Math 141H Homework 1 (Section 6.1. Work) Solution

Show all your work. Jumping to the right answer without minimum reasoning deserves no credit.

1. Find the volume of the solid whose base is the region given by the ellipse \( 4x^2 + y^2 = 1 \) such that
   (1) each cross-section perpendicular to the x axis is a solid square.
   (2) each cross-section perpendicular to the y axis is a solid square.

Solution: (1) Since the side length is \( 2y \), the area is \( A(x) = 4y^2 = 4(1 - 4x^2) \). Moreover, the integration end points are given by \( a = -\frac{1}{2} \) and \( b = \frac{1}{2} \). Therefore,

\[
V = \int_{-\frac{1}{2}}^{\frac{1}{2}} A(x)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} (4 - 16x^2)dx = \left( 4x - \frac{16x^3}{3} \right) \bigg|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{8}{3}.
\]

(2) Since the side length is \( 2x \), the area is \( A(y) = 4x^2 = 1 - y^2 \). Moreover, the integration end points are given by \( c = -1 \) and \( d = 1 \). Therefore,

\[
V = \int_{-1}^{1} A(y)dy = \int_{-1}^{1} (1 - y^2)dy = \left( y - \frac{y^3}{3} \right) \bigg|_{-2}^{2} = \frac{4}{3}.
\]

2. Let \( R \) be the region between the x axis and the graph of \( y = x(x^3 + 1)^\frac{1}{2} \) for \( 1 \leq x \leq 2 \). Calculate the volume \( V \) of the solid generated by revolving \( R \) around the x axis.

Solution [the disc method]: The volume is given by

\[
V = \int_{1}^{2} \pi x^2 \sqrt{x^3 + 1} dx.
\]

Let \( u = x^3 + 1 \implies du = 3x^2 dx \). Therefore,

\[
V = \int_{1}^{2} \pi x^2 \sqrt{x^3 + 1} dx = \frac{\pi}{3} \int_{u(1)}^{u(2)} \sqrt{u} du = \frac{\pi}{3} \int_{2}^{9} \sqrt{u} du = \frac{2\pi}{9} u^{\frac{3}{2}} \bigg|_{2}^{9} = \frac{2\pi}{9} \left( 27 - 2^\frac{3}{2} \right).
\]

3. Let \( R \) be the region bounded by the curve \( y = x^2 \) and \( y = 2x \). Find the volume of the solid obtained by revolving the region \( R \)
   (1) around the x axis.
   (2) around the y axis.

Solution: Two graphs meet when \( x^2 = 2x \implies x = 0, x = 2 \). On the interval \([0, 2]\), \( 2x \geq x^2 \).

(1) [the washer method]

\[
V = \int_{0}^{2} \pi ((2x)^2 - (x^2)^2) dx = \pi \int_{0}^{2} (4x^2 - x^4)dx = \pi \left( \frac{4x^3}{3} - \frac{x^5}{5} \right) \bigg|_{0}^{2} = \frac{64}{15}\pi.
\]
(2) [the shell method]

\[ V = \int_0^2 2\pi x(2x - x^2)dx = 2\pi \int_0^2 (2x^2 - x^3)dx = 2\pi \left( \frac{2x^3}{3} - \frac{x^4}{4} \right) \bigg|_2^0 = \frac{8}{3}\pi. \]

4. Find the volume \( V \) of the solid generated by revolving about the \( x \) axis the region between the graphs of \( y = \frac{1}{2}x^2 + 3 \) and \( y = 12 - \frac{1}{2}x^2 \).

**Solution [the washer method]:** Two curves meet when

\[ \frac{1}{2}x^2 + 4 = 12 - \frac{1}{2}x^2 \implies x^2 = 9 \implies x = \pm 3. \]

And \( 12 - \frac{1}{2}x^2 \geq \frac{1}{2}x^2 + 3 \geq 0 \) on the interval \([-3, 3]\). Therefore,

\[ V = \int_{-3}^3 \pi \left( (12 - \frac{1}{2}x^2)^2 - (\frac{1}{2}x^2 + 3)^2 \right) dx = \pi \int_{-3}^3 (135 - 15x^2)dx = \pi (135x - 5x^3) \bigg|_{-3}^3 = 540\pi. \]

5. Let \( f(x) = e^{x^2} \) and \( g(x) = e^{-x^2} \). Find the volume of the solid obtained by revolving the region between the graph of \( f \) and \( g \) for \( 0 \leq x \leq 1 \) around the \( y \) axis.

**Solution [the shell method]:** Since \( e^{x^2} \geq -e^{x^2} \) on the interval \([0, 1]\),

\[ V = \int_0^1 2\pi x \left( e^{x^2} - (-e^{x^2}) \right) dx = 4\pi \int_0^1 xe^{x^2} dx. \]

Let \( u = x^2 \implies du = 2xdx \). Therefore,

\[ V = 4\pi \int_0^1 xe^{x^2} dx = 2\pi \int_{u(0)}^{u(1)} e^u du = 2\pi (e - 1). \]

6. Find the volume of the ellipsoid obtained by rotating the ellipse \( x^2 + \frac{y^2}{4} = 1 \)

(1) around the \( x \) axis.

(2) around the \( y \) axis.

**Solution [the disc method]:** (1) Let \( y = f(x) = \sqrt{4(1-x^2)} \) and \( y = g(x) = 0 \) on the interval \([-1, 1]\). Then, \( A(x) = \pi y^2 = 4\pi(1-x^2) \). Therefore,

\[ V = 4\pi \int_{-1}^1 (1-x^2)dx = 4\pi \left( x - \frac{x^3}{3} \right) \bigg|_{-1}^1 = \frac{16}{3}\pi. \]

(2) We now exchange the role of \( x \) and \( y \). Let \( x = h(y) = \sqrt{1-\frac{y^2}{4}} \) and \( x = l(y) = 0 \) on the interval \([-2, 2]\). Then, \( A(y) = \pi x^2 = \pi \left( 1 - \frac{y^2}{4} \right) \). Therefore,

\[ V = \pi \int_{-2}^2 \left( 1 - \frac{y^2}{4} \right) dy = \pi \left( y - \frac{y^3}{12} \right) \bigg|_{-2}^2 = \frac{8}{3}\pi. \]
7. Find the volume $V$ of the solid generated by revolving about the $x$ axis the region between the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ on the interval $[0, \frac{\pi}{4}]$.

**Solution [the washer method]:** Since $\cos x \geq \sin x$ on the interval $[0, \frac{\pi}{4}]$,

$$V = \int_{0}^{\frac{\pi}{4}} \pi (\cos^2 x - \sin^2 x) \, dx = \pi \int_{0}^{\frac{\pi}{4}} \cos(2x) \, dx,$$

where we use the identity: $\cos^2 x - \sin^2 x = \cos(2x)$. Let $u = 2x \implies du = 2 \, dx$. Therefore,

$$V = \pi \int_{0}^{\frac{\pi}{4}} \cos(2x) \, dx = \frac{\pi}{2} \left[ \sin u \right]_{0}^{\frac{\pi}{4}} = \frac{\pi}{2}.$$

Alternatively, the volume can be obtained as follows.

$$V = \int_{0}^{\frac{\pi}{4}} \pi (\cos^2 x - \sin^2 x) \, dx = \pi \int_{0}^{\frac{\pi}{4}} \cos(x + \sin x) (\cos x - \sin x) \, dx.$$

Let $u = \cos x + \sin x \implies du = (\cos x - \sin x) \, dx$. Therefore,

$$V = \pi \int_{u(0)}^{u\left(\frac{\pi}{4}\right)} u \, du = \pi \int_{1}^{\sqrt{2}} u \, du = \frac{u^2}{2} \bigg|_{1}^{\sqrt{2}} = \frac{\pi}{2}.$$