Compactness for nonlinear continuity equations

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Abstract. We prove compactness and hence existence for solutions to a class of non linear transport equations. The corresponding models combine the features of linear transport equations and scalar conservation laws. We introduce a new method which gives quantitative compactness estimates compatible with both frameworks.
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1 Introduction

Recent developments in the modeling of various complex transport phenomena (from bacteria to pedestrians' flows) have produced new and challenging equations. In particular those models have a very different behaviour from the usual fluid dynamics when the density is locally high, usually as a consequence of a strict bound on the maximum number of indivivuals that one can have at a given point.

The mathematical theory for well posedness and particularly existence is still however lacunary for those equations. The aim of this article is thus to provide a unified framework for a general class of conservation laws, including

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many of these recent additions. More precisely, we study equations of the form,

$$\partial_t n(t,x) + \mathrm{div}\,(a(t,x)\, f(n(t,x))) = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \tag{1.1}$$

where n usually represents a density of individuals. $f \in W^{1,\infty}(\mathbb{R},\ \mathbb{R})$ is a given function which takes local, non linear effects into account. A typical example for f is the logistic $f(n) = n\,(1 - n/\bar{n})_+$, which limits the velocity of individuals when their density is too high thus ensuring that the density never exceeds a critical value $\bar{n}$. The field $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ provides the direction for the movement of individuals.

Depending on the exact model, a can either be given or be related to n. Many such models have been introduced in the past few years in various contexts from chemotaxis for cells and bacteria to pedestrian flow models. We only give here a few such examples.

Typically a incorporates some non local effects on the density such as with a convolution $a = K \star n$ or a Poisson eq.

$$a(t,x) = -\nabla_x \phi(t,x), \quad -\Delta_x \phi(t,x) = g(n(t,x)), \tag{1.2}$$

where g is another given function of n. Such a model was introduced in two dimensions and in the context of swarming in [38]. The same kind of models was studied in [8] and [17] for chemotaxis (and typically for $g(n) = n$).

More complicated relations between a and n are possible, for instance a Hamilton-Jacobi equation as in [19]

$$a(t,x) = -\nabla_x \phi(t,x), \quad -\Delta_x \phi(t,x) + \alpha\,|\nabla \phi|^2 = g(n(t,x)), \tag{1.3}$$

with $\alpha \geq 0$ (possibly vanishing) and again g a given non linear function.

Eq. (1.1) can be seen as a hybrid model, combining features of usual linear transport equation and scalar conservation laws.

Let us briefly discuss the main difficulty in obtaining existence of distributional solutions to (1.1). With reasonable assumptions (like $f \sim n(1-n)$), it is easy to show that the density n is bounded in every L^p spaces. However contrary to linear continuity equations, a bound on n is not enough to pass to the limit in the nonlinear term $f(n)$ (or $g(n)$ if (1.2) or (1.3) is used).

With Eq. (1.2) or (1.3) and $n \in L^1 \cap L^\infty$, one can easily get $a \in W^{1,p}$ for any $1 < p < \infty$. From that one may obtain compactness on a in L^1_{loc}.

Hence as a non linear model, the main difficulty in obtaining existence of solutions to (1.1) is to prove compactness for the density n. Below we briefly indicate why the usual methods for conservation laws do not work in this setting (see [15] or [37] for more on conservation laws).

When a is regular enough (Lipschitz more precisely), then the usual method of compactness for scalar conservation laws work and one can for example show propagation of BV bounds on n. Unfortunately this Lipschitz bound does not hold here in general (only $W^{1,p}$, $p < \infty$ as explained above). Such BV bounds on n can in fact only be propagated for short times (see [8] for instance).

For scalar conservation laws, another way to obtain compactness is either by compensated compactness or other regularizing effects. Notice here that Eq. (1.1) fits within the framework of conservation laws with discontinuous flux, see for example [4], [24] and [34]. However in dimension larger than 1, those methods cannot be used for (1.1) as the flux cannot be genuinely non linear (it is in only one direction, the one given by a). The 1-dimensional case is quite particular (not only in this respect) and many well posedness results have already been obtained (see for instance [19]).

As far as we know, [17] is for the moment the only result showing existence to an equation like (1.1) over any time interval and in dimension larger than 1. The authors use a kinetic formulation of (1.1), which simply generalizes the kinetic formulation of scalar conservation laws introduced in [29] (see also [30] and [36]). A rigidity property inherent to the kinetic formulation then provides compactness. However a precise connection between a and n is needed; more precisely the result is obtained only for the case of (1.2) (with $g = Id$ though it can obviously be extended to any g suitably smooth).

Comparing the results in this paper with [17], we can allow for much more general equations linking a and n (see assumption (1.7) and remark 5. after Theorem 1.2). Moreover, we develop a general theory with explicit quantitative estimates that exhibit the critical regularity one can expect on the solution of (1.1).

We conclude this brief summary of the various techniques already in use by mentioning gradient flows. In the context of the non linear model (1.1), the theory is essentially still in development. It requires a lot of structure on the equations and that essentially means for the moment Eq. (1.2) *with* $g = Id$ (any generalization to non linear g would be problematic). We refer in particular to [21] where the right metric for the problem and its properties are introduced and studied.

Gradient flows techniques were also used in [32] for a related problem. In that case the corresponding transport is linear but associated with a constraint on the maximal density. In the framework of (1.1) that would correspond to $f(\xi) = \xi \, \mathbb{I}_{\xi<1}$.

Let us now formulate the main results of the paper. Consider a vanishing viscosity approximation

$$\begin{aligned} &\partial_t n_\varepsilon(t,x) + \mathrm{div}\,(a_\varepsilon(t,x)\, f(n_\varepsilon(t,x))) - \varepsilon^2\, \Delta_x n_\varepsilon = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d, \\ &n_\varepsilon(t=0,x) = n_\varepsilon^0(x). \end{aligned} \tag{1.4}$$

Instead of assuming a precise form or relation between a_ε and n_ε, we make very general assumptions on a_ε. Assume that on $[0,\ T]$

$$\exists\, p > 1, \qquad \sup_\varepsilon \sup_{t\in[0,\ T]} \|a_\varepsilon(t,.)\|_{W^{1,p}(\mathbb{R}^d)} < \infty, \tag{1.5}$$

$$\sup_\varepsilon \|\mathrm{div}_x a_\varepsilon\|_{L^\infty([0,\ T]\times\mathbb{R}^d)} < \infty. \tag{1.6}$$

As for linear transport equations, an additional condition is needed on the divergence to obtain compactness. In order to be compatible with (1.2) or (1.3), we assume

$$\begin{cases} \mathrm{div}_x a_\varepsilon = d_\varepsilon + r_\varepsilon \quad \text{with } d_\varepsilon \text{ compact and} \\ \exists C > 0,\ s.t.\ \forall \varepsilon > 0, \forall x, y, \\ |r_\varepsilon(x) - r_\varepsilon(y)| \le C\, |n_\varepsilon(t,x) - n_\varepsilon(t,y)|. \end{cases} \tag{1.7}$$

Then one can prove

Theorem 1.1 *Assume* (1.5), (1.6), (1.7), *that a_ε is compact in L^p, that n_ε^0 is uniformly bounded in $L^1 \cap L^\infty(\mathbb{R}^d)$ and is compact in $L^1(\mathbb{R}^d)$. Then the solution $n_\varepsilon(t,x)$ to* (1.4) *is compact in $L^1_{loc}([0,\ T]\times\mathbb{R}^d)$.*

This in particular implies existence results like

Corollary 1.1 *Assume that $f \in W^{1,\infty}$, $g \in C^2$, $f(0) = g(0) = 0$ and that $f(\xi)\, g(\xi) \ge -C\,|\xi|$ for some given constant C. Let $n^0 \in L^1 \cap L^\infty(\mathbb{R}^d)$, $\alpha \ge 0$, then $\exists n \in L^\infty([0,\ T],\ L^1 \cap L^\infty(\mathbb{R}^d))$ solution in the sense of distribution to* (1.1) *with* (1.2). *Moreover n is an entropy solution to* (1.1) *in the usual sense that $\forall \phi \in C^2$ convex, $\exists q \in C^1$ s.t.*

$$\partial_t(\phi(n(t,x))) + \mathrm{div}_x\,(a(t,x)\, q(n(t,x))) + (\phi'(n)\, f(n) - q(n))\, \mathrm{div}_x\, a \le 0.$$

Note that this is just one example of possible results, it can for instance easily be generalized to (1.3) under corresponding assumptions. Once a is given and in $W^{1,p}$ the uniqueness of the entropy solution to (1.1) is actually not very difficult. However uniqueness for a coupled system like (1.1)-(1.3) is more delicate and left open here.

To prove Th. 1.1, we develop a new method which is a sort of quantified version of the theory of renormalized solution and compatible with the usual L^1 contractivity argument (see [26]) for scalar laws.

Renormalized solutions were introduced in [20] to prove uniqueness to solutions of linear transport equations

$$\partial_t n + \mathrm{div}\,(a\,n) = 0.$$

The compactness of a sequence of bounded solutions is obtained as a consequence of the uniqueness (by proving for instance that $w - \lim_k n_k^2 = (w - \lim_k n_k)^2$). The theory was developped in [20] for $a \in W^{1,1}$ with $\mathrm{div}\, a \in L^\infty$. It was later extended to $a \in BV$, first for the particular case of kinetic equations in [5] (see also [10] for the kinetic case with less than one derivative on a). The general case was dealt with in [1] (see also [13]). For more about renormalized solutions we refer to [2] and [18].

The usual proof of the renormalization property relies on a commutator estimate. It is this estimate that we have to quantify somehow here. More precisely we try to bound quantities like

$$\|n_\varepsilon\|_{p,h}^p = \int_{\mathbb{R}^{2d}} \frac{\mathbb{I}_{|x-y|\leq 1}}{(|x-y|+h)^d}\, |n_\varepsilon(t,x) - n_\varepsilon(t,y)|^p\, dx\, dy, \tag{1.8}$$

uniformly in h. Those norms can be seen as a generalization of usual Sobolev norm, in particular we recall that

$$\int_{\mathbb{R}^{2d}} \frac{\mathbb{I}_{|x-y|\leq 1}}{|x-y|^{d+2s}}\, |n_\varepsilon(t,x) - n_\varepsilon(t,y)|^2\, dx\, dy$$

is equivalent to the usual $\dot{H}_s$ norm for $s \in]0,\ 1[$. This is wrong though for $s = 0$, *i.e.* $\|.\|_{2,0}$ is actually stronger than L^2. In this case $p = 2$, it is in fact easy to see in Fourier that $\|.\|_{2,0}$ more or less controls the log of a derivative and thus provides compactness. Note that the behaviour of fractional Sobolev norms $W^{s,p}$ near $s = 1$ is for instance studied in [7].

We can prove explicit estimates for the norms (1.8)

Theorem 1.2 *Assume* (1.5), (1.6), (1.7), *that* n^0_ε *is uniformly bounded in* $L^1 \cap L^\infty(\mathbb{R}^d)$ *and is compact in* $L^1(\mathbb{R}^d)$. $\exists C > 0$ *only depending on the uniform bounds in* ε *s.t. the solution* $n_\varepsilon(t,x)$ *to* (1.4) *satisfies for any* $t \leq T$

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \frac{\mathbb{I}_{|x-y|\leq 1}}{(|x-y|+h)^d}\, |n_\varepsilon(t,x) - n_\varepsilon(t,y)|\, dx\, dy \\
&\leq e^{C\,t} \Bigg\{ \int_{\mathbb{R}^{2d}} \frac{\mathbb{I}_{|x-y|\leq 2}}{(|x-y|+h)^d}\, |n^0_\varepsilon(x) - n^0_\varepsilon(y)|\, dx\, dy \\
&\qquad + \int_0^t \int_{\mathbb{R}^{2d}} \frac{\mathbb{I}_{|x-y|\leq 1}}{(|x-y|+h)^d}\, |d_\varepsilon(s,x) - d_\varepsilon(s,y)|\, dx\, dy\, ds \\
&\qquad + C\, \frac{\varepsilon^2}{h^2} + C\, |\log h|^{1/\bar p} \Bigg\},
\end{aligned}$$

where $\bar p = \min(2,p)$ *and* $1/p^* + 1/p = 1$.

Remarks.
1. Lemma 3.1 below shows that Theorem 1.2 in fact implies Theorem 1.1 but its proof is of course more complicated.
2. In addition of providing an explicit rate, Theorem 1.2 does not require the compactness of the sequence a_ε. Of course as it is uniformly in $L^\infty_t W^{1,p}_x$, it is always compact in space but not necessarily in time.
3. It is possible to replace (1.5) by

$$\sup_\varepsilon \int_0^T \|a_\varepsilon(t,.)\|_{W^{1,p}(\mathbb{R}^d)} < \infty.$$

The estimate then uses the exponential of this quantity instead of e^{Ct}.
4. If the sequence ∇a_ε is equiintegrable then some kind of rate can also be obtained.
5. Assumption (1.7) can also be extended by asking r_ε to satisfy only

$$\|r_\varepsilon\|_{h,1} \leq C\, \|n_\varepsilon\|_{h,1}.$$

The norms defined by (1.8) are in fact critical for the problem (1.1). Indeed (1.1) contains the case of the linear transport equation (take $f = Id$). In this last case, one may use the characteristics and it was proved in [14] that one indeed propagates a sort of log of derivative on them. If $n^0 \in W^{1,p}$ then this implies a result like Th. 1.2. At the level of the characteristics, it is not complicated to obtain examples showing that this logarithmic gain is the best that one can hope for.

Let us point out though that even in this linear case, the theory behind Th. 1.2 is new and interesting as it proves those quantitative estimates at the level of the PDE without using the connection with the characteristics. Of course contrary to [14], we have to work here at the level of the PDE; because of the shocks, the characteristics cannot be used when f is non linear. The corresponding proof is considerably more complicated and in particular it forces us to carefully track every cancellation in the commutator estimate; we also refer to [6] for an example in a different linear situation where a problem of similar nature is found.

Th. 1.2 gives a rate in $|\log h|^{1/\bar{p}}$ which is probably not optimal. In the linear case $f = Id$, [14] shows that the optimal rate is 1. In our non linear situation, it seems reasonable to conjecture that it should be the same (at least for $p \geq 2$) but it is obviously a difficult question.

The proof of Th. 1.2 requires the use of multilinear singular integrals. This has been an important field of study in itself (we quote only some results below) but quite a few open questions remain, making the optimality of Th. 1.2 unclear.

The first contributions for multilinear singular integrals were essentially in dimension 1, see [9], [11] or [12]. The theory was later developed for instance in [22], [25], [27]. In dimension 1, an almost complete answer was finally given in [31]. In higher dimension, the most complete result that we know of, [33], unfortunately does not contain the case that we have to deal with here.

Let us conclude this introduction by mentioning two important and still open problems. Of course many technical issues are still unresolved: The optimal rate, the case where $a_\varepsilon \in BV$ instead of at least $W^{1,1}$...

First of all, in many situations a bound on the divergence of a_ε is not available. However when f is a logistic function for example, Eq. (1.1) still controls the maximal compression, contrary to a linear transport equation. It means that this case should actually be easier to handle in the non linear setting.

Second some models do not provide any additional derivative on the velocity field a. For instance in porous media, one finds the classical coupling

$$a = -\nabla\phi, \quad \mathrm{div}_x(\alpha(n)\,\nabla\phi) = g,$$

but one could also consider the non viscous equivalent of (1.3). Of course the method presented here fails in those cases...

The next section gives a quick proof of Corollary 1.1. The section after that is devoted to Th. 1.1 and the last one to Th. 1.2.

In the rest of the paper, C will denote a generic constant, which may depend on the time interval $[0,\ T]$ considered, uniform bounds on the initial data n_ε^0 or on a_ε but which never depends on ε or the parameter h that we will introduce.

2 Proof of Corollary 1.1

Define a sequence of approximations n_ε, a_ε where n_ε solves (1.4) with initial data n^0 and a_ε is obtained through n_ε by solving (1.3).

As (1.4) is conservative then one obviously has

$$\|n_\varepsilon(t,.)\|_{L^1(\mathbb{R}^d)} = \|n^0\|_{L^1}.$$

By the maximum principle

$$\frac{d}{dt}\|n_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq \|(f(n_\varepsilon)\,\mathrm{div} a_\varepsilon)_-\|_{L^\infty},$$

where $(.)_-$ denotes the negative part. Using (1.2) implies that

$$\frac{d}{dt}\|n_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq \|(fg(n_\varepsilon))_-\|_{L^\infty} \leq C\,\|n_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^d)},$$

by the assumption in Corollary 1.1. Hence by Gronwall's lemma, the sequence n_ε is uniformly bounded in $L^\infty([0,\ T],\ L^1 \cap L^\infty(\mathbb{R}^d))$ for any $T > 0$.

Thanks to $g(0) = 0$, the usual estimate for (1.2) then gives that a_ε is uniformly in $L^\infty([0,\ T],\ \dot{W}^{1,p}(\mathbb{R}^d))$ for any $1 < p < \infty$. (1.5), (1.6), (1.7) are hence obviously satisfied.

To apply Th. 1.1, it only remains to obtain the compactness of a_ε (note that the refined Th. 1.2 does not require it). First we need an additional bound on n_ε. Multiplying Eq. (1.4) by n_ε and integrating, one finds

$$\varepsilon\int_0^T\int_{\mathbb{R}^d}|\nabla n_\varepsilon|^2\,dx \leq \int_{\mathbb{R}^d}|n^0(x)|^2\,dx + \frac{1}{2}\int_0^T\int_{\mathbb{R}^d}|n_\varepsilon|^2\,\mathrm{div}\,a_\varepsilon\,dx.$$

Thus the previous bounds show that $\varepsilon^{1/2}\,\nabla n_\varepsilon$ is uniformly bounded in L^2.

Now using the transport equation (1.4) and the relation (1.2) implies for $h' = f'g'$

$$\partial_t a_\varepsilon = \nabla\Delta^{-1}\partial_t(g(n_\varepsilon(t,x))) = -\nabla\Delta^{-1}\mathrm{div}\,(a_\varepsilon h(n_\varepsilon)) \\ - \nabla\Delta^{-1}(g'f-h)(n_\varepsilon)\,\mathrm{div}\,a_\varepsilon + \varepsilon\nabla g(n_\varepsilon) - \varepsilon\nabla\Delta^{-1}\,g''(n_\varepsilon)\,|\nabla n_\varepsilon|^2.$$

This proves that $\partial_t a_\varepsilon$ is uniformly bounded in $L^2([0,\ T])$ with values in some negative Sobolev space. Therefore a_ε is locally compact in $L^p(\mathbb{R}^d)$ with p large enough, more precisely $p > (1-1/d)^{-1}$ by Sobolev embeddings.

It only remains to control the behaviour at ∞ of n_ε and hence a_ε. By De la Vallée Poussin, since $n^0 \in L^1$, there exists $\psi \in C^\infty$, convex with $\psi(x) \to +\infty$ as $|x| \to +\infty$, $\nabla\psi \in L^\infty$ and s.t.

$$\int_{\mathbb{R}^d} \psi(x)\,|n^0(x)|\,dx < \infty.$$

By the convexity of ψ, one obtains

$$\frac{d}{dt}\int_{\mathbb{R}^d} \psi(x)\,|n_\varepsilon(t,x)|\,dx \le \int_{\mathbb{R}^d} |\nabla\psi|\,|\mathrm{div}\,a_\varepsilon|\,|f(n_\varepsilon)|\,dx \le C.$$

This implies that $\forall t \in [0,\ T]$

$$\int_{\mathbb{R}^d} \psi(x)\,|n_\varepsilon(t,x)|\,dx \le C. \tag{2.1}$$

By (1.2), it has for first consequence that a_ε is globally compact in L^p, $(1-1/d)^{-1} < p < \infty$. Applying Th. 1.1, one deduces that n_ε is locally compact in L^1 and by (2.1) that n_ε is compact in L^1 and so in any L^p, $1 \le p < \infty$.

Let us now extract two converging subsequences (still denoted by ε)

$$a_\varepsilon \longrightarrow a, \quad n_\varepsilon \longrightarrow n.$$

We may now easily pass to the limit in every term of (1.4) and (1.2) to deduce that n and a are solutions, in the sense of distributions, to (1.1) coupled with (1.2).

Proving that n is an entropy solution to (1.1) follows the usual procedure. For any $\phi \in C^2$ convex, we first note that

$$\partial_t\phi(n_\varepsilon) + \mathrm{div}_x\,(a_\varepsilon q(n_\varepsilon)) + (\phi'(n_\varepsilon)f(n_\varepsilon) - q(n_\varepsilon))\,\mathrm{div}_x\,a_\varepsilon \le 0,$$

with $q' = \phi'\,f'$. With the compactness of n_ε, one may pass to the limit in each term and obtain the same property for n, which concludes the proof of Corollary 1.1.

3 Proof of Theorem 1.1

3.1 The compactness criterion

We first introduce the compactness criterion that we use. Define a family $K_h(x) = 1/(|x|^2 + h^2)^{d/2}$ for $|x| \leq 1$ and K_h non negative, independent of h, with support in $B(0,2)$ and in $C^\infty(\mathbb{R}^d \setminus B(0,1))$.

Lemma 3.1 *A sequence of functions u_k, uniformly bounded in $L^p(\mathbb{R}^d)$ is compact in L^p_{loc} if*

$$\limsup_k \; |\log h|^{-1} \int_{\mathbb{R}^{2d}} K_h(x-y)\, |u_k(x) - u_k(y)|^p \, dx\, dy \longrightarrow 0 \quad \textit{as } h \to 0.$$

Conversely if u_k is globally compact in L^p then the previous limit holds.

Proof. We recall that if u_k is compact in L^p then

$$\delta(\eta) = \eta^{-d} \sup_k \int_{|x-y|\leq \eta} |u_k(x) - u_k(y)|^p \, dx\, dy \longrightarrow 0 \quad \text{as } \eta \to 0.$$

So assuming u_k is compact, one simply decomposes

$$\begin{aligned}
&\sup_k \int_{\mathbb{R}^{2d}} K_h(x-y)\, |u_k(x) - u_k(y)|^p \, dx\, dy \leq C \\
&+ C \sum_{n \leq |\log h|} \sup_k \int_{2^{-n-1} \geq |x-y| \leq 2^{-n}} 2^{dn} |u_k(x) - u_k(y)|^p \, dx\, dy \\
&\leq C + C \sum_{n \leq |\log h|} \delta(2^{-n}),
\end{aligned}$$

which gives the result.

Conversely assume that

$$\alpha(h) = \limsup_k |\log h|^{-1} \int_{\mathbb{R}^{2d}} K_h(x-y)\, |u_k(x) - u_k(y)|^p \, dx\, dy \longrightarrow 0 \text{ as } h \to 0.$$

Denote $\tilde{K}_h(x) = C_h |\log h|^{-1} \, K_h(x-y)$, with C_h s.t.

$$\int \tilde{K}_h(x) \, dx = 1,$$

and therefore $\tilde K_h$ a convolution kernel. Note that C_h is bounded from below and from above uniformly in h. Now

$$\begin{aligned}\|u_k-\tilde K_h\star_x u_k\|_{L^p}^p\leq&|\log h|^{-p}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}K_h(x-y)|u_k(x)-u_k(y)|\,dy\right)^p\,dy\\ \leq&|\log h|^{-p}\,\|K_h\|_{L^1}^{p-1}\int_{\mathbb{R}^{2d}}K_h(x-y)|u_k(x)-u_k(y)|^p\,dy\,dx\\ \leq&C\,|\log h|^{-1}\int_{\mathbb{R}^{2d}}K_h(x-y)|u_k(x)-u_k(y)|^p\,dy\,dx\end{aligned}$$

is converging to 0 uniformly in k as the lim sup is 0 and it is converging for any fixed k by the usual approximation by convolution in L^p. On the other hand for a fixed h, $\tilde K_h\star u_k$ is compact in k and this proves that u_k also is.

3.2 The main argument given for a linear transport equation

Before proving Theorem 1.1, we wish to explain the main idea behind the proof in a simple and wellknown setting. Let us consider a sequence u_ε of solutions to the transport equation

$$\begin{aligned}&\partial_t u_\varepsilon(t,x)+v_\varepsilon\cdot\nabla u_\varepsilon(t,x)=0,\quad t\in[0,\ T],\ x\in\mathbb{R}^d,\\ &u_\varepsilon(t=0,x)=u_\varepsilon^0(x),\end{aligned}\tag{3.1}$$

for a given velocity field. The following result was originally proved in [20]

Theorem 3.1 *Assume that u_ε^0 is uniformly bounded in $L^1\cap L^\infty$ and compact. Assume moreover that v_ε is compact in L^p, uniformly bounded in $L^\infty_t W^{1,p}_x$ for some $p>1$ and that $\operatorname{div} v_\varepsilon=0$. Then the sequence of solutions u_ε to* (3.1) *is compact in L^1.*

Proof of Theorem 3.1.

First of all notice that u_ε is uniformly bounded in $L^\infty_t(L^1_x\cap L^\infty_x)$. Moreover as v_ε is compact, one may freely assume that it converges in $L^p_{t,x}$ toward a limit $v\in L^\infty_t W^{1,p}_x$ (by extracting a subsequence).

Now define

$$Q_\varepsilon(t)=\int_{\mathbb{R}^{2d}}K_h(x-y)\,|u_\varepsilon(t,x)-u_\varepsilon(t,y)|^2\,dx\,dy.$$

From Equation (3.1) and the divergence free condition on v_ε, one simply computes

$$\frac{dQ_\varepsilon}{dt} = \int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\,(v_\varepsilon(t,y)-v_\varepsilon(t,x))|u_\varepsilon(t,x)-u_\varepsilon(t,y)|^2\,dx\,dy.$$

Therefore by introducing the limit v

$$\begin{aligned}\frac{dQ_\varepsilon}{dt} \leq & C\,\|v_\varepsilon - v\|_{L^p}\,\|\nabla K_h\|_{L^1}\\ &+\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\,(v(t,y)-v(t,x))|u_\varepsilon(t,x)-u_\varepsilon(t,y)|^2\,dx\,dy.\end{aligned}$$

The second term is equal to

$$\int_0^1\int_{\mathbb{R}^{2d}} (x-y)\otimes\nabla K_h(x-y) : \nabla v(t,\theta x+(1-\theta)y)\,|u_\varepsilon(t,x)-u_\varepsilon(t,y)|^2\,dx\,dy,$$

with $A:B$ denoting the full contraction of the two matrices. Note that for $|x|>1$, ∇K_h is bounded and for $|x|<1$,

$$x\otimes\nabla K_h(x) = \frac{x\otimes x}{(|x|+h)^{d+1}|x|}.$$

Define

$$\tilde K_h(x) = x\otimes\nabla K_h(x) - \lambda\frac{\mathrm{Id}\,|x|}{(|x|+h)^{d+1}}\,\mathbb{I}_{|x|\leq 1},$$

with $\lambda = \int_{S^{d-1}} \omega_1^2\,d\omega$.

Thanks to the definition of λ, $\tilde K_h$ is now a Calderon-Zygmund operator, meaning that for any $1<q<\infty$, there exists a constant C independent of h s.t.

$$\|\tilde K_h \star g\|_{L^q} \leq C\,\|g\|_{L^q}.$$

As v is divergence free, one may simply replace by $\tilde K_h$

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^{2d}} (x-y)\otimes\nabla K_h(x-y) : \nabla v(t,\theta x+(1-\theta)y)\,|u_\varepsilon(t,x)-u_\varepsilon(t,y)|^2\,dx\,dy\\ &= \int_0^1\int_{\mathbb{R}^{2d}} \tilde K_h(x-y) : \nabla v(t,\theta x+(1-\theta)y)\,|u_\varepsilon(t,x)-u_\varepsilon(t,y)|^2\,dx\,dy\\ &\leq C\int_0^1\int_{\mathbb{R}^{2d}} K_h(x-y)\,|\nabla v(t,\theta x+(1-\theta)y)-\nabla v(t,x)|\,dx\,dy\\ &\quad+\|\tilde K_h\star(\nabla v u_\varepsilon^2)\|_{L^1} + 2\,\|u_\varepsilon\,\tilde K_h\star(\nabla v u_\varepsilon)\|_{L^1} + \|u_\varepsilon^2\,\tilde K_h\star\nabla v\|_{L^1}.\end{aligned}$$

Thanks to the uniform bounds on u_ε, and changing variables, one immediately deduce that

$$\int_0^1 \int_{\mathbb{R}^{2d}} (x-y)\otimes\nabla K_h(x-y) : \nabla v(t,\theta x+(1-\theta)y)\,|u_\varepsilon(t,x)-u_\varepsilon(t,y)|^2\,dx\,dy$$
$$\leq C + C\int_{\mathbb{R}^{2d}} K_h(x-y)\,|\nabla v(t,x)-\nabla v(t,y)|\,dx\,dy.$$

Putting together all the terms in the estimate, we have

$$\begin{aligned}\frac{dQ_\varepsilon}{dt} &\leq C + C\,\frac{\|v_\varepsilon - v\|_{L^p}}{h}\\ &+ C\int_{\mathbb{R}^{2d}} K_h(x-y)\,|\nabla v(t,x)-\nabla v(t,y)|\,dx\,dy\end{aligned}$$

or

$$\begin{aligned}Q_\varepsilon(t) \leq & C + C\,\frac{\|v_\varepsilon - v\|_{L^p}}{h}\\ &+ C\int_0^T\int_{\mathbb{R}^{2d}} K_h(x-y)\,|\nabla v(t,x)-\nabla v(t,y)|\,dx\,dy\\ &+ C\int_{\mathbb{R}^{2d}} K_h(x-y)\,|n_\varepsilon^0(x)-n_\varepsilon^0(y)|^2\,dx\,dy.\end{aligned}$$

As n_ε^0 is compact and v is independent of ε then the previous estimate shows that

$$\lim_{h\to 0} |\log h|^{-1} \limsup_\varepsilon \sup_t \int_{\mathbb{R}^{2d}} K_h(x-y)\,|n_\varepsilon(t,x)-n_\varepsilon(t,y)|^2\,dx\,dy = 0.$$

Lemma 3.1 then proves that u_ε is compact in space. However by Eq. (3.1), $\partial_t u_\varepsilon$ is uniformly bounded in $L^\infty_t(W_x^{-1,p})$. Therefore compactness in time follows and the theorem is proved.

3.3 A simple proof for Theorem 1.1

We first give here a simple proof of the compactness. This proof is not optimal in the sense that it does not give an explicit rate for how the norm in our compactness criterion behaves

$$\int_0^T \int K_h(x-y)|n_\varepsilon(t,x)-n_\varepsilon(t,y)|\,dx\,dy\,dt.$$

This is however a more difficult problem, which is partially dealt with in the next section.

As a_ε is compact in L^p, by extracting a subsequence (still denoted by ε), a_ε converges strongly in L^p to some $a \in W^{1,p}$. By the compactness of d_ε and n^0_ε and by Lemma 3.1, we may assume without loss of generality that there exists a continuous function $\delta(h)$ with $\delta(0) = 0$, independent of ε and a function $\alpha(\varepsilon)$, s.t.

$$\begin{aligned}
&|\log h|^{-1} \int_{\mathbb{R}^d} K_h(x-y)\,|n^0_\varepsilon(y) - n^0_\varepsilon(x)|\,dx\,dy \le \delta(h),\\
&|\log h|^{-1} \int_0^T \int_{\mathbb{R}^d} K_h(x-y)\,|d_\varepsilon(t,y) - d_\varepsilon(t,x)|\,dx\,dy\,dt \le \delta(h),\\
&|\log h|^{-1} \int_0^T \int_{\mathbb{R}^d} K_h(x-y)\,|\nabla a(t,y) - \nabla a(t,x)|^p\,dx\,dy\,dt \le \delta^p(h),\\
&\int_0^T \int_{\mathbb{R}^d} |a_\varepsilon(t,x) - a(t,x)|^p\,dx\,dt \le \alpha^p(\varepsilon).
\end{aligned} \tag{3.2}$$

Note that the estimate is written for ∇a and not for the sequence ∇a_ε as no compactness can be assumed on ∇a_ε.

Then one proves

Proposition 3.1 *Let n_ε be a sequence of solutions to* (1.4) *with initial data n^0_ε uniformly bounded in $L^1 \cap L^\infty$ and compact in L^1. Assume* (1.5), (1.6), (1.7) *and hence* (3.2). *Then for some constant C uniform in h and ε*

$$\int_0^T \int_{\mathbb{R}^{2d}} K_h(x-y)\,|n_\varepsilon(t,x) - n_\varepsilon(t,y)|\,dx\,dy\,dt \le C\,\frac{\varepsilon^2}{h^2} + C\,\delta(h)\,|\log h| + C\,\frac{\alpha(\varepsilon)}{h}.$$

The disappointing part of Prop. 3.1 is that the rates $\delta(h)$ and $\alpha(\varepsilon)$ are not explicit but depend intrinsically on the sequence a_ε. See the next section for a more explicit (but much more complicated) result.

Prop. 3.1 proves the compactness in space of n_ε by Lemma 3.1. The compactness in time is then straightforward since n_ε solves a transport equation (1.1).

Hence Theorem 1.1 follows.

Proof of Prop. 3.1.

The proof mostly follows the steps of the proof of Theorem 3.1. The main differences are the nonlinear flux, the vanishing viscosity terms and the fact that now the field a_ε is not assumed to be divergence free (only bounded).

First of all, by condition (1.6), for any $T > 0$, $n_\varepsilon(t,x)$ is bounded in $L^1 \cap L^\infty([0,\ T] \times \mathbb{R}^d)$, uniformly in ε.

We start with Kruzkov's usual argument of doubling of variable. If n_ε is a solution to (1.4) then

$$\begin{aligned}
&\partial_t |n_\varepsilon(t,x) - n_\varepsilon(t,y)| + \mathrm{div}_x\big(a_\varepsilon(t,x) F(n_\varepsilon(t,x), n_\varepsilon(t,y))\big) \\
&\quad + \mathrm{div}_y\big(a_\varepsilon(t,y) F(n_\varepsilon(t,y), n_\varepsilon(t,x))\big) + \mathrm{div}_x a_\varepsilon(t,x)\; G(n_\varepsilon(t,x), n_\varepsilon(t,y)) \\
&\quad + \mathrm{div}_y a_\varepsilon(t,y)\; G(n_\varepsilon(t,y), n_\varepsilon(t,x)) - \varepsilon^2\,(\Delta_x + \Delta_y)\,|n_\varepsilon(t,x) - n_\varepsilon(t,y)| \leq 0.
\end{aligned}$$

This computation is formal but can easily be made rigourous by using a suitable regularisation of $|.|$. Here F satisfies

$$F'(\xi,\zeta) = f'(\xi)\,\mathrm{sign}(\xi - \zeta),\ \ F(\xi,\zeta) = 0,$$

which means that

$$F(\xi,\zeta) = (f(\xi) - f(\zeta))\,\mathrm{sign}(\xi - \zeta) = F(\zeta,\xi).$$

And as for G

$$G(\xi,\zeta) = f(\xi)\,\mathrm{sign}(\xi - \zeta) - F(\xi,\zeta) = \bar{G}(\xi,\zeta) - \frac{1}{2}\,F(\xi,\zeta),$$

with $\bar{G}(\xi,\zeta) = \frac{1}{2}(f(\xi) + f(\zeta))\,\mathrm{sign}(\xi - \zeta) = -\bar{G}(\zeta,\xi)$. Now define

$$Q(t) = \int_{\mathbb{R}^{2d}} K_h(x-y)\,|n_\varepsilon(t,x) - n_\varepsilon(t,y)|\,dx\,dy.$$

Remark that

$$\begin{aligned}
\varepsilon^2 \int_{\mathbb{R}^{2d}} & K_h(x-y)\,(\Delta_x + \Delta_y)\,|n_\varepsilon(t,x) - n_\varepsilon(t,y)|\,dx\,dy \\
&= \varepsilon^2 \int_{\mathbb{R}^{2d}} \Delta K_h(x-y)\,|n_\varepsilon(t,x) - n_\varepsilon(t,y)|\,dx\,dy \\
&\leq C\,\varepsilon^2 \|\Delta K_h\|_{L^1} \leq C\,\frac{\varepsilon^2}{h^2}.
\end{aligned}$$

Using this and because of the symmetry of F and the antisymmetry of $\bar{G}$

$$\begin{aligned}\frac{d}{dt}Q(t) \leq & C\,\frac{\varepsilon^2}{h^2} + \int_{|x-y|\leq 1} \frac{x-y}{(|x-y|+h)^{d+2}} \cdot (a_\varepsilon(t,y)-a_\varepsilon(t,x)) \\ & \qquad\qquad F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy \\ & + \int_{|x-y|\geq 1} \nabla K(x-y)\cdot (a_\varepsilon(t,y)-a_\varepsilon(t,x))\,F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy \\ & + \int_{\mathbb{R}^{2d}} K_h(x-y)\,(\operatorname{div} a_\varepsilon(t,x)-\operatorname{div} a_\varepsilon(t,y))\,\bar{G}(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy \\ & - \int_{\mathbb{R}^{2d}} K_h(x-y)\,(\operatorname{div} a_\varepsilon(t,x)+\operatorname{div} a_\varepsilon(t,y))\,F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy \\ & = C\,\frac{\varepsilon^2}{h^2} + A + B + D + C\,\|\operatorname{div} a_\varepsilon\|_{L^\infty}\,Q(t).\end{aligned}$$

Let us begin with the last term. Use (1.7) to decompose

$$\begin{aligned}D \leq & \int_{\mathbb{R}^{2d}} K_h(x-y)\,|d_\varepsilon(t,x)-d_\varepsilon(t,y)|\,|\bar{G}(n_\varepsilon(t,y),n_\varepsilon(t,x))|\,dx\,dy \\ & + C\int_{\mathbb{R}^{2d}} K_h(x-y)\,|n_\varepsilon(t,x)-n_\varepsilon(t,y)|\,|\bar{G}(n_\varepsilon(t,y),n_\varepsilon(t,x))|\,dx\,dy.\end{aligned}$$

As $G(n_\varepsilon(t,x),n_\varepsilon(t,y))$ is uniformly bounded in L^∞, one gets from (3.2)

$$\int_0^T D\,dt \leq |\log h|\,\delta(h) + C\int_0^T Q(t)\,dt. \tag{3.3}$$

For the second term B, just note that $\nabla K \in C_c^\infty(\mathbb{R}^d \setminus B(0,1))$, and that $|F(n_\varepsilon(t,x),n_\varepsilon(t,x))| \leq |f(n_\varepsilon(t,x))| + |f(n_\varepsilon(t,y))|$ is uniformly bounded in L^1. So one simply has

$$B \leq C.$$

The main term is hence A. Using again (3.2) and the bound on $|F(.,.)|$, one gets

$$\begin{aligned}\int_0^T A\,dt \leq & C\,\frac{\alpha(\varepsilon)}{h} + \int_0^T\int_{|x-y|\leq 1} \frac{x-y}{(|x-y|+h)^{d+1}\,|x-y|}\cdot (a(t,y)-a(t,x)) \\ & \qquad F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy\,dt \\ \leq & C\,\frac{\alpha(\varepsilon)}{h} + \int_0^T\int_0^1\int_{|x-y|\leq 1} \frac{(x-y)\otimes(x-y)}{(|x-y|+h)^{d+1}\,|x-y|} : \nabla a(t,\theta x+(1-\theta)y) \\ & \qquad F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy\,d\theta\,dt.\end{aligned}$$

Still using (3.2),

$$\int_0^T A\,dt \le C\,\left(\frac{\alpha(\varepsilon)}{h} + |\log h|\,\delta(h)\right)$$
$$+\int_0^T \int_{|x-y|\le 1} \frac{(x-y)\otimes(x-y)}{(|x-y|+h)^{d+1}\,|x-y|}\ :\nabla a(t,x) F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy\,dt$$
$$\le C\,\left(\frac{\alpha(\varepsilon)}{h} + |\log h|\,\delta(h)\right) + \int_0^T E(t)\,dt.$$

Denote as in the proof of Theorem 3.1

$$\lambda = \int_{S^{d-1}} \omega_1^2\,dS(\omega), \quad \bar K_h(x) = \left(\frac{x\otimes x}{(|x|+h)^{d+1}\,|x|} - \lambda\frac{|x|}{(|x|+h)^{d+1}}\,Id\right)\,\mathbb{I}_{|x|\le 1}.$$

By the definition of λ, $\bar K_h$ is a Calderon-Zygmund operator bounded on any L^p for $1<p<\infty$. Now write

$$E = \int_{\mathbb{R}^{2d}} \bar K_h(x-y)\,\nabla a(x)\,F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy$$
$$+\lambda\int_{|x-y|\le 1}\frac{|x-y|}{(|x-y|+h)^{d+1}}\,\mathrm{div}\,a(t,x)\ F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy$$
$$\le \int_{\mathbb{R}^{2d}} \bar K_h(x-y)\,\nabla a(x)\,F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy + C\,Q(t),$$

as the divergence of a is bounded.

Introduce

$$\chi_\varepsilon(t,x,\xi) = \mathbb{I}_{0\le\xi\le n_\varepsilon(t,x)}.$$

Then note that χ_ε is compactly supported in ξ and that

$$F(n_\varepsilon(t,y),n_\varepsilon(t,x)) = \int_0^\infty f'(\xi)\,|\chi_\varepsilon(t,x,\xi)-\chi_\varepsilon(t,y,\xi)|^2\,d\xi.$$

Hence as $\nabla a\in L^p$, and χ_ε is uniformly bounded in $L^\infty_{t,\xi}(L^1_x\cap L^\infty_x)$, for $1/p+1/p^*=1$,

$$\int_{\mathbb{R}^{2d}} \bar K_h(x-y)\,\nabla a(x)\,F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy$$
$$=\int_{\mathbb{R}_+} f'(\xi)\,\int_{\mathbb{R}^{2d}} \bar K_h(x-y)\,\nabla a(x)\,|\chi_\varepsilon(t,x,\xi)-\chi_\varepsilon(t,y,\xi)|^2\,d\xi\,dx\,dy$$
$$\le \int_{\mathbb{R}_+} f'(\xi)\,\left(\|\bar K_h\star(\nabla a\chi_\varepsilon^2)\|_{L^1} + \|\bar K_h\star\chi_\varepsilon^2)\|_{L^{p^*}} + 2\|\bar K_h\star(\nabla a\chi_\varepsilon)\|_{L^1}\right)d\xi$$
$$\le C.$$

Combining all estimates we conclude that

$$Q(t) \leq Q(0) + C\,\frac{\varepsilon^2}{h^2} + C\,|\log h|\,\delta(h) + C\,\frac{\alpha(\varepsilon)}{h} + C\int_0^t Q(s)\,ds.$$

The initial data $Q(0)$ is bounded by (3.2) and finally by Gronwall lemma we obtain on any finite interval

$$Q(t) \leq C\,\frac{\varepsilon^2}{h^2} + C\,|\log h|\,\delta(h) + C\,\frac{\alpha(\varepsilon)}{h},$$

which proves the proposition.

4 An explicit estimate : Proof of Theorem 1.2

Checking carefully the proof of Prop. 3.1, one sees that to get an explicit rate, it would be necessary to bound a term like

$$\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\,(a_\varepsilon(x)-a_\varepsilon(y))\,|g_\varepsilon(x)-g_\varepsilon(y)|^2\,dx\,dy \tag{4.1}$$

only in terms of the $W^{1,p}$ norm of a_ε and the $L^1\cap L^\infty$ norms of g_ε.

Here we do not aim at optimal estimates, just explicit ones. We present a very elementary proof of

Proposition 4.1 *Let $1 < p < \infty$, $\exists C_p < \infty$ s.t. $\forall a(x)$, $g(x)$ smooth and compactly supported*

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\,(a(x)-a(y))\,|g(x)-g(y)|^2\,dx\,dy \\ \leq C_p\,\|g\|_{L^\infty}\,\|g\|_{L^1\cap L^{p^*}}\,\|\nabla a\|_{L^p\cap L^1}\,|\log h|^{1/\bar p} \\ + C_p\,(\|\mathrm{div}\,a\|_{L^\infty} + \|\nabla a\|_{L^p})\int_{\mathbb{R}^{2d}} K_h(x-y)|g(x)-g(y)|^2\,dx\,dy,\end{aligned}$$

with $1/p^* + 1/p = 1$ and $\bar p = \min(p,2)$.

Note that the rate $|\log h|^{1/\bar p}$ is most probably not optimal. A way to obtain a better rate could be to combine Lemma 4.1 below with the estimates in [33] as we suggest below.

The kind of Calderon-like estimate like Prop. 4.1 has been extensively studied in dimension 1, see for instance [9] or [11], [12]. The situation in higher dimension is however more complicated. In particular it seems necessary to use the bound on the divergence of a_ε to estimate (4.1) (as was already suggested by the proof of Prop. 3.1).

Following the previous section, a simple idea would be to estimate (4.1) by

$$\begin{aligned} &C\,\|\mathrm{div}\, a\|_{L^\infty} \int_{\mathbb{R}^{2d}} K_h(x-y)\,|g_\varepsilon(x)-g_\varepsilon(y)|^2\,dx\,dy \\ &+\int_0^1\int_{\mathbb{R}^{2d}} L_h(x-y):\nabla a(\theta x+(1-\theta)y)\,|g_\varepsilon(x)-g_\varepsilon(y)|^2\,dx\,dy, \end{aligned}$$

where L_h is now a Calderon-Zygmund operator. Expanding the square, one sees that it would be enough to bound in some L^q space

$$\int_0^1\int_{\mathbb{R}^d} L_h(x-y):\nabla a(\theta x+(1-\theta)y)\,g_\varepsilon(y)\,dy.$$

Using Fourier transform (we denote by $\mathcal{F}$ the Fourier transform) and an easy change of variable, this term is equal to

$$\int_{\mathbb{R}^{2d}} e^{ix\cdot(\xi_1+\xi_2)}\,m(\xi_1,\xi_2)\,\mathcal{F}\nabla a(\xi_1)\,\mathcal{F}g(\xi_2)\,d\xi_1\,d\xi_2,$$

with

$$m(\xi_1,\xi_2)=\int_0^1 \mathcal{F}L_h(\theta\,\xi_1+\xi_2)\,d\theta.$$

We now have a multi-linear operator in dimension d of the kind studied in Muscalu, Tao, Thiele [33]. Unfortunately m does not satisfy the assumptions of this last article as it does not have the right behaviour on the subspace $\xi_1 \parallel \xi_2$. Instead it would be necessary to have a multi-dimensional equivalent of [31] (which, as far as we know, is not yet proved) or to use Lemma 4.1.

4.1 Proof of Theorem 1.2 given Prop. 4.1

For the moment let us assume Prop. 4.1. Define

$$Q(t)=\int_{\mathbb{R}^{2d}} K_h(x-y)\,|n_\varepsilon(t,x)-n_\varepsilon(t,y)|\,dx\,dy.$$

We follow the same first steps as in the proof of Prop. 3.1, with the same notations. We obtain

$$\begin{aligned}\frac{dQ}{dt} \leq & C + C\,\frac{\varepsilon^2}{h^2} + C\,Q(t) + C\,\int_{\mathbb{R}^{2d}} K_h(x-y)\,|d_\varepsilon(t,x)-d_\varepsilon(t,y)|\,dx\,dy \\ & + \int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\cdot(a_\varepsilon(t,x)-a_\varepsilon(t,y))\,F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy.\end{aligned}$$

We only have to bound the last term. Let us introduce again

$$\chi_\varepsilon(t,x,\xi) = \mathbb{I}_{0\leq\xi\leq n_\varepsilon(t,x)}.$$

Note that χ_ε is supported in ξ in $[0,\ \|n_\varepsilon^0\|_{L^\infty}]\subset[0,\ C]$.

Now write

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\cdot(a_\varepsilon(t,x)-a_\varepsilon(t,y))\,F(n_\varepsilon(t,y),n_\varepsilon(t,x))\,dx\,dy \\ &\quad = \int_0^C f'(\xi)\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\cdot(a_\varepsilon(t,x)-a_\varepsilon(t,y)) \\ &\qquad\qquad\qquad |\chi_\varepsilon(t,y,\xi)-\chi_\varepsilon(t,x,\xi)|^2\,dx\,dy\,d\xi \\ &\quad \leq C\,|\log h|^{1-2/\bar p} + C\,\int_{\mathbb{R}^{2d}} K_h(x-y)\,\int_0^C |\chi_\varepsilon(t,y,\xi)-\chi_\varepsilon(t,x,\xi)|^2\,d\xi\,dx\,dy,\end{aligned}$$

using Prop. 4.1 and the uniform bounds on $\|a_\varepsilon\|_{L^\infty_t L^p_x}$ and $\|\chi_\varepsilon\|_{L^1\cap L^\infty}$. Now simply note that because of the definition of χ_ε

$$\int_0^C |\chi_\varepsilon(t,y,\xi)-\chi_\varepsilon(t,x,\xi)|^2\,d\xi \leq |n_\varepsilon(t,x)-n_\varepsilon(t,y)|,$$

and the last term in the previous inequality is hence simply bounded by Q. One finally obtains

$$\begin{aligned}\frac{dQ}{dt} \leq & C + +C\,\frac{\varepsilon^2}{h^2} + C\,Q(t) + C\,|\log h|^{1-2/\bar p} \\ & + C\,\int_{\mathbb{R}^{2d}} K_h(x-y)\,|d_\varepsilon(t,x)-d_\varepsilon(t,y)|\,dx\,dy.\end{aligned}$$

To conclude the proof of Theorem 1.2, it is now enough to apply Gronwall's lemma.

4.2 Beginning of the proof of Prop. 4.1

As before we will control $a(x) - a(y)$ with ∇a. Contrary to the previous case though, it is not enough to integrate over the segment. Instead use the lemma

Lemma 4.1

$$a_i(x) - a_i(y) = |x-y| \int_{B(0,1)} \psi\left(z, \frac{x-y}{|x-y|}\right) \cdot \nabla a_i\left(x + |x-y|\, z\right) \frac{dz}{|z|^{d-1}}$$
$$+ |x-y| \int_{B(0,1)} \psi\left(z, \frac{x-y}{|x-y|}\right) \cdot \nabla a_i\left(y + |x-y|\, z\right) \frac{dz}{|z|^{d-1}},$$

where $|z|\,\psi$ is Lipschitz on $B(0,1)\times S^{d-1}$ and for a given constant α,

$$\int_{B(0,1)} \psi\left(z, \frac{x-y}{|x-y|}\right) \frac{dz}{|z|^{d-1}} = \alpha \frac{x-y}{|x-y|}.$$

Proof of Lemma 4.1. We refer to [10] for a complete, detailed proof. Let us simply mention that the idea is to integrate along many trajectories between x and y instead of just the segment. □

Lemma 4.1 gives two terms that are completely symmetric and it is enough to deal with one of them. After an easy change of variable, one finds

$$\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\,(a(x)-a(y))\,|g(x)-g(y)|^2\,dx\,dy$$
$$= \int_{\mathbb{R}^d}\int_0^1 \frac{r^d}{(r+h)^{d+1}} \int_{B(0,1)}\int_{S^{d-1}} \psi(z,\omega)\otimes\omega \,:\, \nabla a(x+rz)$$
$$|g(x)-g(x+r\omega)|^2\,d\omega\,\frac{dz}{|z|^{d-1}}\,dr\,dx + \quad \textit{symmetric},$$

where $A:B$ denotes the total contraction of two matrices $\sum_{i,j} A_{ij}\,B_{ij}$.

Now define

$$L(z,\omega) = \psi(z,\omega)\otimes\omega - \lambda\,\mathrm{Id},$$

for $\lambda = \int_{B(0,1)}\int_{S^{d-1}} \omega_1^2\,d\omega\,\frac{dz}{|z|^{d-1}}$.

Note that

$$\int_{\mathbb{R}^d}\int_0^1 \frac{r^d}{(r+h)^{d+1}}\int_{B(0,1)}\int_{S^{d-1}} \psi(z,\omega)\otimes\omega\ :\ \nabla a(x+rz)$$

$$|g(x)-g(x+r\omega)|^2\,d\omega\,\frac{dz}{|z|^{d-1}}\,dr\,dx$$

$$\leq \int_0^1\int_{\mathbb{R}^d\times B(0,1)\times S^{d-1}} \frac{r^d\,L(z,\omega)}{(r+h)^{d+1}}\ :\ \nabla a(x+rz)\,|g(x)-g(x+r\omega)|^2\,\frac{1}{|z|^{d-1}}$$

$$+\,C\,\|\mathrm{div}\,a\|_{L^\infty}\int_0^1\int_{\mathbb{R}^d\times S^{d-1}} \frac{r^d}{(r+h)^{d+1}}\,|g(x)-g(x+r\omega)|^2$$

By the definition of K_h, the second term is bounded by

$$C\,\|\mathrm{div}\,a\|_{L^\infty}\int_{\mathbb{R}^{2d}} K_h(x-y)|g(x)-g(y)|^2\,dx\,dy.$$

and it only remains to bound the first one. In order to get the optimal rate for $\nabla a\in L^p$ with $p>2$, we need to introduce an additional decomposition of ∇a. For $p>2$ as L^p may be obtained by interpolating between L^2 and L^∞, let

$$\nabla a = A+\bar A,\quad \|\bar A\|_{L^\infty}\leq 2\|\nabla a\|_{L^p},\quad \|A\|_{L^2}\leq 2\|\nabla a\|_{L^p}.$$

If $p<2$ then we simply put $A=\nabla a$. In both cases, if ∇a is smooth and compactly supported then one may of course assume the same of A and $\bar A$.

Define

$$Q(A,g)=\int_0^1\int_{B(0,1)\times S^{d-1}} \frac{r^d\,L(z,\omega)}{(r+h)^{d+1}}\ :\ A(x+rz)\,g(x+r\omega)\,d\omega\,\frac{dz}{|z|^{d-1}}\,dr.$$

The term with $\bar A$ may be bounded directly by using the L^∞ norm of $\bar A$; for the other one simply by expanding the square $|g(x)-g(y)|^2$, one obtains

$$\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\,(a(x)-a(y))\,|g(x)-g(y)|^2\,dx\,dy$$

$$\leq C\,(\|\mathrm{div}\,a\|_{L^\infty}+\|\nabla a\|_{L^p})\int_{\mathbb{R}^{2d}} K_h(x-y)\,|g(x)-g(y)|^2\,dx\,dy$$

$$+\int_{\mathbb{R}^d}(-2g\,Q(A,g)+g^2\,Q(A,1))\,dx.$$

Note that bounding $Q(A,1)$ is in fact easy as it is an ordinary convolution and $\frac{1}{r}\,L$ defines a Calderon-Zygmund operator. However the control of $Q(A,g)$ essentially requires to rework Calderon-Zygmund theory.

Of course for r of order h then one has

$$\begin{aligned}\Big\|\int_0^h\int_{B(0,1)\times S^{d-1}} &\frac{r^d\,L(z,\omega)}{(r+h)^{d+1}}\,:\,A(x+rz)\,g(x+r\omega)\,d\omega\,\frac{dz}{|z|^{d-1}}\,dr\Big\|_{L^1}\\ &\leq \frac{1}{h}\int_0^h\int_{B(0,1)\times S^{d-1}}\|A(x+rz)\,g(x+r\omega)\|_{L^1}\;d\omega\,\frac{dz}{|z|^{d-1}}\,dr\\ &\leq \|A\|_{L^1}\,\|g\|_{L^\infty}.\end{aligned}$$

It is hence enough to consider

$$\bar Q(A,g)=\int_h^1\int_{B(0,1)\times S^{d-1}}\frac{r^d\,L(z,\omega)}{(r+h)^{d+1}}\,:\,A(x+rz)\,g(x+r\omega)\,d\omega\,\frac{dz}{|z|^{d-1}}\,dr.$$

We introduce the Littlewood-Paley decomposition of A (see for instance Triebel [39])

$$A(x)=\sum_{i=0}^\infty A_i(x),$$

where for $i>0$, $\hat A_i=\mathcal{F}A\,p(2^{-i}\xi)$ with p compactly supported in the annulus of radii $1/2,\ 2$; and $\hat A_0=\mathcal{F}A\,p_0(\xi)$. The functions p and p_0 determines a partition of unity. We note either $\mathcal{F}g$ or $\hat g$ the Fourier transform of a function g. In the following we denote by $\mathcal{P}_i$ the projection operator $\mathcal{P}_i\phi=\mathcal{F}^{-1}p(2^{-i}\xi)\,\hat\phi$.

There is an obvious critical scale in the decomposition which is where 2^{-i} is of order r. Accordingly we decompose further

$$\begin{aligned}&\bar Q(A,g)=Q_1(A,g)+Q_2(A,g)\\ &\ =\sum_{i\leq|\log h|}\int_h^{2^{-i}}\int_{B(0,1)\times S^{d-1}}\frac{r^d\,L(z,\omega)}{(r+h)^{d+1}}\,:\,A_i(x+rz)\,g(x+r\omega)\,d\omega\,\frac{dz}{|z|^{q-1}}\,dr\\ &\ +\sum_i\int_{\max(h,2^{-i})}^1\int_{B(0,1)\times S^{d-1}}\frac{r^d\,L(z,\omega)}{(r+h)^{d+1}}\,:\,A_i(x+rz)\,g(x+r\omega)\,d\omega\,\frac{dz}{|z|^{q-1}}\,dr.\end{aligned}$$

Each term is bounded in a different way. Note of course that in Q_1 as $r\geq h$ there is of course no frequency i higher than $|\log h|$ (they are all in Q_2).

4.3 Control on Q_1 in L^2

The aim is here is to prove

Lemma 4.2 $\forall 1 < q < \infty$, $\exists C > 0$ *such that for any A and g smooth and compactly supported functions,*

$$\|Q_1(A,g)\|_{L^1} \leq C\,\|A\|_{B^0_{q,1}}\,\|g\|_{L^{q^*}}.$$

where $B^0_{q,1}$ is the usual Besov space and $1/q^* + 1/q = 1$.

As we wish to remain as elementary as possible here, we avoid the use of Besov spaces in the sequel. Instead for $q = 2$ it it possible to obtain directly the Lebesgue space by losing $|\log h|^{1/2}$ namely

Lemma 4.3 $\exists C > 0$ *such that for any A and g smooth and compactly supported functions,*

$$\|Q_1(A,g)\|_{L^1} \leq C\,|\log h|^{1/2}\,\|g\|_{L^2}\,\|A\|_{L^2}.$$

The proof is relatively simple. Indeed in Q_1 since $r < 2^{-i}$, A_i does not change much over a ball of radius r. Therefore, we simply replace $A_i(x+rz)$ by $A_i(x)$ in Q_1. This gives

$$\begin{aligned}
&Q_1(A,g) \leq I + II\\
&\leq \sum_{i\leq|\log h|}\int_h^{2^{-i}}\int_{B(0,1)\times S^{d-1}} \frac{r^d\,L(z,\omega)}{(r+h)^{d+1}} \,:\, A_i(x)\,g(x+r\omega)\,d\omega\,\frac{dz}{|z|^{d-1}}\,dr\\
&\quad+\sum_{i\leq|\log h|}\int_h^{2^{-i}}\frac{1}{r+h}\int_{S^{d-1}}|g(x+r\omega)|\,d\omega\\
&\qquad\qquad\int_{B(0,1)}|A_i(x+rz)-A_i(x)|\,\frac{dz}{|z|^{d-1}}\,dr.
\end{aligned}\tag{4.2}$$

Let us bound the first term. As A_i does not depend on z anymore, this term is simply equal to

$$\sum_{i,j\leq i} A_i(x)\int_h^1\int_{S^{d-1}}\tilde L_i(r\omega)g_j(x+r\omega)\,r^{d-1}d\omega\,dr,$$

where

$$\tilde{L}_i(r\omega) = \frac{r}{(r+h)^{d+1}}\,(\omega\otimes\omega - \tilde{\lambda} I) = \int_{B(0,1)} \frac{r\mathbb{I}_{r\leq 2^{-i}}}{(r+h)^{d+1}}\; L(z,\omega)\,\frac{dz}{|z|^{d-1}}.$$

By the definition of λ, $\tilde{\lambda} = \int_{S^{d-1}} \omega_1^2\, d\omega$ and hence $\tilde{L}_i$ is a Calderon-Zygmund operator with operator norm bounded uniformly in i.

Now write for $1/q^* + 1/q = 1$

$$\begin{aligned}\|I(x)\|_{L^1} &\leq \sum_{i\leq|\log h|} \|A_i\|_{L^q}\, \|\tilde{L}_i \star g\|_{L^{q^*}} \\ &\leq C\,\|g\|_{L^{q^*}} \sum_i \|A_i\|_{L^q} = C\,\|g\|_{L^{q^*}}\,\|A\|_{B^0_{q,1}}.\end{aligned}$$

Let us turn to the second term.

$$\begin{aligned}\|II\|_{L^1} \leq &\sum_{i\leq|\log h|} \int_{S^{d-1}} \int_h^{2^{-i}} \left\| \frac{g(x+r\omega)}{r+h} \int_{B(0,1)} |A_i(x+rz) - A_i(x)|\,\frac{dz}{|z|^{d-1}} \right\|_{L^1} \\ &\leq C\,\|g\|_{L^{q^*}} \sum_{i\leq|\log h|} \int_{B(0,1)} \int_h^{2^{-i}} \|A_i(.+rz) - A_i(.)\|_{L^q}\,\frac{dr}{r+h}\,\frac{dz}{|z|^{d-1}}.\end{aligned}$$

So

$$\begin{aligned}\|II\|_{L^1} \leq C\,\|g\|_{L^{q^*}} &\sum_{i\leq|\log h|} (\|A_i(.)\|_{L^q} + \|A_{i+1}(.)\|_{L^q} + \|A_{i-1}(.)\|_{L^q}) \\ &\int_{B(0,1)} \int_h^{2^{-i}} 2^i\, r\,|z|\,\frac{dr}{r+h}\,\frac{dz}{|z|^{d-1}},\end{aligned}$$

where we used the localization in Fourier space of the A_i and more precisely the well known property

$$\|A_i(.+\eta) - A_i(.)\|_{L^q} \leq C\,2^i\,|\eta|\,(\|A_i(.)\|_{L^q} + \|A_{i+1}(.)\|_{L^q} + \|A_{i-1}(.)\|_{L^q}).$$

One then concludes that

$$\|II\|_{L^1} \leq C\,\|g\|_{L^{q^*}} \sum_{i\leq|\log h|} \|A_i\|_{L^q} = C\,\|g\|_{L^{q^*}}\,\|A\|_{B^0_{q,1}}.$$

Combining the estimates on I and II in (4.2) gives Lemma 4.2.

For the proof of Lemma 4.3, it is enough to observe that in the case $q = 2$ by Cauchy-Schwartz

$$\sum_{i\le|\log h|} \|A_i\|_{L^2} \le |\log h|^{1/2} \left(\sum_i \|A_i\|_{L^2}^2\right) = |\log h|^{1/2} \|A\|_{L^2}.$$

4.4 Control on Q_2 for $A \in L^2$

As for usual Calderon-Zygmund theory, the optimal bound on Q_2 is obtained in a L^2 setting namely

Lemma 4.4 *$\exists C > 0$ s.t. for any g and A smooth with compact support*

$$\|Q_2(A, g)\|_{L^2} \le C \|g\|_{L^\infty} \|A\|_{L^2}.$$

To prove this, first bound

$$|Q_2(A, g)| \le \|g\|_{L^\infty} \int_0^1 \int_{S^{d-1}} \left| \int_{B(0,1)} \frac{r^d L(z,\omega)}{(r+h)^{d+1}} : \sum_{i\ge -\log_2 r} A_i(x + rz) \frac{dz}{|z|^{d-1}} \right| d\omega\ dr.$$

Hence

$$\|Q_2(A, g)\|_{L^2} \le C\, R\, \|g\|_{L^\infty}^2 + \frac{C}{R} \int_0^1 \int_{S^{d-1}} \left\| \int_{B(0,1)} \frac{r^d L(z,\omega)}{(r+h)^{d+1}} : \sum_{i\ge -\log_2 r} A_i(. + rz) \frac{dz}{|z|^{d-1}} \right\|_{L^2}^2 d\omega\ dr.$$

Use Fourier transform and Plancherel equality on the last term to bound it by

$$\sum_{\alpha,\beta=1}^{3} \int_0^1 \frac{1}{r+h} \int_{S^{d-1}} \int_{|\xi|\ge 1/r} |\mathcal{F} A_{\alpha\beta}(\xi)|^2 \int_{B(0,1)\times B(0,1)} L_{\alpha\beta}(z,\omega)\, L_{\alpha\beta}(z',\omega)\, e^{i\xi\cdot r(z-z')} \frac{dz}{|z|^{d-1}} \frac{dz'}{|z'|^{d-1}}\, d\xi\, d\omega\, dr$$

One only has to bound the multiplier

$$m(\xi,\omega,r)=\int_{B(0,1)\times B(0,1)} L_{\alpha\beta}(z,\omega)\,L_{\alpha\beta}(z',\omega)\,e^{i\xi\cdot r(z-z')}\frac{dz}{|z|^{d-1}}\,\frac{dz'}{|z'|^{d-1}}.$$

Define

$$M(\xi,\omega,r,s)=\int_{S^{d-1}} L_{\alpha\beta}(su,\omega)e^{irs\xi\cdot u}\,du,$$

such that

$$m(\xi,\omega,r)=\int_0^1\int_0^1 M(\xi,\omega,r,s)\,\bar M(\xi,\omega,r,s')\,ds\,ds'.$$

Assuming for instance that ξ is along the first axis, by the regularity on ψ and hence L given by Lemma 4.1

$$\begin{aligned}|M(\xi,\omega,r,s)|&=\left|\int_{S^{d-1}} L_{\alpha\beta}(su,\omega)e^{irs|\xi|u_1}\,du\right|\leq\frac{1}{r\,|\xi|}\int_{S^{d-1}}|\nabla_z L_{\alpha\beta}(su,\omega)|\,du\\&\leq\frac{C}{r\,s\,|\xi|}.\end{aligned}$$

As M is also obviously bounded, one deduces that

$$|M(\xi,\omega,r,s)|\leq\frac{C}{\sqrt{r\,s\,|\xi|}}.$$

Introducing this in m immediately gives

$$m(\xi,\omega,r)\leq\frac{C}{r\,|\xi|}.$$

Therefore eventually

$$\begin{aligned}\|Q_2(A,g)\|_{L^2}&\leq C\,R\,\|g\|_{L^\infty}^2+\frac{C}{R}\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^d}|\mathcal{F}A_{\alpha\beta}(\xi)|^2\int_{|\xi|^{-1}}^1\frac{1}{r+h}\,\frac{1}{r\,|\xi|}\,dr\,d\xi\\&\leq C\,R\,\|g\|_{L^\infty}^2+\frac{C}{R}\sum_{\alpha,\beta=1}^{3}\int_{\mathbb{R}^d}|\mathcal{F}A_{\alpha\beta}(\xi)|^2\,d\xi\\&\leq C\,\|g\|_{L^\infty}\,\|A\|_{L^2},\end{aligned}$$

by optimizing in R, which proves the lemma.

4.5 Control on Q for $A \in L^p$

To get an optimal bound, one should now try to obtain weak-type estimates on Q_2, showing for instance that it belongs to $L^1 - weak$ if $A \in L^1$; and then use interpolation. Additionally, we would have to use the bound given by Lemma 4.2 with Besov spaces.

However here we will be satisfied with any explicit rate, even if it is not optimal. We hence completely avoid some (not negligible) technical difficulties and obtain instead

Lemma 4.5 *$\forall 1 < q < \infty$, $\exists C > 0$ s.t. for any smooth g and A with compact support*

$$\|Q(A,g)\|_{L^1+L^q} \leq C\,|\log h|^{1/\bar q}\,\|g\|_{L^\infty\cap L^2}\,\|A\|_{L^q},$$

where $\bar q = \min(q, q^*)$ with $1/q^* + 1/q = 1$.

Remark. Note that thanks to our decomposition of ∇a, we only use Lemma 4.5 for $q \leq 2$.

Proof of Lemma 4.5. Fix g and consider $Q(A,g)$ as a linear operator on A. The easy control for $r \leq h$, Lemmas 4.3 and 4.4 imply that this operator is bounded from L^2 to $L^1 + L^2$ with norm $C\,(\|g\|_{L^\infty} + |\log h|^{1/2}\,\|g\|_{L^2})$.

On the other hand, one has the easy estimate

$$|\bar Q(A,g)| \leq C\,\|g\|_{L^\infty} \int_0^1 \frac{1}{r+h} \int_{B(0,1)} |A(x+rz)|\,\frac{dz}{|z|^{d-1}}.$$

Therefore for any $1 \leq q \leq \infty$, $\bar Q$ is bounded on L^q with norm less than $C\,\|g\|_{L^\infty}\,|\log h|$. By usual interpolation, one deduces the lemma.

4.6 Conclusion on the proof of Prop. 4.1

By subsection 4.2

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \nabla K_h(x-y)\,(a(x)-a(y))\,|g(x)-g(y)|^2\,dx\,dy \\
&\quad\leq C\,(\|\mathrm{div} a\|_{L^\infty} + \|\nabla a\|_{L^p}) \int_{\mathbb{R}^{2d}} K_h(x-y)\,|g(x)-g(y)|^2\,dx\,dy \\
&\quad\quad + 2\|g\|_{L^1\cap L^\infty}\,\|Q(A,g)\|_{L^1+L^\infty} + 2\,\|g^2\|_{L^{p^*}}\,\|Q(A,1)\|_{L^p}.
\end{aligned}$$

Bound directly $Q(A,g)$ by Lemma 4.5 and observe that $Q(A,1)$ is bounded on any L^p with $1 < p < \infty$. This completes the proof of the proposition.

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