Midterm test for Advanced calculus, 410, on 11/11/2011

The test lasts 50 minutes. No documents are allowed. The use of a calculator, cell phone or other equivalent electronic device is not allowed.

1) (4pts) Study the convergence of the following series

\[ \sum_{n \geq 1} \frac{1}{(n + 1)^2}, \quad \sum_{n \geq 1} \frac{e^{-1^n}}{n}. \]

2) (3pts) We define \( f(x) = -x^2 \) if \( x < 0 \), \( f(x) = x^2 \) if \( x > 0 \) and \( f(0) = 0 \). Show that \( f \) is differentiable at 0 and that \( f'(0) = 0 \).

3) (2pts) Let \( f : \mathbb{R} \to \mathbb{R} \) satisfying

\[ \forall x, y \in \mathbb{R}, \quad |f(x) - f(y)| \leq |x - y|^{1/2}. \]

Use the \( \varepsilon - \delta \) criterion to prove that \( f \) is uniformly continuous on \( \mathbb{R} \).
4) (1pt) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Use the mean value theorem to prove that there exists a sequence $(x_n)$ with $x_n \neq 0$, $\forall n$, $x_n \to 0$ and

$$f'(x_n) \longrightarrow f'(0).$$

5) (1 bonus pt) Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\forall x, y \in \mathbb{R}, \quad |f(x) - f(y)| \leq |x - y|^3.$$ 

Show that $f$ is constant on $\mathbb{R}$. 