Homework 5. Due Wed. Oct.10 by noon in my mailbox

Turn in your codes through ELMS, including testing scripts (with names including the string 'test' in their name) and drivers (including the string 'dr' in their name). Comment all your codes. Hand in your answers with print-outs of the graphs.

In two dimensions, a plane wave is a function defined for

- a given frequency $\omega \in \mathbb{R}$, such that $\omega > 0$,
- a given direction $d \in \mathbb{R}^2$, such that $|d| = 1$,

as

$$\phi_d : x \mapsto exp(i\omega d \cdot x).$$

We want to investigate some properties of a set of $N$ plane waves $\{\phi_\ell\}_{1 \leq \ell \leq N}$ with directions $\{d_\ell\}_{1 \leq \ell \leq N}$ and different numbers $N$ of functions. In particular, plane waves are known for their approximation properties of solutions to the Helmholtz equation

$$-\Delta u - \omega^2 u = 0,$$

so for a given solution $u_E$ we want to build a local approximation of $u_E$ as a linear combination of plane waves: around a point $x_0 \in \mathbb{R}^2$

$$u_E(x) \approx u_A(x) \text{ as } x \approx x_0, \text{ where } u_A := \sum_{\ell=1}^{N} \alpha_\ell \phi_\ell.$$

Procedure to build an order $n$ approximation $u_A$ at $x_0$ with $N$ plane waves of directions $\{d_\ell\}_{1 \leq \ell \leq N}$

For a fixed $n \in \mathbb{N}$, we seek for $\{\alpha_\ell\}_{1 \leq \ell \leq N}$ as a solution of the linear system

$$\forall (i_x, i_y) \in \mathbb{N}^2, i_x + i_y \leq n, \sum_{\ell=1}^{N} \alpha_\ell \partial_x^{i_x} \partial_y^{i_y} \phi_\ell(x_0) = \partial_x^{i_x} \partial_y^{i_y} u_E(x_0). \quad (1)$$

1. You will implement a function that computes the matrix of System 1, following these steps.

(a) Identify the inputs and outputs of your function.
(b) Choose an ordering of your equations.
(c) Implement and test the corresponding index function: it computes the index of the $(i_x, i_y)$ equation.
(d) Derive a formula for the entries of the matrix.
(e) Implement and test a function that builds one column of the matrix: it computes the derivatives of one plane wave into a vector, using the ordering from 1b.
(f) Implement and test a function that builds the full matrix.

2. You will implement a function that computes the corresponding right hand side, for the exact solution $u_E(x, y) = \exp\left(i\left(\cos\left(\frac{\pi}{4}\right) x + \sin\left(\frac{\pi}{4}\right) y\right)\right)$.
   
   (a) Derive a formula for the entries of the right hand side vector.
   (b) Implement and test a function that builds the right hand side.

Solving the system condition number

We now choose $\omega = 1$ and $N$ equi-spaced directions $\{d_\ell\}_{1 \leq \ell \leq N}$ as $d_\ell = (\cos \frac{2\pi \ell}{N}, \sin \frac{2\pi \ell}{N})$. We consider the linear system built at $x_0 = (1, -3)$ with matrix from 1. and right hand side from 2. in the following questions.

3. What is the size of this linear system? Justify.

4. We now turn to the rank of the matrix. According to the theory - see for instance Lemma 2 in [1] - the rank of the matrix is $2n + 1$ if $N = 2n + 1$.
   
   (a) Compute the rank of the matrix constructed from your function for $1 \leq n \leq 10$ and $1 \leq N \leq 11$. Present the results in a table as follows

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   (b) Verify that the matrix constructed from your function has rank $2n + 1$ for $1 \leq n \leq 10$ with $N = 2n + 1$, as predicted by the theory (from the entries of the previous table). Present the results in a graph showing the rank of the matrix as a function of $n$.

5. For $1 \leq n \leq 15$ with $N = 2n + 1$, solve the linear system using QR decomposition.

6. (a) In class we presented the Choleski factorization of real symmetric positive definite matrices. State the hypotheses and result of the Choleski factorization of complex matrices.
   (b) For $1 \leq n \leq 15$ with $N = 2n + 1$, solve the linear system using the Choleski decomposition of the normal equations.

7. To compare the two methods, present on a graph the condition number of the system solved as a function of $n$, and on a second graph the norm of the residue as a function of $n$. 

2
References

[1] Imbert-Gérard, Interpolation properties of generalized plane waves