Correction for the Midterm test for Advanced calculus, 410, on 10/5/2011

1) (3pts) Prove that the following limit holds

\[ u_n = \frac{\exp(2 + (-1)^n)}{n} \to 0. \]

2) (3pts) Show that the following sequence is increasing

\[ u_0 = 1, \quad u_1 = 2, \quad \forall n \geq 0 \quad u_{n+2} = 2u_{n+1} - u_n. \]

Hint: Use induction and consider \( u_{n+1} - u_n \).

3) (3pts) Prove that no matter how we choose \( \alpha \in \mathbb{R} \), the following function is not continuous at 0

\[ f(x) = \sin(1/x) \quad \text{if} \quad x \neq 0, \quad f(0) = \alpha. \]

4) (1pts) Let \( (u_n) > 0 \) be a given sequence with \( u_n \to 0 \). Build \( \sigma : \mathbb{N} \to \mathbb{N} \) strictly increasing s.t. for any other \( \sigma' : \mathbb{N} \to \mathbb{N} \) strictly increasing with \( \sigma'(k) \geq \sigma(k), \forall k \), one has

\[ u_{\sigma(n)} \geq u_{\sigma'(n)}, \forall n. \]

Correction.

1) We note that \( 2 + (-1)^n \leq 3 \) and as \( \exp \) is an increasing and positive function

\[ 0 \leq u_n \leq \frac{e^3}{n}. \]

By the archimedian property of \( \mathbb{R} \), one then concludes that

\[ u_n \to 0. \]

2) We show by induction that the following property holds for any \( n \geq 0 \)

\[ P(n) : \quad u_{n+1} - u_n = 1. \]

Indeed \( P(0) \) is true as \( u_1 - u_0 = 2 - 1 = 1 \).

Now assuming that \( P(n) \) is true. We apply the relation defining \( u_{n+2} \) to find

\[ u_{n+2} - u_{n+1} = 2u_{n+1} - u_n - u_{n+1} = u_{n+1} - u_n = 1, \]

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by $P(n)$. Hence $P(n + 1)$ holds.

We have shown that $P(n)$ was true $\forall n \geq 0$. We conclude that $u_n$ is strictly increasing as

$$u_{n+1} - u_n > 0, \forall n \geq 0.$$

Note that one could even compute $u_n$ explicitly: $u_n = 1 + n$.

3) We define the two sequences

$$u_n = \frac{1}{2\pi n}, \quad v_n = \frac{1}{\pi/2 + 2\pi n}.$$

By the archimedian property of $\mathbb{R}$, one has

$$u_n \to 0, \quad v_n \to 0.$$

But

$$f(u_n) = \sin(2\pi n) = 0 \to 0, \quad f(v_n) = \sin(\pi/2 + 2\pi n) = 1 \to 1.$$

Now, by contradiction if $\exists \alpha \in \mathbb{R}$ s.t. $f$ is continuous at 0 then one should have

$$\lim f(u_n) = \alpha = \lim f(v_n),$$

which would imply $0 = 1$, a contradiction.

4) We define $\sigma(n)$ by induction satisfying the property

$$P(n) : \forall \sigma' : \mathbb{N} \to \mathbb{N}, \text{ strictly increasing s.t. } \sigma'(k) \leq \sigma(k) \forall k \leq n,$$

one has $u_{\sigma(l)} \geq u_{\sigma'(l)} \forall l \leq n$.

We start with $n = 0$. We apply the definition of convergence to $u_0/2$:

$$\exists N \in \mathbb{N}, \quad \forall l \geq N, \ |u_l| \leq u_0/2.$$

Hence $u_0 > \sup\{u_l | l \geq N\}$. Therefore

$$\sup\{u_l | l \geq 0\} = \max\{u_l | l < N\},$$

and the first set has a maximum, attained for an index $k$

$$u_k = \max\{u_l | l \geq 0\}.$$

We define $\sigma(0) = k$. It obviously satisfies $P(0)$ as $u_{\sigma(0)} \geq u_l, \forall l$. 2
Now assuming $\sigma(n)$ is defined and satisfies $P(n)$. We apply the definition of convergence to $u_{\sigma(n)+1/2}$:

$$\exists N \in \mathbb{N}, \quad \forall l \geq N, \quad |u_l| \leq u_{\sigma(n)+1}/2.$$ 

Hence $u_{\sigma(n)+1} > \sup\{u_l \mid l \geq N\}$. Therefore

$$\sup\{u_l \mid l \geq \sigma(n) + 1\} = \max\{u_l \mid \sigma(n) + 1 \geq l < N\},$$

and the first set has a maximum, attained for an index $k$

$$u_k = \max\{u_l \mid l \geq \sigma(n) + 1\}.$$ 

We define $\sigma(n + 1) = k$. We have to check that it satisfies $P(k)$. Hence take any

$$\sigma' : \mathbb{N} \to \mathbb{N}, \text{ strictly increasing s.t. } \sigma'(k) \geq \sigma(k) \quad \forall k \leq n + 1.$$ 

First of all, obviously $\sigma'$ satisfies

$$\sigma'(k) \geq \sigma(k) \quad \forall k \leq n.$$ 

Hence by $P(n)$, $u_{\sigma(l)} \geq u_{\sigma'(l)}$, $\forall l \leq n$. And it only remains the case $l = n + 1$. 

Since $\sigma'(n + 1) \geq \sigma(n + 1)$ then $\sigma'(n + 1) \geq \sigma(n) + 1$. But as

$$u_{\sigma(n+1)} \geq u_l, \quad \forall l \geq \sigma(n) + 1,$$

we can conclude.

By induction we have constructed the requested sequence.