Homework 4. Due Tue. Oct. 2 by 4.30pm in my mailbox

1. **(5 pts)** Ref. [1], Chapter 5, Problem 1 (page 123). I am slightly rephrasing it.

The solution to \( Au = b \) may be written \( u = A^{-1}b \). The goal of this exercise is to show that you should NOT write the command \( u = \text{inv}(A) \ast b \) in Matlab as it is more than twice as expensive to execute as \( u = A \backslash b \). In other lower level languages, it is also cheaper to solve \( Au = b \) using Gaussian elimination. Below you will calculate the computational cost of finding \( B = \text{inv}(A) \).

(a) Show that about \((2/3)n^3\) flops reduces \( AB = I \) to \( UB = L^{-1} \).

(b) Show that computing the entries of \( B \) from \( UB = L^{-1} \) by back substitution takes about \( n^3 \) flops.

(c) Use this to verify the claim that computing \( A^{-1} \) is more than twice as expensive as solving \( Au = b \) by LU factorization.

2. **(10 pts)** Ref. [1], Chapter 5, Problem 4 (page 124). **Comments.** In (c), Exercise 11 is the one from Chapter 4.

3. **(5 pts)** Let \( A \) be \( N \times N \) symmetric matrix. Use Householder matrices to show that there exists an orthogonal matrix \( Q \) such that \( T := QT \ \text{AQ} \) is tridiagonal. **Hint:** Let \( N \geq 3 \). **Design a Householder matrix** \( Q_1 \) **such that** \( Q_1 A \) **has all zeros in the first column below row 2. Then show that** \( A_1 := Q_1 AQ_1^T \) **has zeros in the first crow after column 2. If** \( N > 3 \), **design a Householder matrix** \( Q_2 \) **such that** \( Q_2 A_1 \) **has all zeros in column 1 below row 2 and in column 2 below row 3. Set** \( A_2 := Q_2 A_1 Q_2^T \) **and show that it has zeros all zeros in row 1 after column 2 and in row 2 after column 3. And so on. At the end, set** \( Q := Q_{N-2} \ldots Q_2 Q_1 \). **You can look up Householder transformations in [2] (Section 3.4.1)**

**Remark** If an \( N \times N \) matrix \( A \) is symmetric then there exist orthogonal matrix \( V \) and a diagonal matrix \( D \) such that \( A = VDV^T \) (the eigenvalue decomposition). However, if \( N > 4 \), the matrix \( V \) in principle cannot be found exactly at a finite number of steps except for some special cases. This is due to the fact that the roots of a polynomial of degree \( \geq 5 \) cannot be expressed as any finite algebraic expression involving the polynomial coefficients. Contrary to this, if the goal is more modest, i.e., to reduce \( A \) to a tridiagonal matrix rather than to a diagonal one, it can be always achieved in a finite number of iterations \((\leq N - 2)\).

**References**

[1] Bindel and Goodman, Principles of scientific computing