$\frac{d}{dt} d_{ij}(t) \geq -|V_i(t) - V_j(t)|,$$

this entails

$$d_{ij}(t) \;\geq\; \exp\left(-C\,\frac{R^2\,N}{d_{min}(0)^3}\,|V_{single}|\,t\right)\,d_{ij}(0)$$

or

$$d_{min}(t) \;\geq\; \exp\left(-C\,\frac{R^2\,N}{d_{min}(0)^3}\,|V_{single}|\,t\right)\,d_{min}(0)$$

for $0 \leq\; t \;\leq\; t_*$. We conclude by observing that

$$\exp\left(-C\,\frac{R^2\,N}{d_{min}(0)^3}\,|V_{single}|\,t_*\right) \;\geq\; \frac{1}{2}$$

is ensured by

$$C\,\frac{R^2\,N}{d_{min}(0)^3}\,|V_{single}|\,t_* \;\leq\; 1.$$

On the other hand, by definition of τ,

$$\frac{R^2\,N}{d_{min}(0)^3}\,|V_{single}| \;=\; \Lambda^2(0)\,\frac{1}{\tau}.$$

This shows that we can choose t_* at least of order $\Lambda^{-2}(0)\,\tau$.

3 Proof of Proposition 1.1

We begin with a technical lemma, then we present the method we use to prove Proposition 1.1.

3.1 The single-particle solution

The main idea is to express the solution operator for the multiple–particle Stokes problem in terms of the *single*–particle solution operator T_j

$$T_j\colon L_0^\infty(\partial B_j) \to L^\infty(\mathbb{R}^3 - B_j).$$

The single–particle solution operator T_j is defined as follows: If U is a bounded velocity field on ∂B_j with zero average normal component (this defines the space $L_0^\infty(\partial B_j)$), then $u = T_jU$ is the solution of the Stokes equations in $\mathbb{R}^3 - B_j$ with Dirichlet boundary condition U on ∂B_j. The Ansatz of writing the solution operator of an elliptic equation with a composite Dirichlet boundary in terms of the solution operators for the individual components is known as the "method of reflections". It capitalizes on the fact that the solution operator for the individual component is well–understood. In our case, T_j satisfies the following estimates, where $\|\cdot\|_{\partial B_j}$ denotes the L^∞ norm on ∂B_j:

Lemma 3.1 *If $R \ll d_{min}$*

$$\begin{aligned}
\|T_j U\|_{\partial B_i} &\lesssim \frac{R}{d_{ij}}\|U\|_{\partial B_j} \quad \textit{for } i \neq j, && (3.1)\\
\|T_j U - (T_j U)(\cdot + X_k - X_i)\|_{\partial B_i} &\lesssim d_{ik}\,(\frac{R}{d_{ij}^2} + \frac{R}{d_{jk}^2})\,\|U\|_{\partial B_j}, \ \textit{for } i\neq j, i \neq k, &&\\
|\int_{\partial B_j} \sigma(T_j U)\cdot n dS| &\lesssim \mu R \|U\|_{\partial B_j}. && (3.2)
\end{aligned}$$

Proof of Lemma 3.1. This result is shown with the explicit integral formula for u and p. Since the problem is symmetric by translation, we may suppose that $j = 1$ with $X_1 = 0$. As it is explained in [15], we have

$$\begin{aligned}
&u_\alpha(x) = \tfrac{1}{8\pi\mu}\textstyle\int_{\partial B_1} U_\beta(y)\left(\mu\tfrac{\partial T_{\beta\alpha}}{\partial n} - P_\alpha n_\beta\right) dS(y),\\
&p(x) = \tfrac{1}{4\pi}\textstyle\int_{\partial B_1} U_\alpha\left(\mu\tfrac{\partial S_\alpha}{\partial n} + q n_\alpha\right) dS(y).
\end{aligned}$$

Let us denote

$$\begin{aligned}
&r_0 = |x|, \quad \bar{x} = \tfrac{R^2}{r_0^2}x, \quad \bar{r}_0 = |\bar{x}|,\\
&z = y - x, \quad r = |z|, \quad \bar{z} = y - \bar{x}, \quad \bar{r} = |\bar{z}|.
\end{aligned}$$

We can now give the formula for T

$$\begin{aligned}
T_{\alpha\beta} &= \tfrac{\delta_{\alpha\beta}}{r} + \tfrac{z_\alpha z_\beta}{r^3} - \tfrac{R}{r_0\bar{r}}\delta_{\alpha\beta} - \tfrac{R^3}{r_0^3}\tfrac{\bar{z}_\alpha\bar{z}_\beta}{\bar{r}^3} - \tfrac{r_0^2-R^2}{r_0}\Big(\tfrac{\bar{x}_\alpha\bar{x}_\beta}{R^3\bar{r}}\\
&-\tfrac{R}{r_0^2\bar{r}^3}\left(\bar{x}_\alpha\bar{z}_\beta + x_\beta\bar{z}_\alpha\right) - 2\tfrac{\bar{x}_\alpha\bar{x}_\beta}{R^3}\bar{x}\cdot\nabla_y\tfrac{1}{\bar{r}}\Big) - (|y|^2 - R^2)\tfrac{\partial\phi_\beta}{\partial y_\alpha},
\end{aligned}$$

whereas for P and ϕ

$$\begin{aligned}
&P_\alpha = 2\mu\tfrac{z_\alpha}{r^3} - 2\mu\tfrac{R^3}{r_0^3}\tfrac{\bar{z}}{\bar{r}^3} - 2\mu\phi_\alpha - 4\mu y\cdot\nabla_y\phi_\alpha,\\
&\phi_\alpha = \tfrac{r_0^2-R^2}{2r_0^3}\left(\tfrac{3x_\alpha}{R\bar{r}} + \tfrac{R\bar{z}_\alpha}{\bar{r}^3} + 2\tfrac{x_\alpha}{R}\bar{x}\cdot\nabla_y\tfrac{1}{\bar{r}} + 3\tfrac{R}{\bar{r}_0}\tfrac{\partial}{\partial\bar{x}_\alpha}\log\tfrac{\bar{r}_0\bar{r}+y\cdot\bar{x}-r_0^2}{|y|\bar{r}_0+y\cdot\bar{x}}\right).
\end{aligned}$$

And finally the formulas for S, and q are

$$\begin{aligned}
&S_\alpha = \tfrac{\partial}{\partial y_\alpha}\tfrac{1}{r} - \tfrac{R}{r_0^3}\tfrac{x}{\bar{r}} - \tfrac{R^3}{r_0^3}\tfrac{\partial}{\partial y_\alpha}\tfrac{1}{\bar{r}} + \tfrac{2\bar{x}_\alpha}{Rr_0}\bar{x}\cdot\nabla_y\tfrac{1}{\bar{r}} - (|y|^2 - R^2)\tfrac{\partial\psi}{\partial y_\alpha},\\
&q = 2\mu\left(\psi + 2y\cdot\nabla_y\psi\right),
\end{aligned}$$

with
$$\psi = \frac{1}{2Rr_0}\left(\frac{3}{\bar r} + 2\bar x\cdot\nabla_y\frac{1}{\bar r} + \frac{3}{\bar r_0}\log\frac{\bar r_0\bar r + y\cdot\bar x - r_0^2}{|y|\bar r_0 + y\cdot\bar x}\right).$$
With these expressions, it is easy (but lengthy) to check that the kernels of the two convolutions are quite regular away from ∂B_1, thus proving the lemma.

□

3.2 The multiple-particle solution

In order to express the multiple–particle solution operator in terms of the T_j's, it is convenient to introduce the operator
$$A\colon \Pi_{j=1}^N L_0^\infty(\partial B_j) \longrightarrow \Pi_{j=1}^N L_0^\infty(\partial B_j),$$
which is defined through
$$A_{ij}U = \left\{\begin{array}{cc} (T_j(U_{|\partial B_j}))_{|\partial B_i} & \text{for } j\neq i \\ U & \text{for } j=i \end{array}\right\}. \tag{3.3}$$
The next lemma collects the estimates on A and A^{-1} we later need. We denote by A^{-1}_{ij} the component i,j of the inverse of A (and **not** the inverse of A_{ij}).

Lemma 3.2 *A^{-1} exists for $\Lambda \ll 1$, $R \ll d_{min}$, and we have*
$$\begin{aligned}
\|A_{ij}U\|_{\partial B_i} &\lesssim \frac{R}{d_{ij}}\|U\|_{\partial B_j} \quad \textit{for } j\neq i, && (3.4)\\
\|A^{-1}_{ij}U\|_{\partial B_i} &\lesssim \frac{R}{d_{ij}}\|U\|_{\partial B_j} \quad \textit{for } j\neq i, && (3.5)\\
\|A^{-1}_{ii}U - U\|_{\partial B_i} &\lesssim \frac{R^2N^{1/3}}{d_{min}^2}\|U\|_{\partial B_i}, && (3.6)\\
\textit{in particular } \|A^{-1}_{ii}U\|_{\partial B_i} &\lesssim \|U\|_{\partial B_i}. && (3.7)
\end{aligned}$$

We now may express the multiple–particle solution operator and thus also the metric tensor g_{ij} in terms of the single–particle solution operator T_i (and the related A_{ij}). We recall that
$$\int_{\partial B_j}\sigma(T_iU)\cdot n dS = 0 \quad \text{for } j\neq i, \tag{3.8}$$

and

$$\int_{\partial B_i} \sigma(T_i U)\cdot n dS = 6\pi\mu R U \quad \text{for constant } U. \tag{3.9}$$

For a constant vector U, we mean by the notation $A_{ij}U$ or $A^{-1}_{ij}U$ the component i of the operator A or A^{-1} applied to the function equal to U on ∂B_j and to 0 on any ∂B_k with $k \neq j$.

Lemma 3.3 *We have for any constant vector U*

$$g_{ij}U \quad = \quad \int_{\partial B_i} \sigma(T_i A^{-1}_{ij} U)\cdot n dS.$$

Proof of Lemma 3.2.
Estimate (3.4) is an immediate consequence of the definition (3.3) and the estimate (3.1). We now argue that for all positive integers n

$$\|(\mathrm{id}-A)^n_{ij}U\|_{\partial B_i} \le \left\{ \begin{array}{ll} (C\Lambda)^{n-1}\frac{R}{d_{ij}}\|U\|_{\partial B_j} & \text{for } j\neq i \\ (C\Lambda)^{n-1}\frac{\Lambda}{N}\|U\|_{\partial B_i} & \text{for } j = i \end{array}\right\} \tag{3.10}$$

for some universal constant $C<\infty$. Again $(\mathrm{id}-A)^n_{ij}$ is the component ij of the operator $\mathrm{id}-A$ to the power n and not $id - A_{ij}$ to the power n. To establish (3.10), we make the Ansatz

$$\|(\mathrm{id}-A)^n_{ij}U\|_{\partial B_i} \le \left\{ \begin{array}{ll} \alpha_n\Lambda^{n-1}\frac{R}{d_{ij}}\|U\|_{\partial B_j} & \text{for } j\neq i \\ \beta_n\Lambda^{n-1}\frac{\Lambda}{N}\|U\|_{\partial B_i} & \text{for } j = i \end{array}\right\} \tag{3.11}$$

and derive the condition

$$\alpha_1 \ge C_0 \quad \text{and} \quad \alpha_{n+1} \ge 2C_0C_1\alpha_n + C_0\beta_n, \tag{3.12}$$

$$\beta_1 \ge 0 \quad \text{and} \quad \beta_{n+1} \ge C_0C_2\alpha_n, \tag{3.13}$$

where C_0, C_1 and C_2 stand for the universal constants with

$$\|A_{ij}U\|_{\partial B_i} \quad \le \quad C_0\frac{R}{d_{ij}}\|U\|_{\partial B_j} \quad \text{for } j\neq i, \tag{3.14}$$

$$\sum_{j\neq i}\frac{R}{d_{ij}} \quad \le \quad C_1\frac{RN^{2/3}}{d_{min}} = C_1\Lambda, \tag{3.15}$$

$$\sum_{j\neq i}(\frac{R}{d_{ij}})^2 \quad \le \quad C_2\frac{R^2N^{1/3}}{d^2_{min}} = C_2\frac{\Lambda^2}{N}. \tag{3.16}$$

We establish (3.12) and (3.13) by induction in n. It is true for $n = 1$ by (3.14). Assume now it holds for n. Because of

$$(\mathrm{id} - A)^{n+1}_{ij} U = \sum_k (\mathrm{id} - A)_{ik} (\mathrm{id} - A)^n_{kj} U = -\sum_{k \neq i} A_{ik} (\mathrm{id} - A)^n_{kj} U,$$

and (3.14), we have

$$\|(\mathrm{id} - A)^{n+1}_{ij} U\|_{\partial B_i} \leq \sum_{k \neq i} C_0 \frac{R}{d_{ik}} \|(\mathrm{id} - A)^n_{kj} U\|_{\partial B_k}. \tag{3.17}$$

We first treat the case $j \neq i$; we obtain from the assumption (3.11)

$$\|(\mathrm{id} - A)^{n+1}_{ij} U\|_{\partial B_i} / \|U\|_{\partial B_j} \leq \left(\sum_{k \neq i,j} C_0 \frac{R}{d_{ik}} \alpha_n \Lambda^{n-1} \frac{R}{d_{kj}} \right) + C_0 \frac{R}{d_{ij}} \beta_n \Lambda^{n-1} \frac{\Lambda}{N}. \tag{3.18}$$

We now use the triangle inequality $d_{ij} \leq d_{ik} + d_{kj}$ in form of $\frac{1}{d_{ik} d_{kj}} \leq \frac{1}{d_{ij}} (\frac{1}{d_{ik}} + \frac{1}{d_{kj}})$. Hence we have by (3.15)

$$\sum_{k \neq i,j} \frac{R}{d_{ik}} \frac{R}{d_{kj}} \leq 2 C_1 \Lambda \frac{R}{d_{ij}}.$$

Thus (3.18) turns into

$$\begin{aligned} \Lambda^{-n} \|(\mathrm{id} - A)^{n+1}_{ij} U\|_{\partial B_i} / \|U\|_{\partial B_j} &\leq (2 C_0 C_1 \alpha_n + C_0 \frac{1}{N} \beta_n) \frac{R}{d_{ij}} \\ &\leq (2 C_0 C_1 \alpha_n + C_0 \beta_n) \frac{R}{d_{ij}}. \end{aligned}$$

This establishes (3.12). We now treat the case $j = i$. Using (3.17) for $j = i$ and the assumption (3.11), we obtain

$$\|(\mathrm{id} - A)^{n+1}_{ii}\|_{\partial B_i} / \|U\|_{\partial B_i} \leq \sum_{k \neq i} C_0 \frac{R}{d_{ik}} \alpha_n \Lambda^{n-1} \frac{R}{d_{ik}}.$$

Using (3.16), this turns into

$$\Lambda^{-n} \|(\mathrm{id} - A)^{n+1}_{ii} U\|_{\partial B_i} / \|U\|_{\partial B_i} \leq C_0 C_2 \alpha_n \frac{\Lambda}{N},$$

which yields (3.13). This establishes (3.10).

We formally have $A^{-1} = \sum_{n=0}^{\infty}(\mathrm{id} - A)^n$. We now show that this series converges and that the estimates (3.5) and (3.6) hold. Indeed, for $j \neq i$ we formally have

$$A_{ij}^{-1}U = \sum_{n=1}^{\infty}(\mathrm{id} - A)_{ij}^{n}U.$$

and thus according to (3.10)

$$\|A_{ij}^{-1}U\|_{\partial B_i} \leq \sum_{n=1}^{\infty}(C\Lambda)^{n-1}\frac{R}{d_{ij}}\|U\|_{\partial B_j} \lesssim \frac{R}{d_{ij}}\|U\|_{\partial B_j}.$$

For $j = i$ on the other hand we formally have

$$A_{ii}^{-1}U = U + \sum_{n=2}^{\infty}(\mathrm{id} - A)_{ii}^{n}U$$

and thus according to (3.10)

$$\|A_{ii}^{-1}U - U\|_{\partial B_i} \leq \sum_{n=2}^{\infty}(C\Lambda)^{n-1}\frac{\Lambda}{N}\|U\|_{\partial B_i} \lesssim \frac{\Lambda^2}{N}\|U\|_{\partial B_i} = \frac{R^2 N^{1/3}}{d_{min}^2}\|U\|_{\partial B_i}.$$

□

Proof of Lemma 3.3.
Fix a particle j and a constant velocity field U on ∂B_j. Let u denote the solution of the Stokes equations in $\mathbb{R}^3 - \bigcup_k B_k$ with Dirichlet boundary data $\delta_{kj}U$ on ∂B_k. We claim that this solution can be written as

$$u = \sum_{\ell} T_\ell A_{\ell j}^{-1}U. \tag{3.19}$$

Indeed, the r. h. s. of (3.19) is a solution of the Stokes equations in $\bigcap_k(\mathbb{R}^3 - B_k) = \mathbb{R}^3 - \bigcup_k B_k$. Its Dirichlet boundary data are as desired given by

$$u_{|\partial B_k} = \sum_{\ell}(T_\ell A_{\ell j}^{-1}U)_{|\partial B_k} = \sum_{\ell} A_{k\ell}A_{\ell j}^{-1}U = \delta_{kj}U.$$

This establishes (3.19).

We now fix a particle i and observe that by definition of the metric tensor and thanks to (3.19) and (3.8)

$$g_{ij}U = \int_{\partial B_i} \sigma(u) \cdot n dS = \sum_{\ell} \int_{\partial B_i} \sigma(T_\ell A_{\ell j}^{-1} U) \cdot n dS = \int_{\partial B_i} \sigma(T_i A_{ij}^{-1} U) \cdot n dS.$$

$\square$

3.3 Proof of Proposition 1.1

Estimate (1.10) is straightforward: According to Lemma 3.3 and to (3.9) we have

$$g_{11}U - 6\pi\mu R U = \int_{\partial B_1} \sigma(T_1(A_{11}^{-1}U - U)) \cdot n dS.$$

Because of Lemma 3.1 and Lemma 3.2, this yields the desired estimate:

$$|g_{11}U - 6\pi\mu R U| \lesssim \mu R \|A_{11}^{-1}U - U\|_{\partial B_1} \lesssim \mu R \frac{R^2 N^{1/3}}{d_{min}^2} |U|.$$

Also estimate (1.11) is easy: By Lemma 3.3, Lemma 3.1 and Lemma 3.2 we have

$$|g_{12}U| \lesssim \mu R \|A_{12}^{-1}U\|_{\partial B_1} \lesssim \mu R \frac{R}{d_{12}} |U|.$$

We now tackle (1.12). We first derive the representation

$$\begin{aligned} g_{13}U - g_{23}U \quad &= \int_{\partial B_1} \sigma(T_1(u - u(\cdot + X_2 - X_1))) \cdot n dS \\ &\quad + \int_{\partial B_1} \sigma(T_1 v_1) \cdot n dS - \int_{\partial B_2} \sigma(T_2 v_2) \cdot n dS, \end{aligned} \tag{3.20}$$

where

$$u \quad := \quad T_3(-U + \sum_{j \neq 3} A_{3j} A_{j3}^{-1} U) + \sum_{i \neq 1,2,3} T_i \sum_{j \neq i} A_{ij} A_{j3}^{-1} U, \tag{3.21}$$

$$v_1 \quad := \quad T_2 \sum_{j \neq 2} A_{2j} A_{j3}^{-1} U, \tag{3.22}$$

$$v_2 \quad := \quad T_1 \sum_{j \neq 1} A_{1j} A_{j3}^{-1} U. \tag{3.23}$$

Indeed, according to Lemma 3.3, we have

$$g_{13}U - g_{23}U = \int_{\partial B_1} \sigma(T_1 A_{13}^{-1} U) \cdot n dS - \int_{\partial B_2} \sigma(T_2 A_{23}^{-1} U) \cdot n dS.$$

Using

$$A^{-1} = 2\mathrm{id} - A \,+\, (\mathrm{id} - A)^2 A^{-1},$$

we rewrite

$$\begin{aligned} A_{13}^{-1} &= -A_{13} + \sum_{i \neq 1} \sum_{j \neq i} A_{1i} A_{ij} A_{j3}^{-1} \\ &= A_{13}(-\mathrm{id} + \sum_{j \neq 3} A_{3j} A_{j3}^{-1}) + \sum_{i \neq 1,2,3} A_{1i} \sum_{j \neq i} A_{ij} A_{j3}^{-1} + A_{12} \sum_{j \neq 2} A_{2j} A_{j3}^{-1}. \end{aligned}$$

In view of (3.21) & (3.22), this means

$$A_{13}^{-1} U = (u + v_1)_{|\partial B_1}.$$

Likewise, we have $A_{23}^{-1} U = (u + v_2)_{|\partial B_2}$. Hence we obtain

$$\begin{aligned} g_{13}U - g_{23}U &= \int_{\partial B_1} \sigma(T_1 u) \cdot n dS + \int_{\partial B_1} \sigma(T_1 v_1) \cdot n dS \\ &\quad - \int_{\partial B_2} \sigma(T_2 u) \cdot n dS - \int_{\partial B_2} \sigma(T_2 v_2) \cdot n dS. \end{aligned}$$

Because of $\int_{\partial B_2} \sigma(T_2 u) \cdot n dS = \int_{\partial B_1} \sigma(T_1(u(\cdot + X_2 - X_1))) \cdot n dS$, this turns into (3.20).

We now estimate $g_{13}U - g_{23}U$ based on the representation (3.20). Using Lemma 3.1, we have

$$|g_{13}U - g_{23}U| \lesssim \mu R(\|u - u(\cdot + X_2 - X_1)\|_{\partial B_1} + \|v_1\|_{\partial B_1} + \|v_2\|_{\partial B_2}). \quad (3.24)$$

Let us start with $\|u - u(\cdot + X_2 - X_1)\|_{\partial B_1}$. According to Lemma 3.1 and

Lemma 3.2 we obtain

$$\begin{aligned}
&\|u - u(\cdot + X_2 - X_1)\|_{\partial B_1} \\
&\lesssim \; d_{12}(\frac{R}{d_{13}^2} + \frac{R}{d_{23}^2})\| - U + \sum_{j\neq 3} A_{3j}A_{j3}^{-1}U\|_{\partial B_3} \\
&\quad + \sum_{i\neq 1,2,3} d_{12}(\frac{R}{d_{1i}^2} + \frac{R}{d_{2i}^2})\|A_{i3}A_{33}^{-1}U + \sum_{j\neq 3,i} A_{ij}A_{j3}^{-1}U\|_{\partial B_i} \\
&\lesssim \; d_{12}(\frac{R}{d_{13}^2} + \frac{R}{d_{23}^2})(1 + \sum_{j\neq 3}(\frac{R}{d_{j3}})^2)|U| \\
&\quad + \sum_{i\neq 1,2,3} d_{12}(\frac{R}{d_{1i}^2} + \frac{R}{d_{2i}^2})(\frac{R}{d_{3i}} + \sum_{j\neq 3,i}\frac{R}{d_{ij}}\frac{R}{d_{3j}})|U|.
\end{aligned} \tag{3.25}$$

We observe that because of the triangle inequality $d_{3i} \leq d_{ij} + d_{3j}$, which we use in form of $\frac{1}{d_{ij}d_{3j}} \leq \frac{1}{d_{3i}}(\frac{1}{d_{ij}} + \frac{1}{d_{3j}})$, we have

$$\sum_{j\neq 3,i}\frac{R}{d_{ij}}\frac{R}{d_{3j}} \leq \frac{R}{d_{3i}}(\sum_{j\neq i}\frac{R}{d_{ij}} + \sum_{j\neq 3}\frac{R}{d_{3j}}) \lesssim \frac{R}{d_{3i}}\frac{RN^{2/3}}{d_{min}} = \frac{R}{d_{3i}}\Lambda \ll \frac{R}{d_{3i}}. \tag{3.26}$$

Furthermore,

$$\sum_{j\neq 3}(\frac{R}{d_{j3}})^2 \lesssim \frac{R^2N^{1/3}}{d_{min}^2} = \frac{\Lambda^2}{N} \ll 1.$$

Hence (3.25) simplifies to

$$\begin{aligned}
&\|u - u(\cdot + X_2 - X_1)\|_{\partial B_1} \\
&\lesssim \; d_{12}\left((\frac{R}{d_{13}^2} + \frac{R}{d_{23}^2}) + \sum_{i\neq 1,2,3}(\frac{R}{d_{1i}^2} + \frac{R}{d_{2i}^2})\frac{R}{d_{3i}}\right)|U|.
\end{aligned} \tag{3.27}$$

Using the triangle inequality again, we have

$$\begin{aligned}
&\sum_{i\neq 1,2,3}(\frac{R}{d_{1i}^2} + \frac{R}{d_{2i}^2})\frac{R}{d_{3i}} \\
&\lesssim \; (\frac{1}{d_{13}} + \frac{1}{d_{23}})(\sum_{i\neq 1}\frac{R^2}{d_{1i}^2} + \sum_{i\neq 2}\frac{R^2}{d_{2i}^2} + \sum_{i\neq 3}\frac{R^2}{d_{3i}^2}) \\
&\lesssim \; (\frac{1}{d_{13}} + \frac{1}{d_{23}})\frac{R^2N^{1/3}}{d_{min}^2}.
\end{aligned}$$

Therefore (3.27) turns as desired into

$$\|u - u(\cdot + X_2 - X_1)\|_{\partial B_1} \lesssim d_{12} \left((\frac{R}{d_{13}^2} + \frac{R}{d_{23}^2}) + (\frac{1}{d_{13}} + \frac{1}{d_{23}}) \frac{R^2 N^{1/3}}{d_{min}^2} \right) |U|. \tag{3.28}$$

We now turn to $\|v_1\|_{\partial B_1}$. According to Lemma 3.1 and Lemma 3.2 we have

$$\begin{aligned} \|v_1\|_{\partial B_1} &\lesssim \frac{R}{d_{12}} \|A_{23} A_{33}^{-1} U + \sum_{j \neq 2,3} A_{2j} A_{j3}^{-1} U\|_{\partial B_2} \\ &\lesssim \frac{R}{d_{12}} (\frac{R}{d_{23}} + \sum_{j \neq 2,3} \frac{R}{d_{2j}} \frac{R}{d_{j3}}) |U|. \end{aligned}$$

We now use (3.26) for $i = 2$ and so obtain

$$\|v_1\|_{\partial B_1} \lesssim \frac{R}{d_{12}} \frac{R}{d_{23}} |U|. \tag{3.29}$$

Since we trivially have

$$\frac{R}{d_{12}} \frac{R}{d_{23}} \leq d_{12} \frac{1}{d_{23}} \frac{R^2}{d_{min}^2} \leq d_{12} \frac{1}{d_{23}} \frac{R^2 N^{1/3}}{d_{min}^2},$$

also (3.29) turns as desired into

$$\|v_1\|_{\partial B_1} \lesssim d_{12} \frac{1}{d_{23}} \frac{R^2 N^{1/3}}{d_{min}^2} |U|. \tag{3.30}$$

By symmetry, we likewise have

$$\|v_2\|_{\partial B_2} \lesssim d_{12} \frac{1}{d_{13}} \frac{R^2 N^{1/3}}{d_{min}^2} |U|. \tag{3.31}$$

Combining (3.24), (3.28), (3.30) & (3.31), we obtain (1.12).

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